Analysis of a domain decomposition method for a convected Helmholtz like equation

Research School on Domain Decomposition for Optimal Control Problems
Chair Jean-Morlet - CIRM 2022

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Motivation

In this work, we are interested in solving an **Helmholtz like equation:**

$$-\mathrm{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

where:

- A is a 2x2 symmetric positive definite matrix,
- $\mathbf{a}$ is a vecteur of $\mathbb{R}^2$,
- μ is a real constant.

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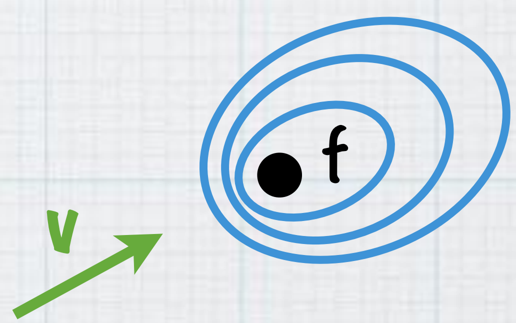
This type of equation occurs in several contexts:

* The **convected Helmholtz** equation $\left(A = c_0 I - \mathbf{v}\mathbf{v}^t,\ \mathbf{a} = -2\omega\mathbf{v},\ \mu = -\omega^2\right)$

$$-\mathrm{div}\left(\left(c_0 - \mathbf{v}\mathbf{v}^t\right)\nabla u\right) - 2i\omega\mathbf{v}\cdot\nabla u + \omega^2 u = f$$



H. Barucq et al, **HDG and HDG+ methods for harmonic wave problems with convections, 2021**

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* The **convected Helmholtz** equation ($A = c_0 I - \mathbf{v}\mathbf{v}^t$, $\mathbf{a} = -2\omega\mathbf{v}$, $\mu = -\omega^2$)
* The **Gröss-Pitaevskii** equation (computation of the ground states)

I. Danaila et al, Computation of ground states of the Gröss-Pitaevskii functional via Riemannian optimization, 2017

Motivation

In this work, we are interested in solving an Helmholtz like equation:

$$-\text{div}(A\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

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* The Gröss-Pitaevskii equation (computation of the ground states)
* The wave-ray equation ($A = I$, $\mathbf{a} = \mathbf{v}$, $\mu = 0$)

  $-\Delta u + i\mathbf{a}\cdot\nabla u = f$

P. Verburg et al, Multi-level wave-ray method for 2d Helmholtz equation, 2010

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Goal: Propose an **efficient iterative algorithm** of resolution

→ In short, it is **as difficult as solving the Helmholtz equation** !!

O.G. Ernst et al, **Why is it difficult to solve the Helmholtz equation ? 2012**

In the next...

1. Motivation
2. Link with Helmholtz equation
3. Convergence analysis on a toy problem
4. An alternated iterative algorithm
5. Conclusion

Link with the Helmholtz equation

Let us consider u solution to

$$-\mathrm{div}(A\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

Then, setting $u = e^{ik\cdot x}v'$ with $k = \frac{1}{2}A^{-1}\mathbf{a}$ one get that

$$-\mathrm{div}(A\nabla v') + \left(\mu - \frac{\|\mathbf{a}\|^2_{A^{-1}}}{4}\right)v' = f'$$

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Remark: Even if $\mu \geq 0$, we see that the problem is **not coercive** if $\|\mathbf{a}\|$ is large.

Link with the Helmholtz equation

Let us consider u solution to

$$-\mathsf{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla\mathsf{u} + \mu\mathsf{u} = \mathsf{f}$$

Then, setting $\mathsf{u} = e^{ik\cdot x}\mathsf{v}'$ with $k = \frac{1}{2}A^{-1}\mathbf{a}$ one get that

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Now, taking the change of variables $(x,y) \leftarrow T(x,y)$ where T is a matrix, we get

$$-\mathsf{div}(TAT^t\,\nabla \mathsf{v}) + \left(\mu - \frac{\|\mathbf{a}\|^2_{A^{-1}}}{4}\right)\mathsf{v} = \widetilde{\mathsf{f}}'$$

Link with the Helmholtz equation

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A simple idea then to obtain the Helmholtz equation is to take $T = G^{-1}$ where $A = GG^t$.

Link with the Helmholtz equation

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A simple idea then to obtain the Helmholtz equation is to take $T = G^{-1}$ where $A = GG^t$.

Remark: The choice of the transformation is not unique !

F.Q. Hu et al, On the use of Prandtl-Glauert-Lorentz transformation for acoustic scattering by rigid bodies with a uniform flow, 2019

Y. Gao et al, Wave scattering in layered orthotropic media I: a stable PML and a high-accuracy boundary integral equation method, 2021

Link with the Helmholtz equation

Cartesian PML formulation:

$$-\mathrm{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

Convected Helmholtz equation

Link with the Helmholtz equation

Cartesian PML formulation:

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla \mathsf{u} + \mu \mathsf{u} = \mathsf{f}$$

Convected Helmholtz equation

Coordinates transformation →

$$-\Delta \mathsf{v} + \widetilde{\varpi}^2 \mathsf{u} = \mathsf{f}$$

Helmholtz equation

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Convected Helmholtz equation

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$$-\Delta \mathsf{v} + \widetilde{\varpi}^2 \mathsf{u} = \mathsf{f}$$

Helmholtz equation

↓ Complex stretching $(x \to \alpha(x)x)$

$$-\text{div}(D_{PML}\,\nabla \mathsf{v}_{PML}) + \rho_{PML}\widetilde{\varpi}^2 \mathsf{u}_{PML} = \mathsf{f}$$

PML Helmholtz equation

Link with the Helmholtz equation

Cartesian PML formulation:

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla \mathsf{u} + \mu \mathsf{u} = \mathsf{f}$$

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$$-\Delta \mathsf{v} + \widetilde{\varpi}^2 \mathsf{u} = \mathsf{f}$$

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$$-\mathrm{div}(D_{PML}\nabla \mathsf{v}_{PML}) + \rho_{PML}\widetilde{\varpi}^2 \mathsf{u}_{PML} = \mathsf{f}$$

PML Helmholtz equation

← Coordinates transformation

$$-\mathrm{div}(A_{PML}\nabla \mathsf{u}_{PML}) + \frac{i}{2}\mathbf{a}_{PML}\cdot\nabla \mathsf{u}_{PML} + \frac{i}{2}\mathrm{div}(\mathbf{a}_{PML}\mathsf{u}_{PML}) + \mu_{PML}\mathsf{u}_{PML} = \mathsf{f}$$

PML Convected Helmholtz equation

P. Marchner et al, Stable Perfectly Matched Layers with Lorentz transformation for the convected Helmholtz equation, 2019

E. Becache et al, Perfectly matched layers for the convected Helmholtz equation, 2004

J. Diaz et al., Stabilized Perfectly Matched layer for advective acoustics, 2003

Link with the Helmholtz equation

Cartesian PML formulation:

$$-\text{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

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PML Helmholtz equation

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$$-\text{div}(A_{PML}\,\nabla u_{PML}) + \frac{i}{2}\mathbf{a}_{PML}\cdot\nabla u_{PML} + \frac{i}{2}\text{div}(\mathbf{a}_{PML}u_{PML}) + \mu_{PML}u_{PML} = f$$

PML Convected Helmholtz equation

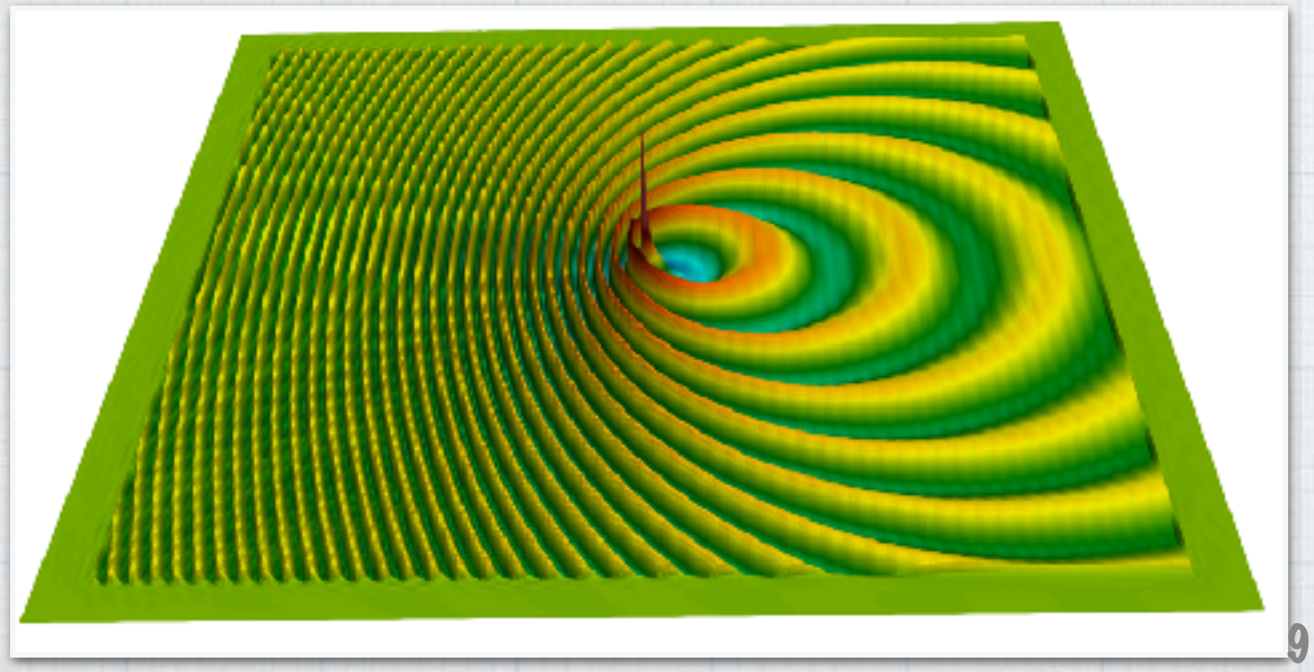

Illustration (Convected Hemlholtz):

$\mathbf{a} = 2\omega\mathbf{v}, \quad \omega = 20$

$\mathbf{v} = [0.8,\ 0]^t$

$A = Id - \mathbf{v}\mathbf{v}^t$

Link with the Helmholtz equation

Cartesian PML formulation:

$-\mathrm{div}(A\,\nabla u) + i\,a\cdot\nabla u + \mu u = f$

Convected Helmholtz equation

→ Coordinates transformation →

$-\Delta v + \widetilde{\omega}^2 u = f$

Helmholtz equation

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$-\mathrm{div}(D_{PML}\,\nabla v_{PML}) + \rho_{PML}\,\widetilde{\omega}^2 u_{PML} = f$

