Semi-implicit high-resolution numerical schemes for some conservation laws and level set problems

Peter FROLKOVIČ *

When solving numerically time dependent partial differential equations, main focus is often given to spatial (semi-)discretization methods combined afterwards with some straightforward time discretization methods. In this talk, we present a class of high-resolution semi-implicit methods for some representative partial differential equations used in conservation laws or in level set methods. Our motivation is similar to fully implicit time discretization methods as presented, e.g., for conservation laws in [1, 2, 3], that is, to offer unconditionally stable numerical schemes. Opposite to the quoted methods where the spatial and temporal discretizations are applied separately, we couple the both discretizations to obtain algebraic systems that are much easier to solve than in the case of fully implicit schemes.

The idea is based on "Inflow-Implicit/Outflow-Explicit" (I^2OE) methods as published in [5, 6], where such a coupling is clearly stated in its name. Related variants of the method were also successfully applied to nonlinear equations of the advective level set equation in [4, 7] including polyhedral meshes [9].

We extend the idea to a class of numerical schemes having one free parameter. The schemes are derived for the linear advection equation with variable velocity in non-conservative form in [8, 10] and for conservation laws in [10, 11] where several properties can be proved. The high-resolution forms having TVD (Total Variation Diminishing) property [1] are published in [11] and the WENO (Weighted Essentially Non-Oscillatory) forms [12] are in progress.

The methods can be formally derived using a partial Lax-Wendroff (or Cauchy-Kowalevski) procedure where the time derivatives in Taylor series are replaced by mixed derivatives exploiting the partial differential equations. This is a main modification of the standard L-W procedure in which the time derivatives are replaced by spatial derivatives only. Using some appropriate approximations of the mixed derivatives, one obtains a convenient form of the Jacobian for the resulting nonlinear algebraic equations (or the matrix in the case of linear equations) when the system of equations can be solved efficiently using, for example, a fast sweeping strategy [2].

References

[1] K. Duraisamy, J. D. Baeder. Implicit scheme for hyperbolic conservation laws using nonoscillatory reconstruction in space and time. *SIAM J. Sci. Comp.*, 29: 2607-2620, 2007.

[2] E. Lozano, T. D. Aslam. Implicit fast sweeping method for hyperbolic systems of conservation laws. *J. Comput. Phys.*, 430: 110039, 2021.

[3] G. Puppo, M. Semplice, G. Visconti. Quinpi: integrating conservation laws with CWENO implicit methods. *Commun. Appl. Math. Comput.*, 2022.

[4] K. Mikula, M. Ohlberger. A new level set method for motion in normal direction based on a semi-implicit forward-backward diffusion approach. *SIAM Journal on Scientific Computing*, 32: 1527–1544, 2010.

[5] K. Mikula, M. Ohlberger. Inflow-Implicit/Outflow-Explicit Scheme for Solving Advection Equations. *In: in Finite Volumes in Complex Applications VI*, Springer, 683–692, 2011.

[6] K. Mikula, M. Ohlberger, J.Urban. Inflow-Implicit/Outflow-Explicit finite volume methods for solving advection equations. *Appl. Num. Math.*, 85: 16-37, 2014.

[7] P. Frolkovič, K. Mikula, J. Urbán. Semi-implicit finite volume level set method for advective motion of interfaces in normal direction. *Appl. Num. Math.*, 95: 214–228, 2015.

[8] P. Frolkovič, K. Mikula. Semi-implicit second order schemes for numerical solution of level set advection equation on Cartesian grids. *Appl. Math. Comput.*, 329: 129–142, 2018.

[9] J. Hahn, K. Mikula, P. Frolkovič, M. Medl'a, B. Basara. Iterative inflow-implicit outflow-explicit finite volume scheme for level-set equations on polyhedron meshes. *Comp. Math. Appl.*, 77(6): 1639–1654, 2019.

[10] P. Frolkovič, S. Krišková, M, Rohová, M. Žeravý. Semi-implicit methods for advection equations with explicit forms of numerical solution. *arXiv:2106.15474, accepted to SIAM JJIAM*, 2022.

[11] P. Frolkovič, M. Žeravý. Semi-implicit high resolution numerical scheme for conservation laws. *arXiv preprint arXiv:2206.09425*, 2022.

[12] C.-W. Shu. Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws. *In: Advanced Numerical Approximation of Nonlinear Hyperbolic Equations, Lecture Notes in Mathematics*, Springer, 325–432, 1998.

*Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia. Email: peter.frolkovic@stuba.sk. This research has been supported by the projects VEGA 1/0709/19 and APVV 19-0460.