PML Helmholtz equation

← Coordinates transformation ←

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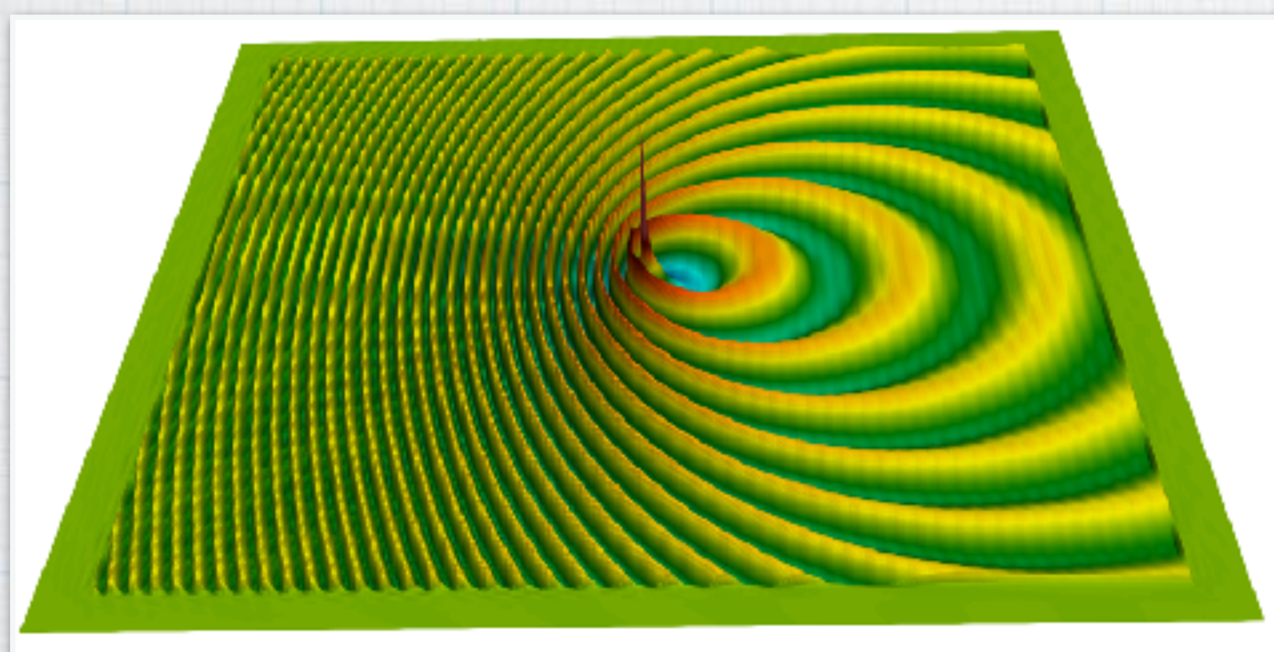
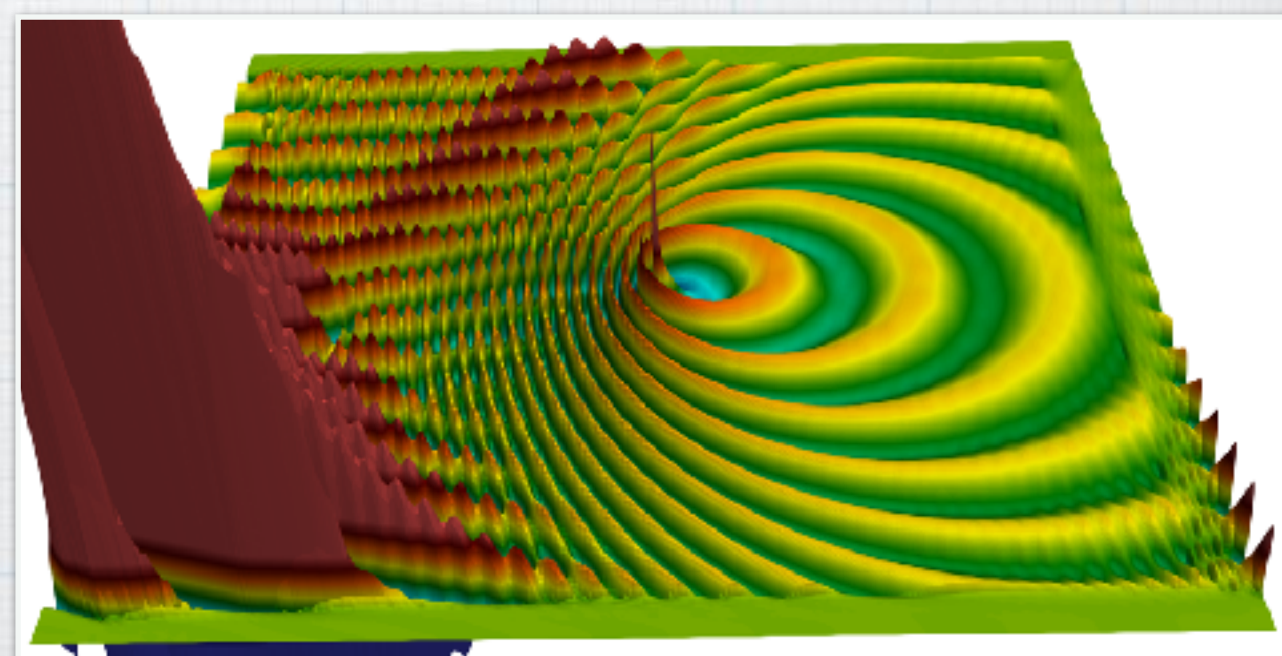
$-\mathrm{div}(A_{PML}\,\nabla u_{PML}) + \frac{i}{2}a_{PML}\cdot\nabla u_{PML}$

$+\frac{i}{2}\mathrm{div}(a_{PML}u_{PML}) + \mu_{PML}u_{PML} = f$

PML Convected Helmholtz equation





Link with the Helmholtz equation

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Coordinates transformation →

$$-\Delta v + \widetilde{\omega}^2 u = f$$

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PML CH equation

ABC (Absorbing Boundary Conditions):

N. Rouxelin et al, **Prandtl-Glauert-Lorentz based Absorbing Boundary Conditions for the convected Helmholtz equation**, 2021

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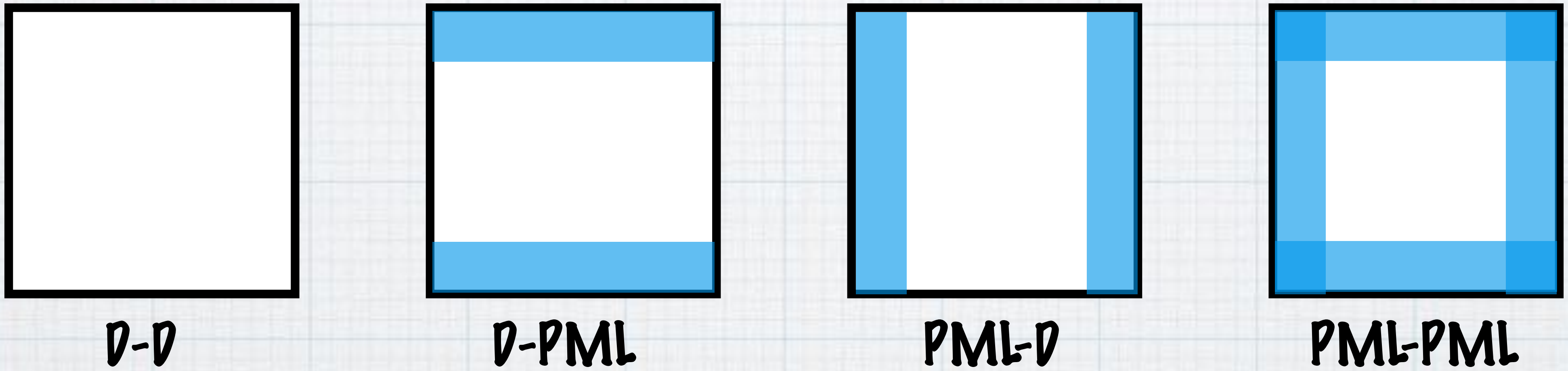
Convergence analysis on a toy problem

Formulation of the problem

$$\begin{cases} -\text{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

We will consider four configurations :



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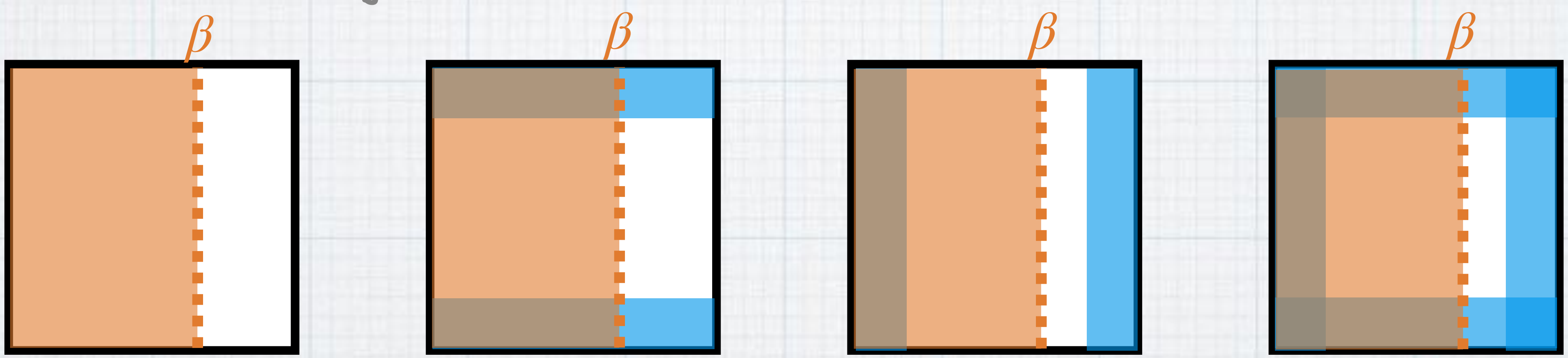

D-D D-PML PML-D PML-PML

In each case, we will consider a Schwarz iterative algorithm of resolution with 2 subdomains.

$$\begin{cases} \mathscr{L}_{CH} u^{1,n} = f_1 & \text{in } \Omega_1 \\ u^{1,n} = 0 & \text{on } \partial\Omega \\ (\partial_x + p_{1,2}) u^{1,n} = (\partial_x + p_{1,2}) u^{2,n-1} & \text{on } \Gamma_{12} \end{cases}$$

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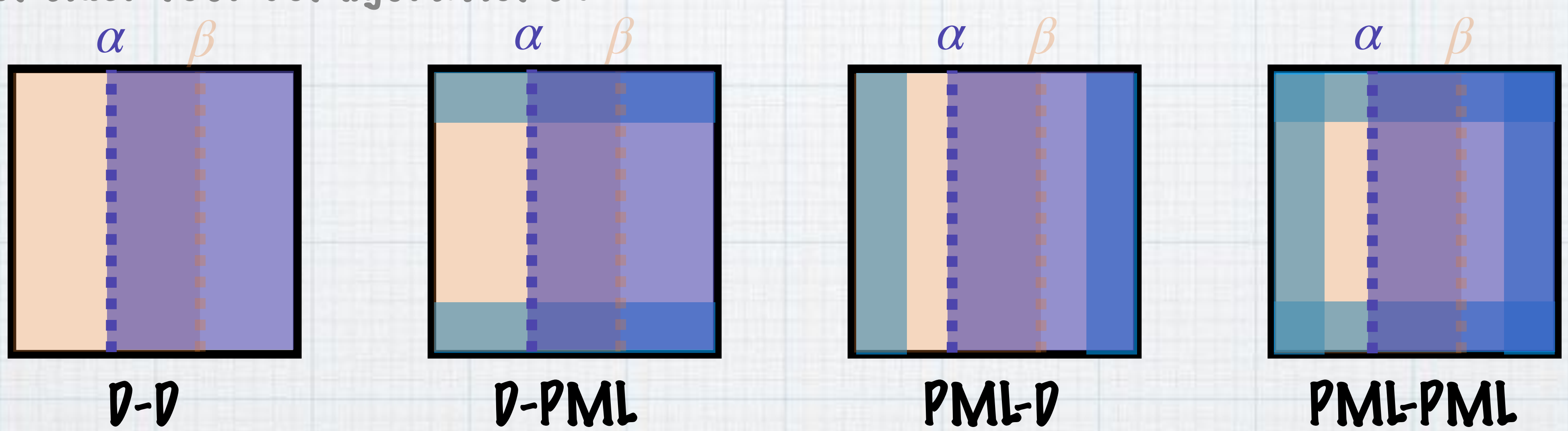

In each case, we will consider a **Schwarz iterative algorithm** of resolution with **2 subdomains**.

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$$\begin{cases} \mathscr{L}_{CH} u^{2,n} = f_2 & \text{in} \quad \Omega_2 \\ u^{2,n} = 0 & \text{on} \quad \partial\Omega \\ (\partial_x + p_{2,1})u^{2,n} = (\partial_x + p_{2,1})u^{1,n-1} & \text{on} \quad \Gamma_{21} \end{cases}$$

Convergence analysis on a toy problem

Schwarz algorithm:

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Equivalent Schwarz algorithm for Helmholtz equation:

$$\begin{cases} \mathscr{L}_{H}\mathsf{v}^{1,n} = \widetilde{\mathsf{f}}_1 & \text{in} \quad \widetilde{\Omega}_1 \\ \mathsf{v}^{1,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathsf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathsf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

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Convergence analysis on a toy problem

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Remarks: * The convergence analysis can be done only for the Helmholtz equation.

Convergence analysis on a toy problem

Schwarz algorithm:

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Remarks:

* The convergence analysis can be done only for the Helmholtz equation.
* To preserve the separable geometry in the Helmholtz case, we need to assume that A is diagonale.

Convergence analysis on a toy problem

Schwarz algorithm:

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Remarks: * Optimized TC can be derived from optimized TC for the Helmholtz equation

M.J. Gander et al, Optimized schwarz methods with overlap for the helmholtz equation 2016

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases}$$

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Remark: For simplicity, we will assume that $A = Id$ s.t. $\widetilde{\Omega} = [0,1]^2$, $\widetilde{\Gamma}_{1,2} = \{\beta\} \times [0,1]$ and $\widetilde{\Gamma}_{2,1} = \{\beta\} \times [0,1]$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \sin(\xi \pi y)\left(\mathbf{A}^{i,n}(\xi) e^{i\mathscr{S}(\xi)x} + \mathbf{B}^{i,n}(\xi) e^{-i\mathscr{S}(\xi)x}\right) \quad \text{where } \mathscr{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}.$$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \sin(\xi \pi y)\left(\mathbf{A}^{i,n}(\xi) e^{i\mathcal{S}(\xi)x} + \mathbf{B}^{i,n}(\xi) e^{-i\mathcal{S}(\xi)x}\right) \quad \text{where } \mathcal{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}.$$

The boundary conditions implies: $\mathbf{B}^{1,n}(\xi) = -\mathbf{A}^{1,n}(\xi)$ and $\mathbf{B}^{2,n}(\xi) = -e^{2i\mathcal{S}(\xi)}\mathbf{A}^{2,n}(\xi)$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases}$$

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The TC implies:

$$\mathbf{A}^{1,n}(\xi) = \frac{e^{i\mathcal{S}(\xi)\beta}(i\mathcal{S}(\xi) + \widetilde{p}_{1,2}) + e^{2i\mathcal{S}(\xi)} e^{-i\mathcal{S}(\xi)\beta}(i\mathcal{S}(\xi) - \widetilde{p}_{1,2})}{e^{i\mathcal{S}(\xi)\beta}(i\mathcal{S}(\xi) + \widetilde{p}_{1,2}) + e^{-i\mathcal{S}(\xi)\beta}(i\mathcal{S}(\xi) - \widetilde{p}_{1,2})}\mathbf{A}^{2,n-1}(\xi)$$

$$\mathbf{A}^{2,n}(\xi) = \frac{e^{i\mathcal{S}(\xi)\alpha}(i\mathcal{S}(\xi) + \widetilde{p}_{2,1}) + e^{-i\mathcal{S}(\xi)\alpha}(i\mathcal{S}(\xi) - \widetilde{p}_{2,1})}{e^{i\mathcal{S}(\xi)\alpha}(i\mathcal{S}(\xi) + \widetilde{p}_{2,1}) + e^{2i\mathcal{S}(\xi)} e^{-i\mathcal{S}(\xi)\alpha}(i\mathcal{S}(\xi) - \widetilde{p}_{2,1})}\mathbf{A}^{1,n-1}(\xi)$$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

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$$\mathbf{v}^{i,n} = \sum_{\xi\in\mathbb{N}} N_\xi \sin(\xi\pi y)\left(\mathbf{A}^{i,n}(\xi)e^{i\mathcal{S}(\xi)x} + \mathbf{B}^{i,n}(\xi)e^{-i\mathcal{S}(\xi)x}\right) \quad \text{where } \mathcal{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}.$$

The boundary conditions implies: $\mathbf{B}^{1,n}(\xi) = -\mathbf{A}^{1,n}(\xi)$ and $\mathbf{B}^{2,n}(\xi) = -\,e^{2i\mathcal{S}(\xi)}\mathbf{A}^{2,n}(\xi)$

The TC implies: $\mathbf{A}^{1,n}(\xi) = \rho_1^{DD}(\xi)\mathbf{A}^{2,n-1}(\xi)$

$\mathbf{A}^{2,n}(\xi) = \rho_2^{DD}(\xi)\mathbf{A}^{1,n-1}(\xi)$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases} \qquad \begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

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The boundary conditions implies: $\mathbf{B}^{1,n}(\xi) = -\mathbf{A}^{1,n}(\xi)$ and $\mathbf{B}^{2,n}(\xi) = -\,e^{2i\mathcal{S}(\xi)}\mathbf{A}^{2,n}(\xi)$

The TC implies:

$$\mathbf{A}^{1,n}(\xi) = \rho_1^{DD}(\xi)\mathbf{A}^{2,n-1}(\xi) \longrightarrow \mathbf{A}^{1,n}(\xi) = \rho_1^{DD}(\xi)\rho_2^{DD}(\xi)\mathbf{A}^{1,n-2}(\xi)$$

$$\mathbf{A}^{2,n}(\xi) = \rho_2^{DD}(\xi)\mathbf{A}^{1,n-1}(\xi) \longrightarrow \mathbf{A}^{2,n}(\xi) = \rho_1^{DD}(\xi)\rho_2^{DD}(\xi)\mathbf{A}^{2,n-2}(\xi)$$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\left|\begin{array}{lll} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in} & \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right.$$

$$\left|\begin{array}{lll} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in} & \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

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$$\mathbf{A}^{2,n}(\xi) = \rho_2^{DD}(\xi)\mathbf{A}^{1,n-1}(\xi) \longrightarrow \mathbf{A}^{2,n}(\xi) = \rho_1^{DD}(\xi)\rho_2^{DD}(\xi)\mathbf{A}^{2,n-2}(\xi)$$

$$\rho_1^{DD}(\xi)\rho_2^{DD}(\xi) := \rho^{DD}(\xi)$$

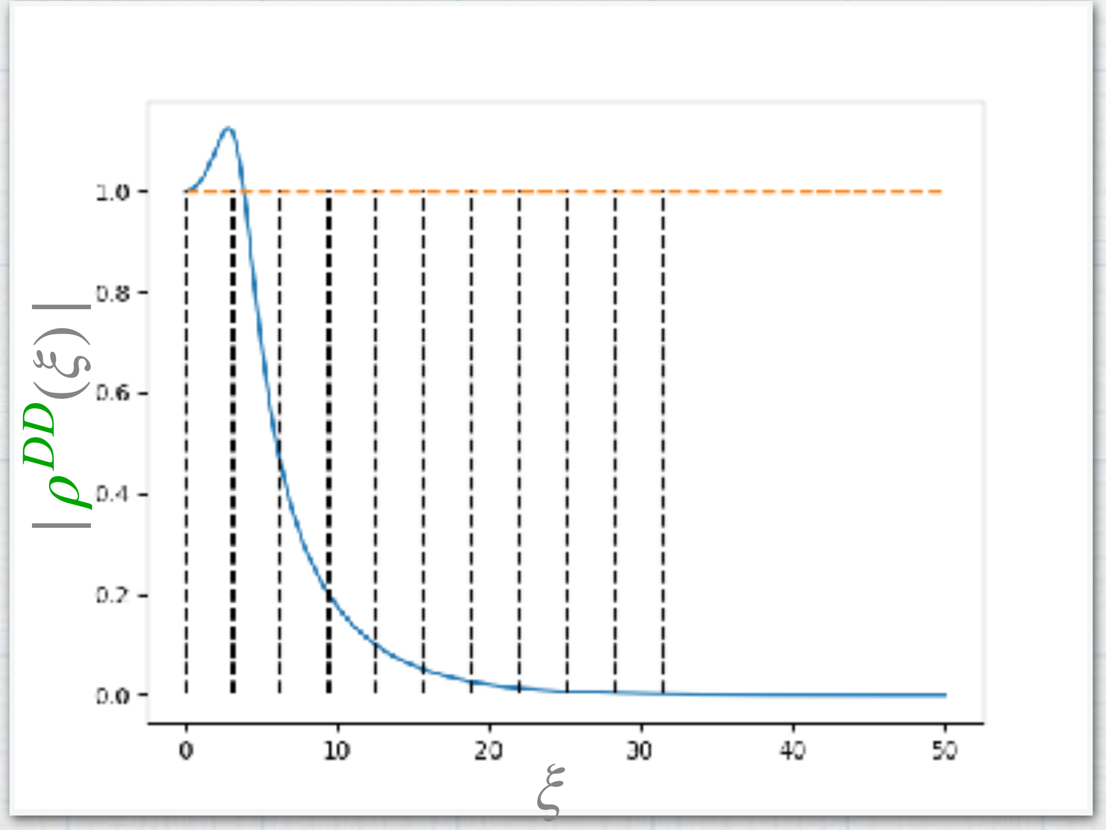
Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in} \quad \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in} \quad \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} \quad \widetilde{\Gamma}_{21} \end{cases}$$

Illustration on an example (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\widetilde{p}_{1,2} = \widetilde{p}_{2,1} = i\omega$)



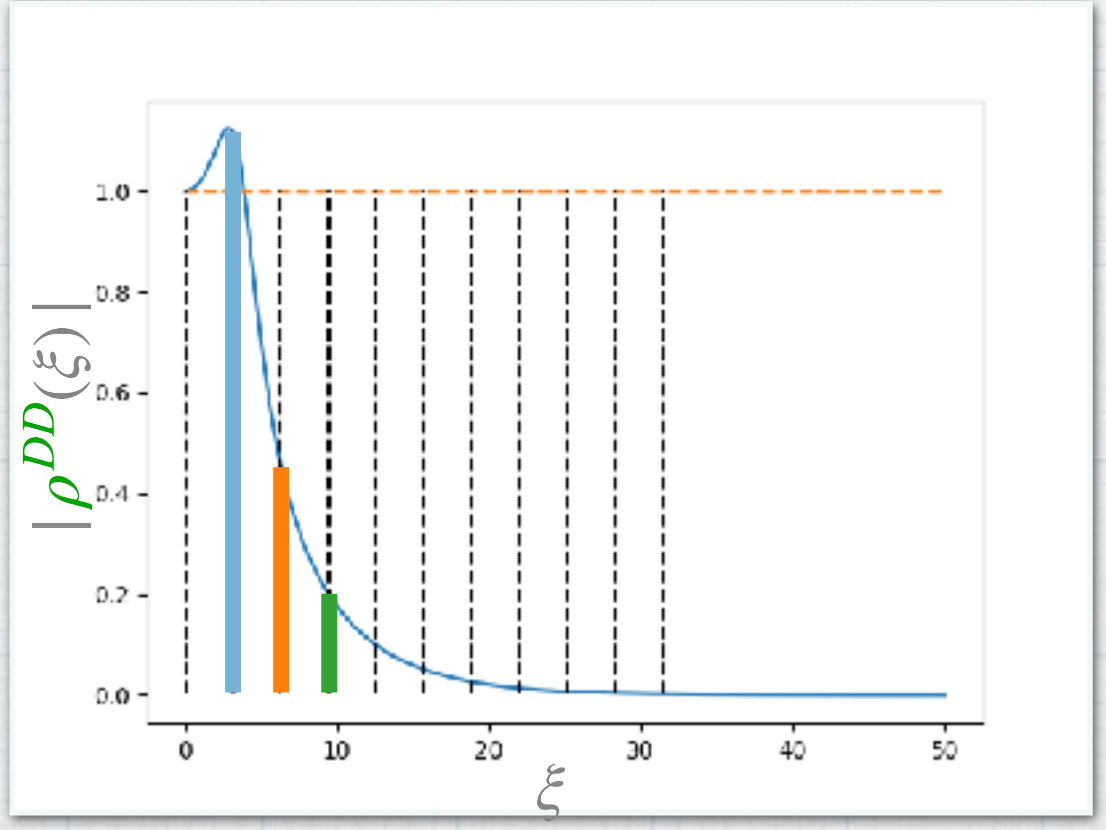
Convergence analysis on a toy problem

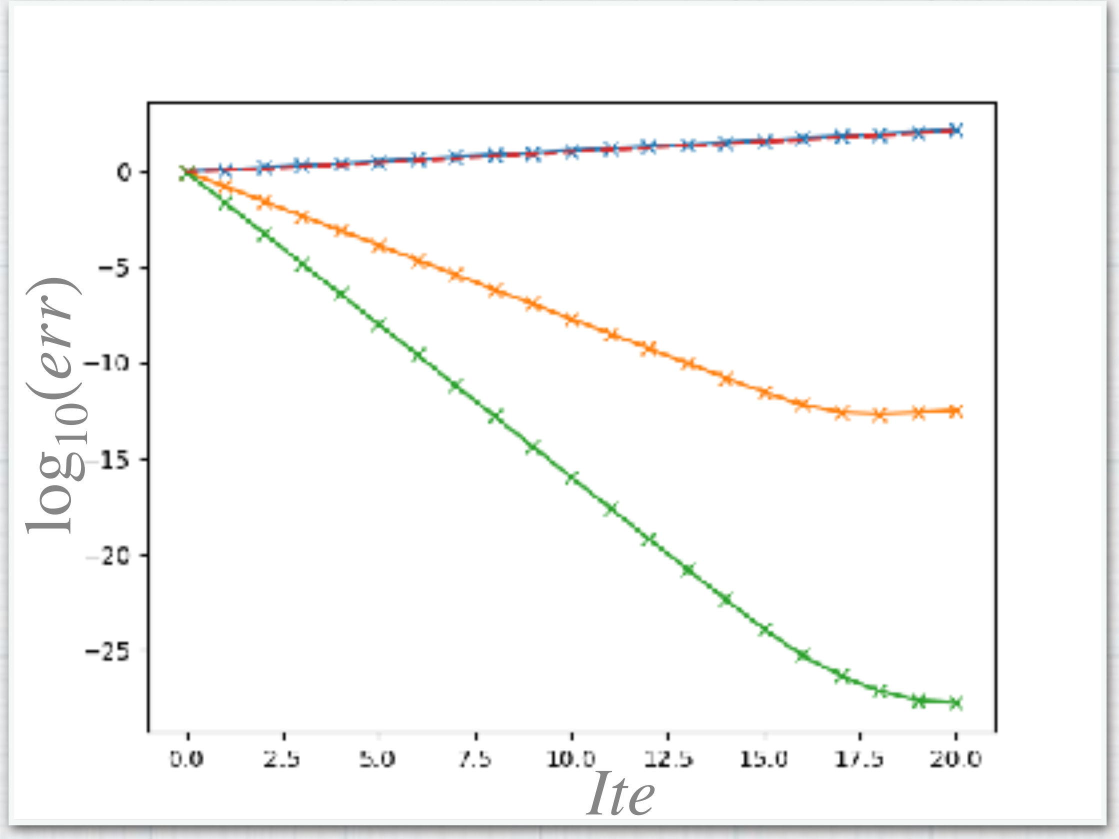
Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Illustration on an example (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\widetilde{p}_{1,2} = \widetilde{p}_{2,1} = i\omega$)





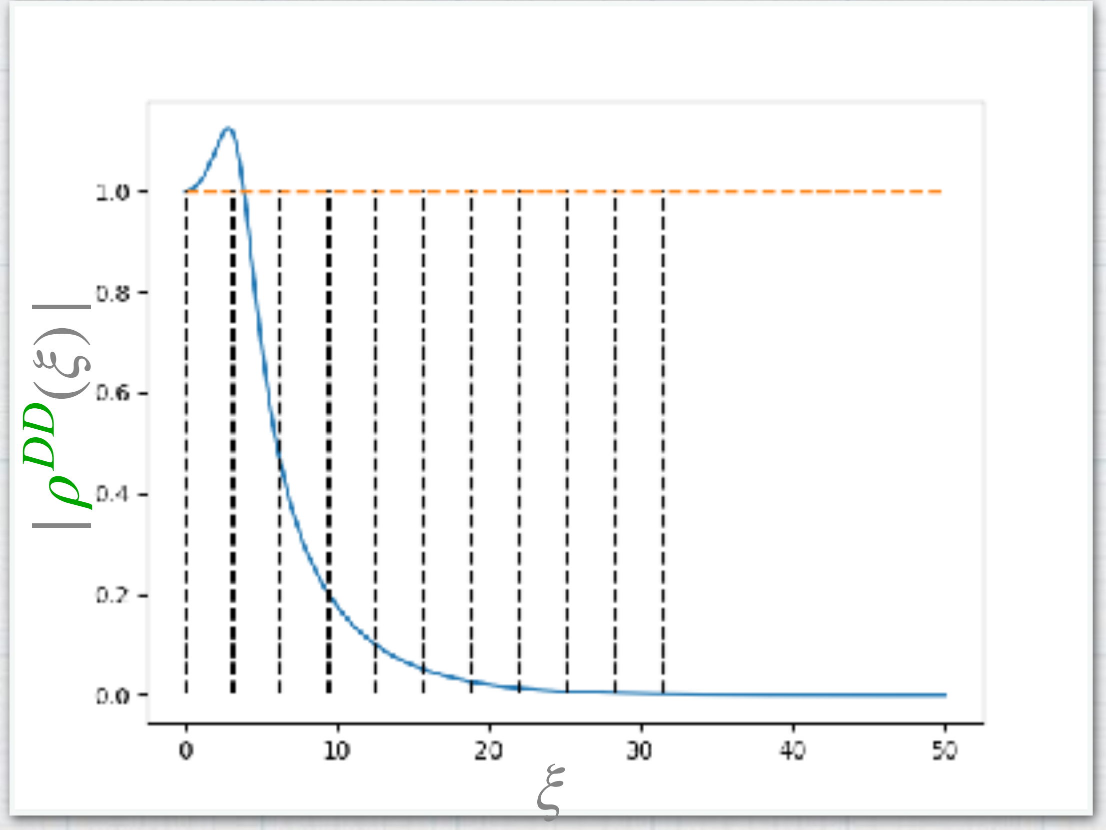
Convergence analysis on a toy problem

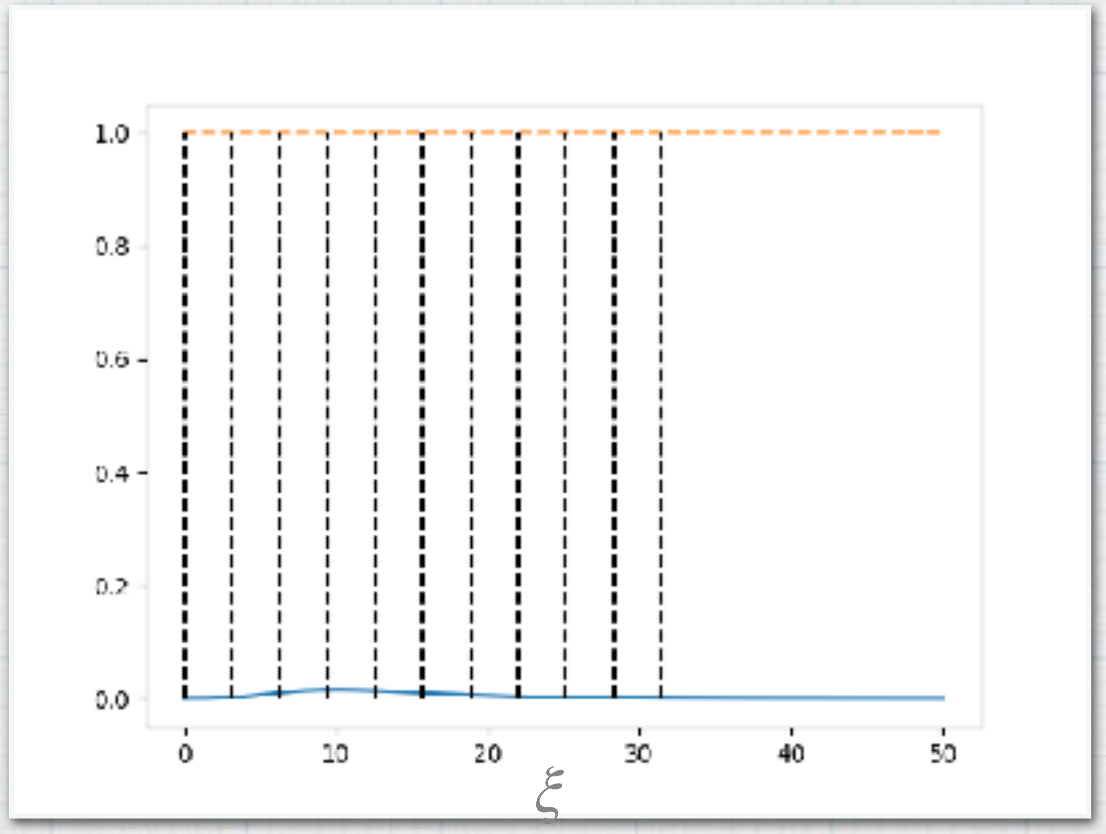
Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Illustration on an example (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$ vs Advection-Diffusion)





Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\begin{cases} \mathcal{L}_H^{PML} \mathbf{v}^{1,n} = 0 & \text{in} \quad \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathcal{L}_H^{PML} \mathbf{v}^{2,n} = 0 & \text{in} \quad \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} \quad \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \Psi_\xi(y) \left(\mathbf{A}^{i,n}(\xi) e^{i\mathcal{S}(\lambda_\xi)x} + \mathbf{B}^{i,n}(\xi) e^{-i\mathcal{S}(\lambda_\xi)x} \right)$$

where $\left(\Psi_\xi, \lambda_\xi\right)$ are the eigenfunctions and eigenvalues of

$$\begin{cases} -\partial_{\widetilde{y}}^2 \Psi_\xi = {\lambda_\xi}^2 \Psi_\xi, & y \in [0,1] \\ \Psi_\xi = 0, & y \in \{0,1\} \end{cases}$$

where $\widetilde{y}$ is the complex stretched coordinate.

Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\begin{cases} \mathscr{L}_H^{PML}\mathbf{v}^{1,n} = 0 & \text{in} \quad \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H^{PML}\mathbf{v}^{2,n} = 0 & \text{in} \quad \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} \quad \widetilde{\Gamma}_{21} \end{cases}$$

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$$\begin{cases} -\partial_{\widetilde{y}}^2 \Psi_\xi = \lambda_\xi^{\ 2}\, \Psi_\xi, & y \in [0,1] \\ \Psi_\xi = 0, & y \in \{0,1\} \end{cases}$$

where $\widetilde{y}$ is the complex stretched coordinate.

Remark: The eigenfunctions $\left(\Psi_\xi\right)_\xi$ form a complete basis of $L^2([\ell, 1-\ell])$.

L.F. Knockaert et al, On the completeness of eigenmodes in a parallel plate waveguide with a perfectly matched layer termination, 2002

Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\begin{cases} \mathscr{L}_H^{PML} \mathbf{v}^{1,n} = 0 & \text{in} \quad \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2}) \mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2}) \mathbf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H^{PML} \mathbf{v}^{2,n} = 0 & \text{in} \quad \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1}) \mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1}) \mathbf{v}^{1,n-1} & \text{on} \quad \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \Psi_\xi(y) \Big(\mathbf{A}^{i,n}(\xi) e^{i\mathcal{S}(\lambda_\xi)x} + \mathbf{B}^{i,n}(\xi) e^{-i\mathcal{S}(\lambda_\xi)x} \Big).$$

Similar calculations as before show that

$$\mathbf{A}^{1,n}(\xi) = \rho_1^{DPML}(\xi) \mathbf{A}^{2,n-1}(\xi)$$

$$\mathbf{A}^{2,n}(\xi) = \rho_2^{DPML}(\xi) \mathbf{A}^{1,n-1}(\xi)$$

Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\begin{cases} \mathcal{L}_H^{PML}\mathbf{v}^{1,n} = 0 & \text{in} \quad \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathcal{L}_H^{PML}\mathbf{v}^{2,n} = 0 & \text{in} \quad \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} \quad \widetilde{\Gamma}_{21} \end{cases}$$

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Similar calculations as before show that

$$\mathbf{A}^{1,n}(\xi) = \rho_1^{DPML}(\xi)\mathbf{A}^{2,n-1}(\xi) \longrightarrow \mathbf{A}^{1,n}(\xi) = \rho_1^{DPML}(\xi)\rho_2^{DPML}(\xi)\mathbf{A}^{1,n-2}(\xi)$$

$$\mathbf{A}^{2,n}(\xi) = \rho_2^{DPML}(\xi)\mathbf{A}^{1,n-1}(\xi) \longrightarrow \mathbf{A}^{2,n}(\xi) = \underbrace{\rho_1^{DPML}(\xi)\rho_2^{DPML}(\xi)}_{:= \rho^{DPML}(\xi)}\mathbf{A}^{2,n-2}(\xi)$$

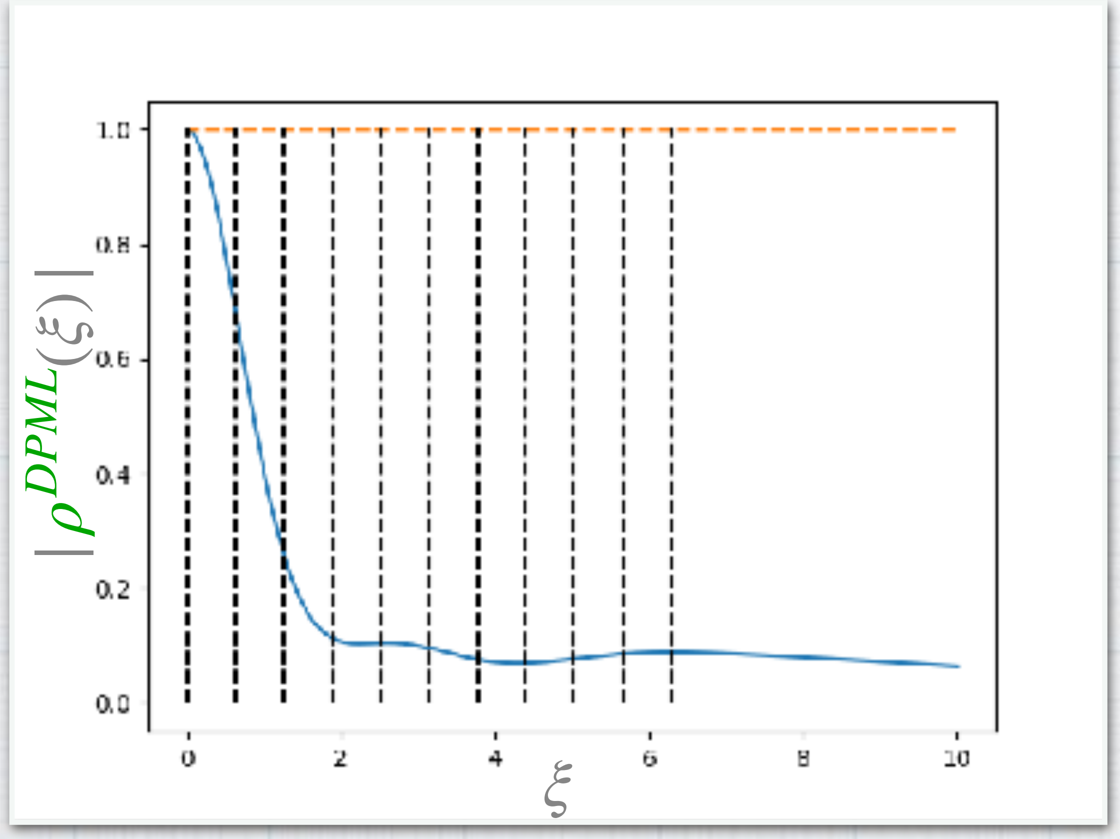
Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\left| \begin{array}{lll} \mathscr{L}_H^{PML}\mathbf{v}^{1,n} = 0 & \text{in} & \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array} \right.$$

$$\left| \begin{array}{lll} \mathscr{L}_H^{PML}\mathbf{v}^{2,n} = 0 & \text{in} & \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array} \right.$$

Illustration on an example (Wave Ray: $-\Delta\mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\widetilde{p}_{1,2} = \widetilde{p}_{2,1} = i\omega$)



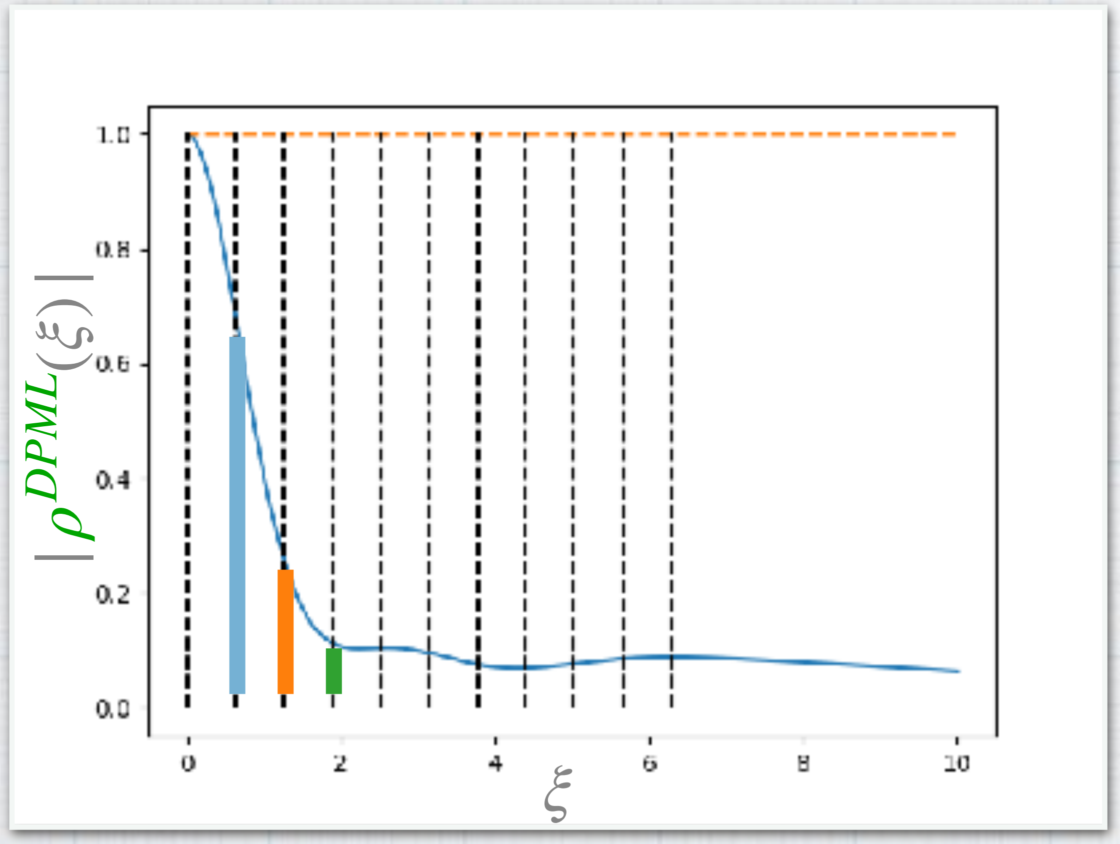
Convergence analysis on a toy problem

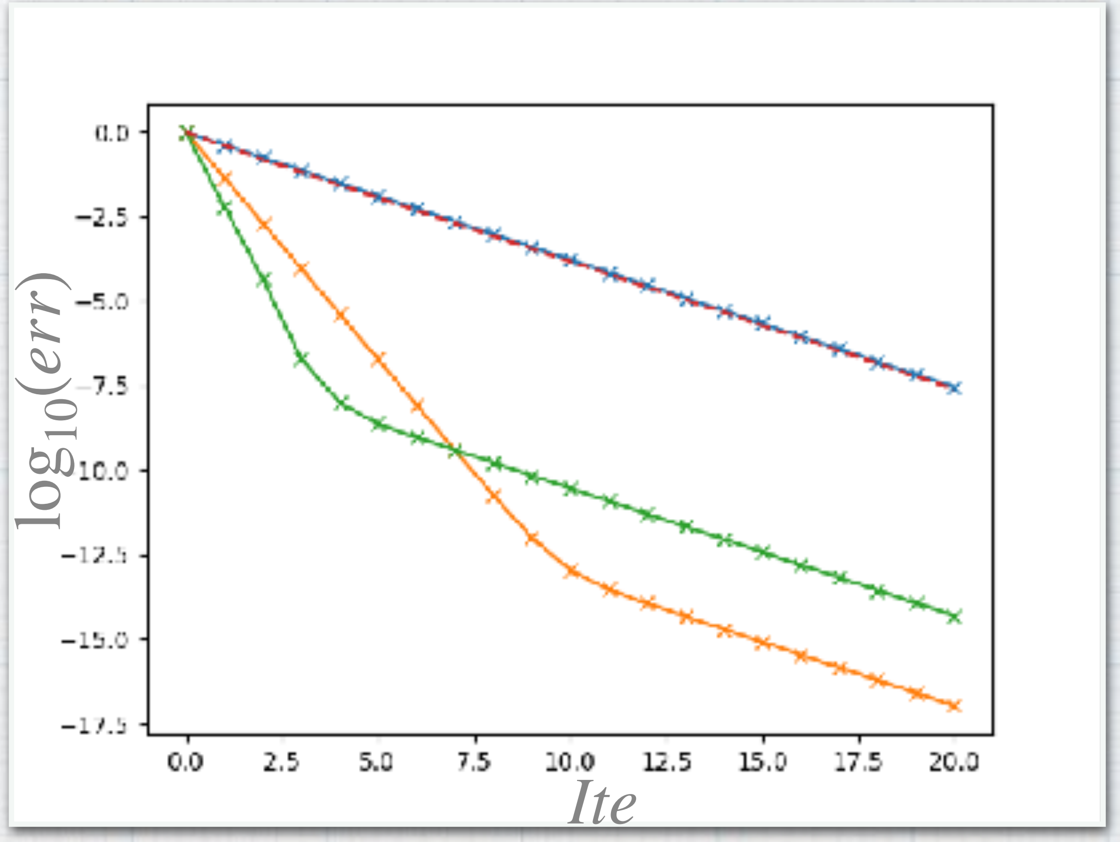
Convergence analysis (D-PML case)

$$\begin{cases} \mathscr{L}_H^{PML}\mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H^{PML}\mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

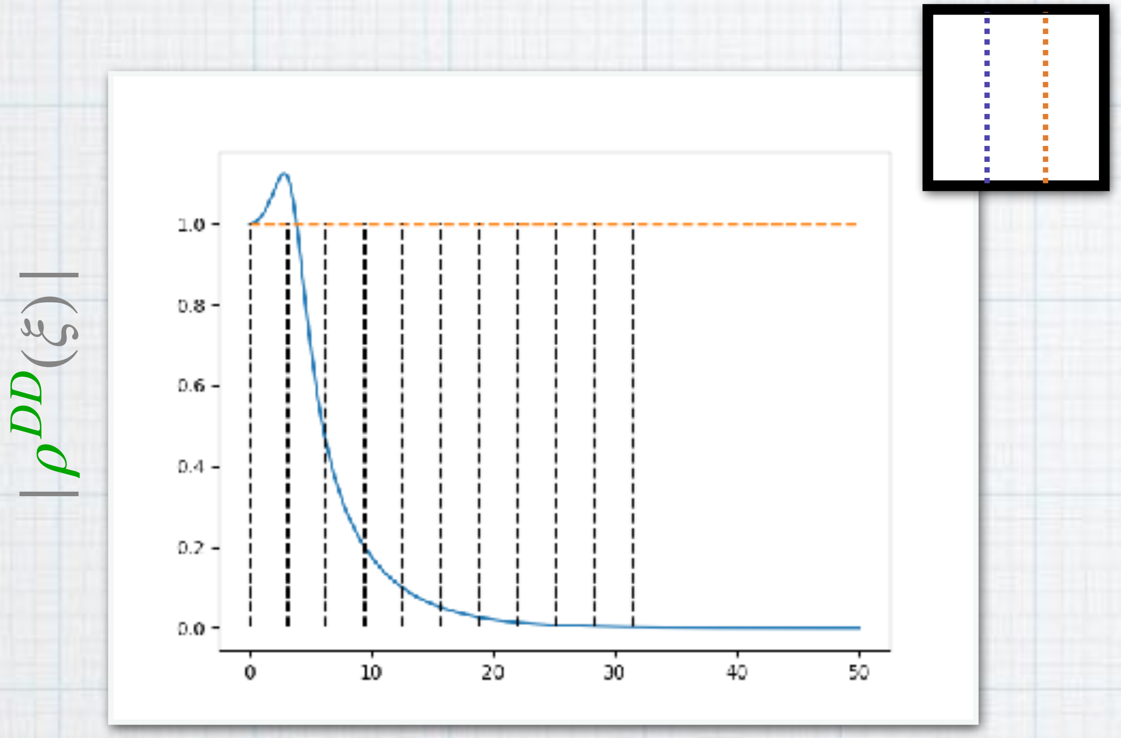
Illustration on an example (Wave Ray: $-\Delta\mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\widetilde{p}_{1,2} = \widetilde{p}_{2,1} = i\omega$)

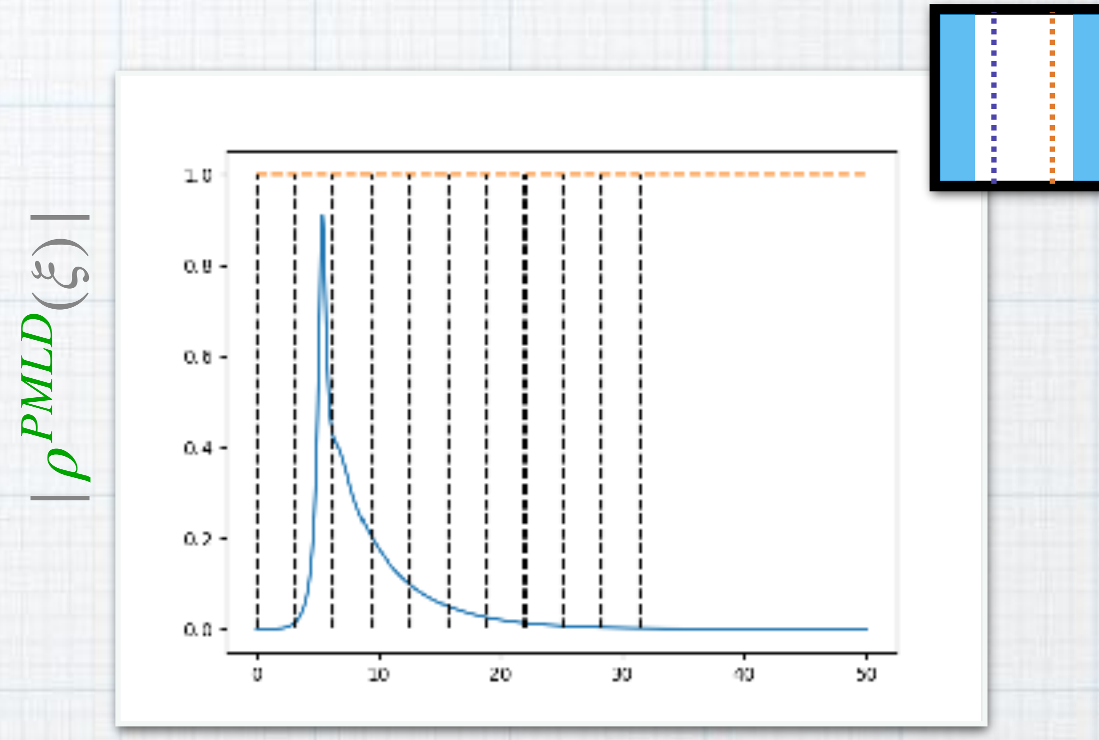


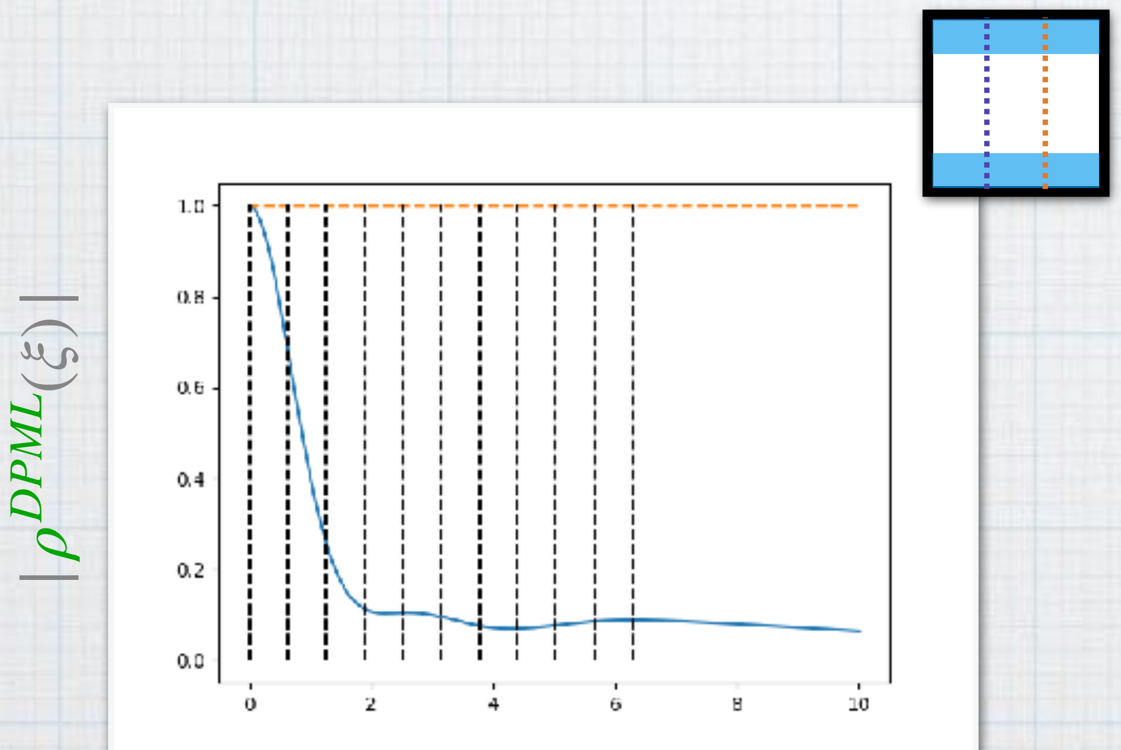


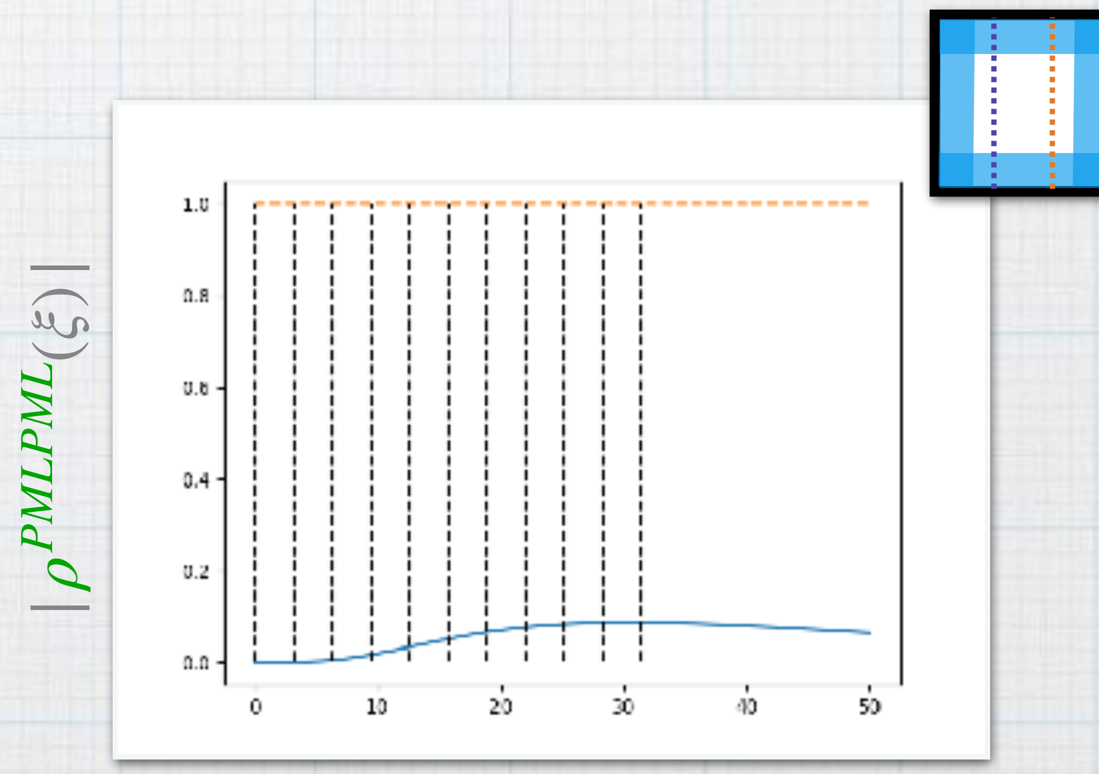
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\sigma_{PML} = 10$)



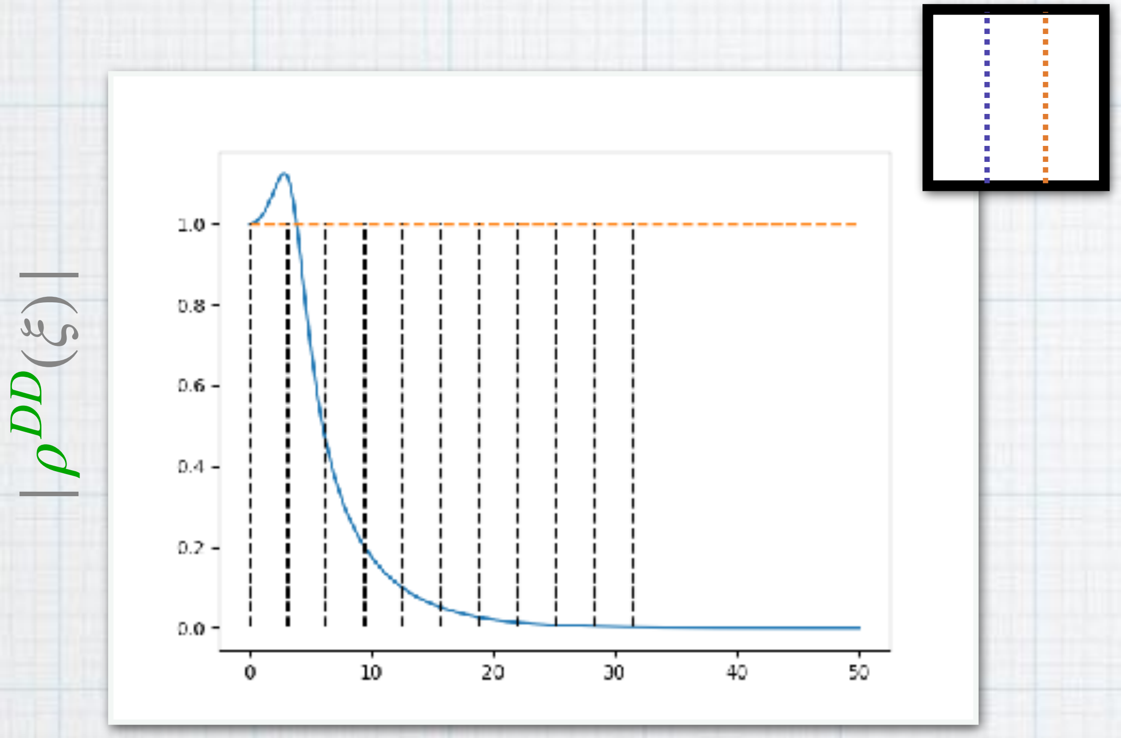


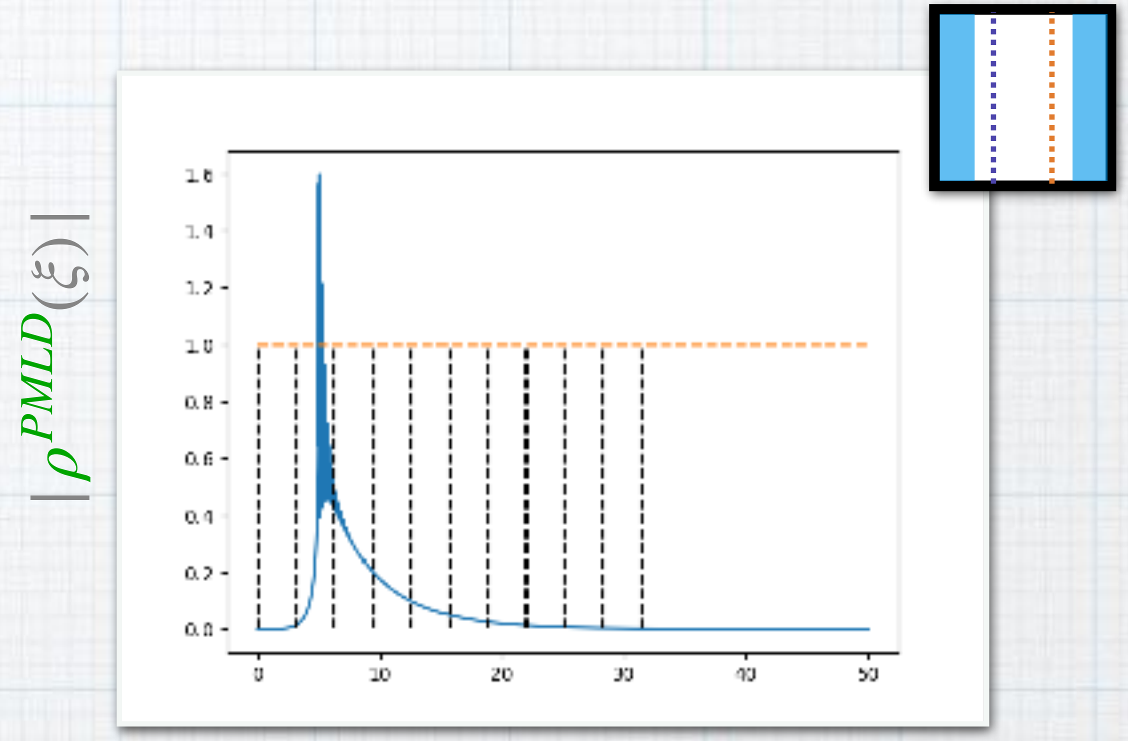


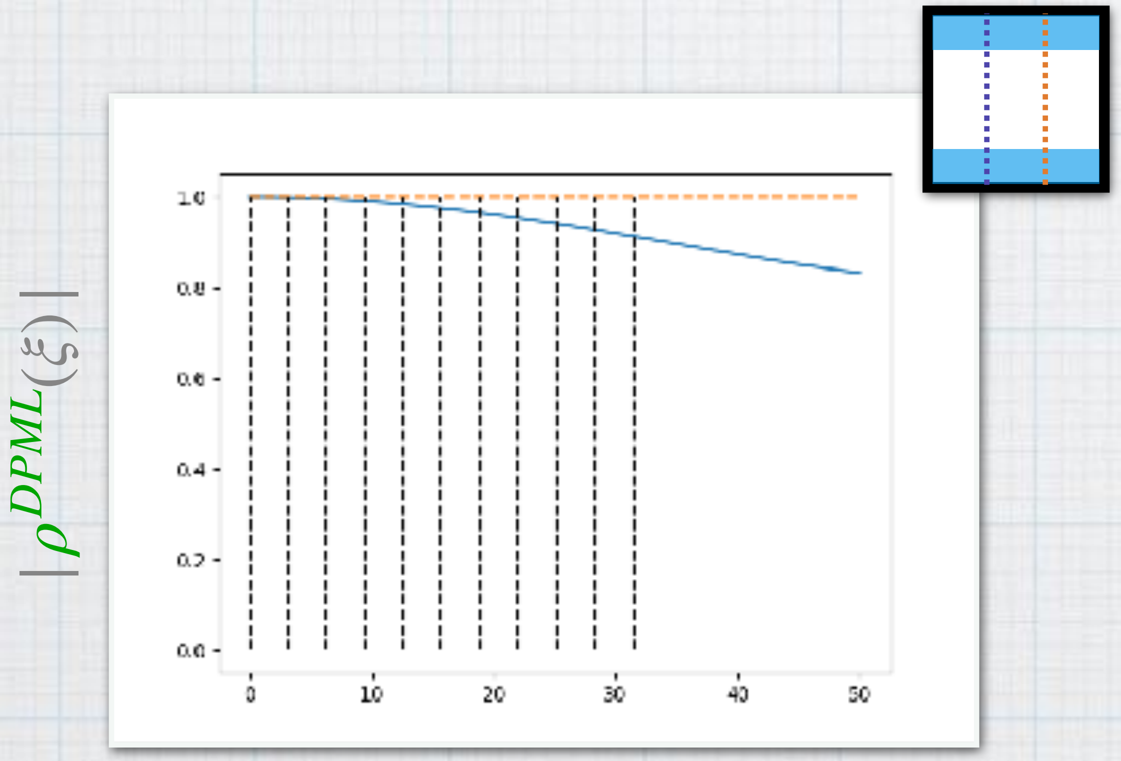


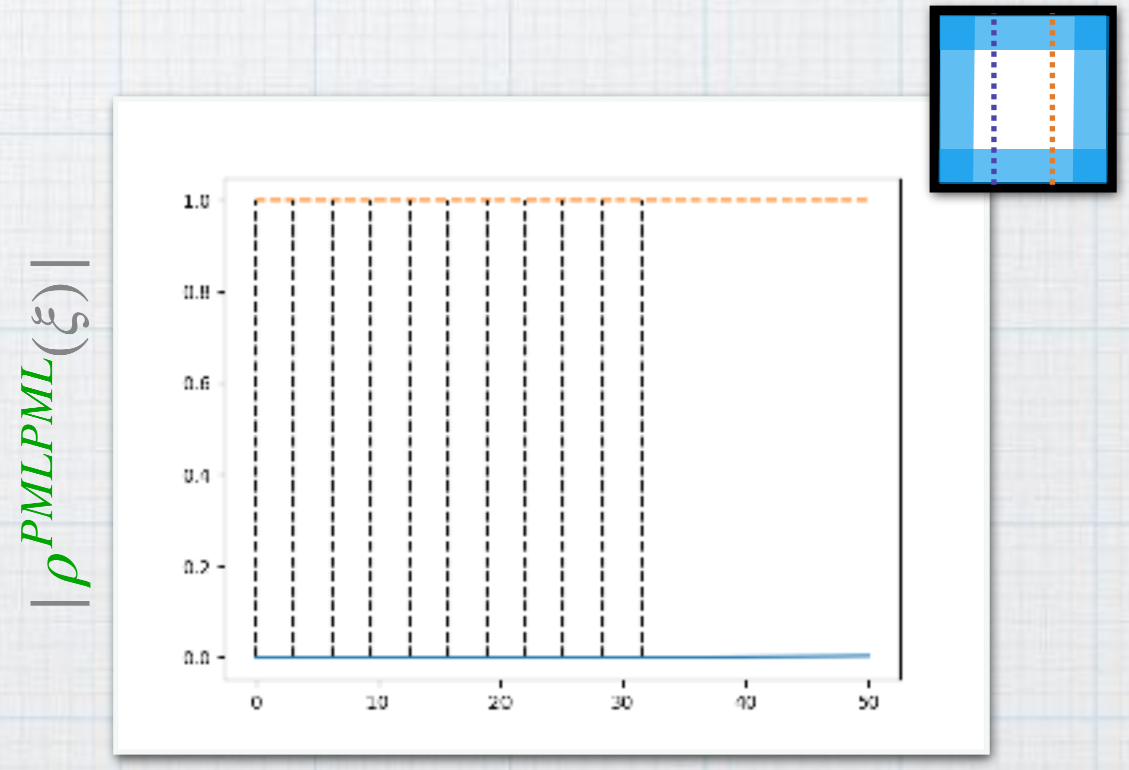
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta\mathbf{u}+i\mathbf{a}\cdot\nabla\mathbf{u}=0,\ \widetilde{\omega}=5,\ \sigma_{PML}=50$)





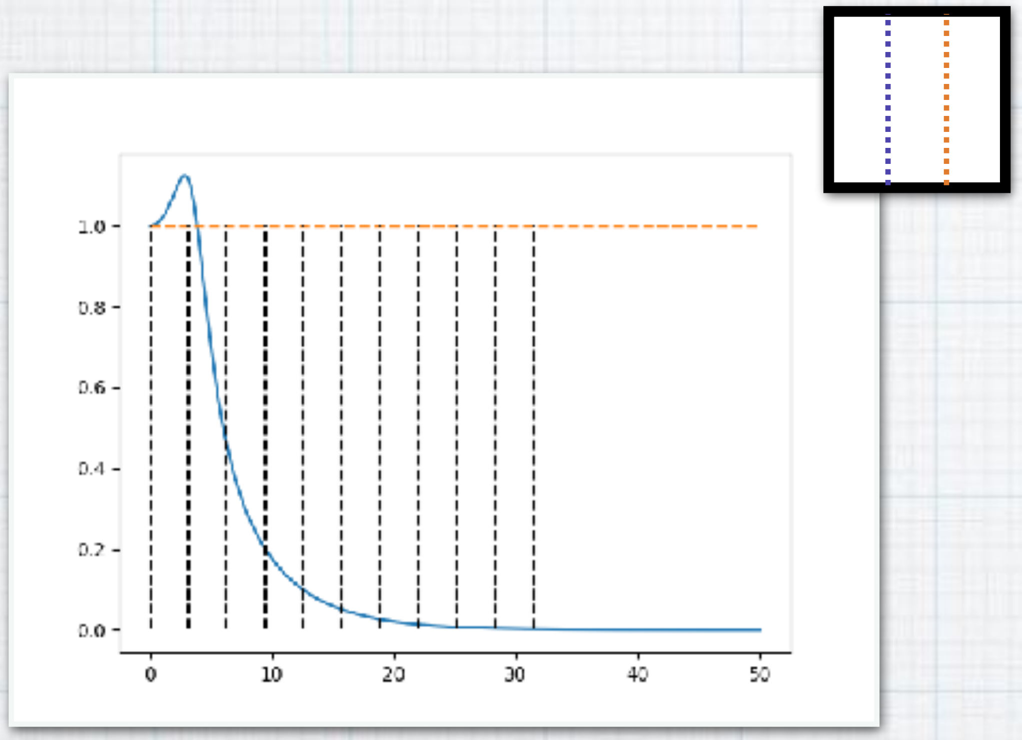




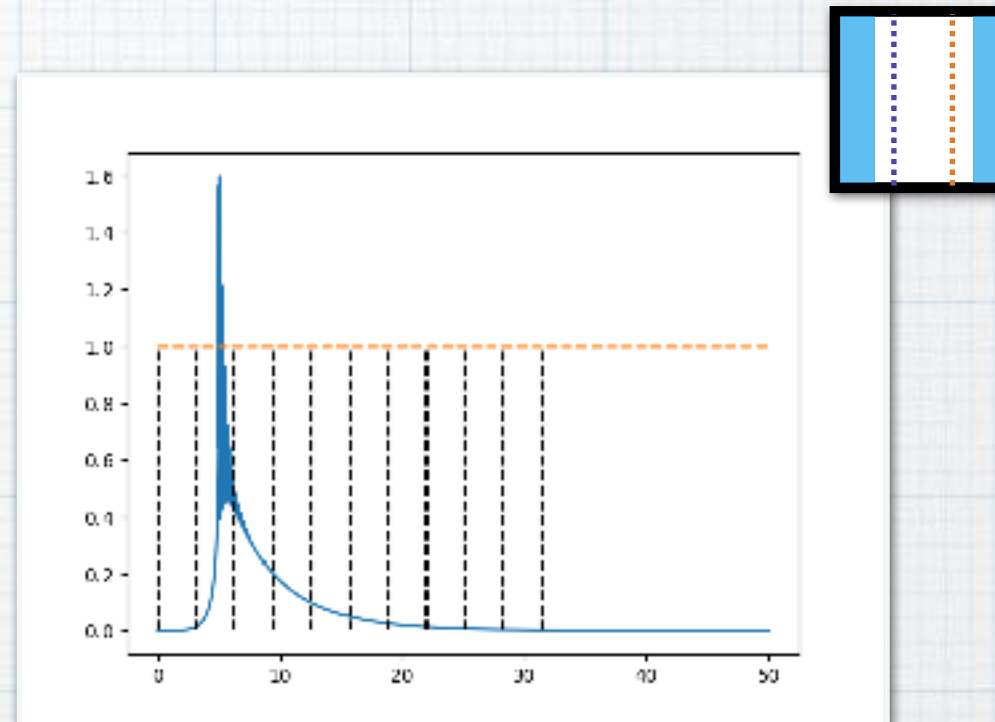
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\sigma_{PML} = 50$)

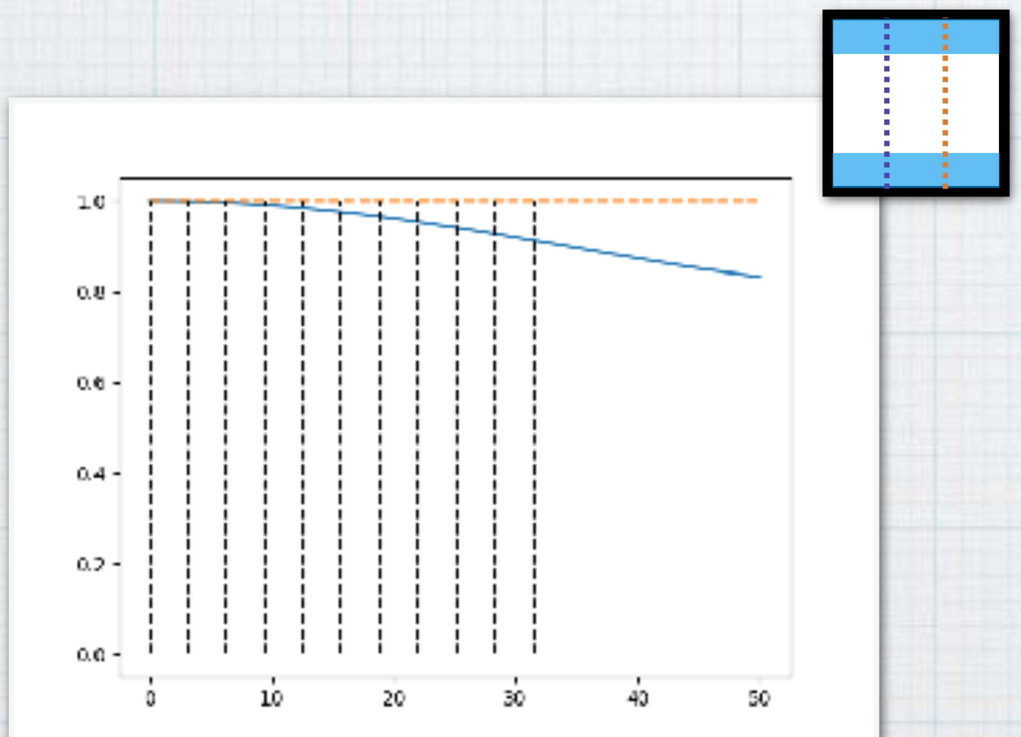
$|\rho^{DD}(\xi)|$



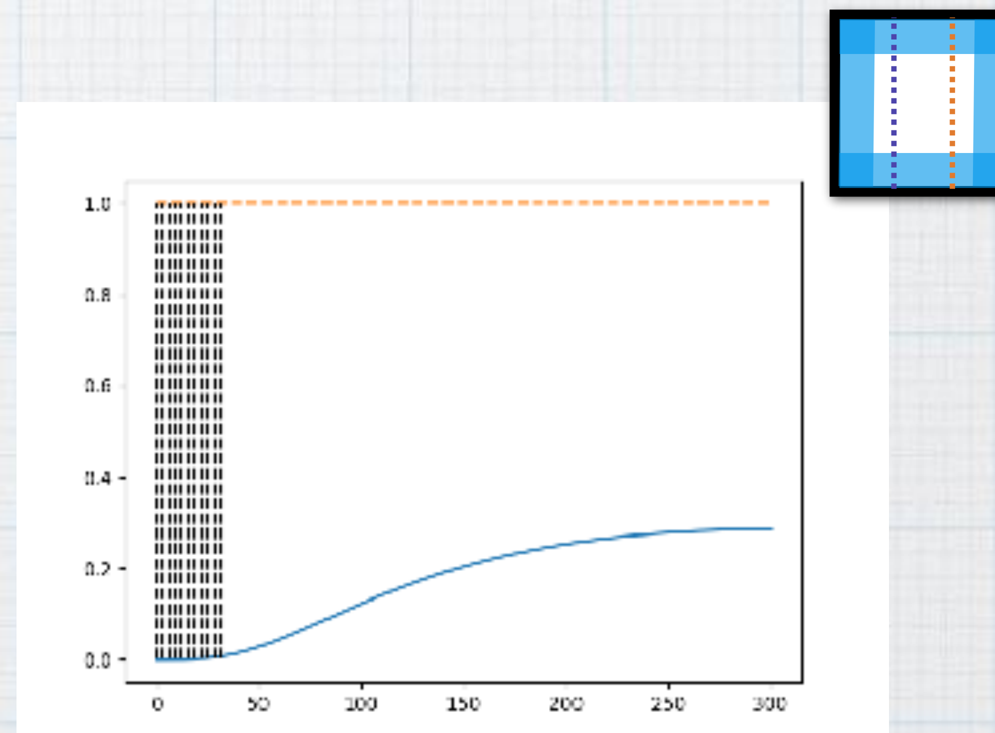
$|\rho^{PMLD}(\xi)|$



$|\rho^{DPML}(\xi)|$

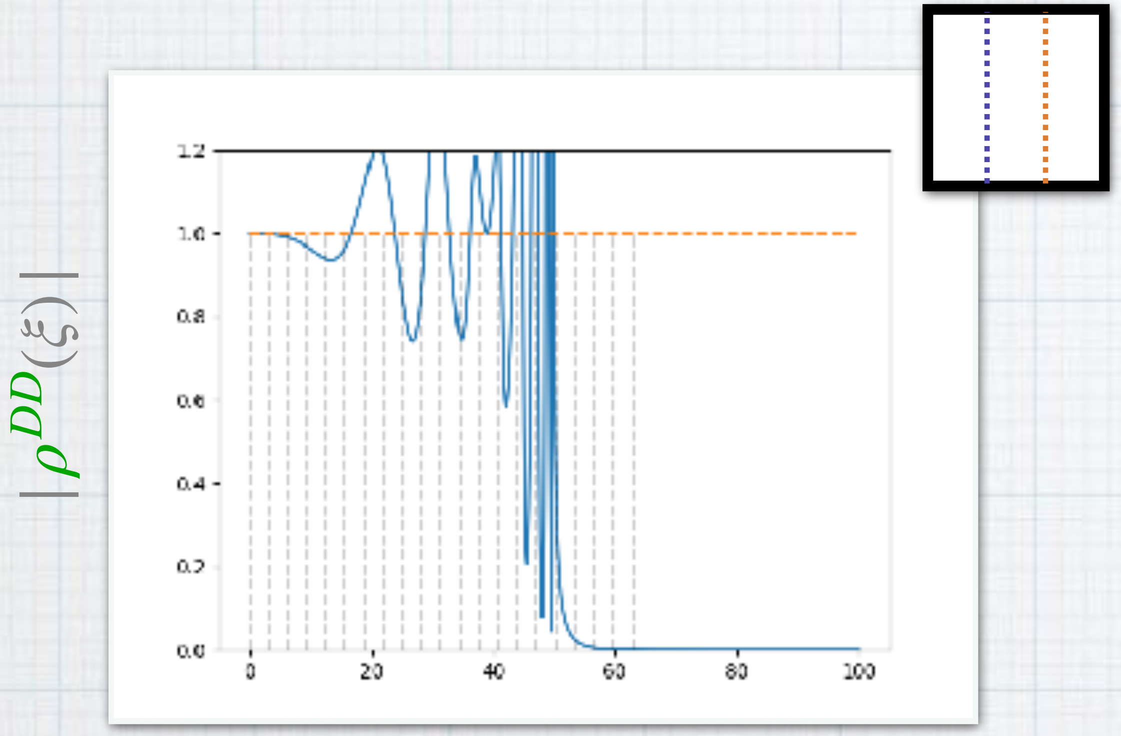


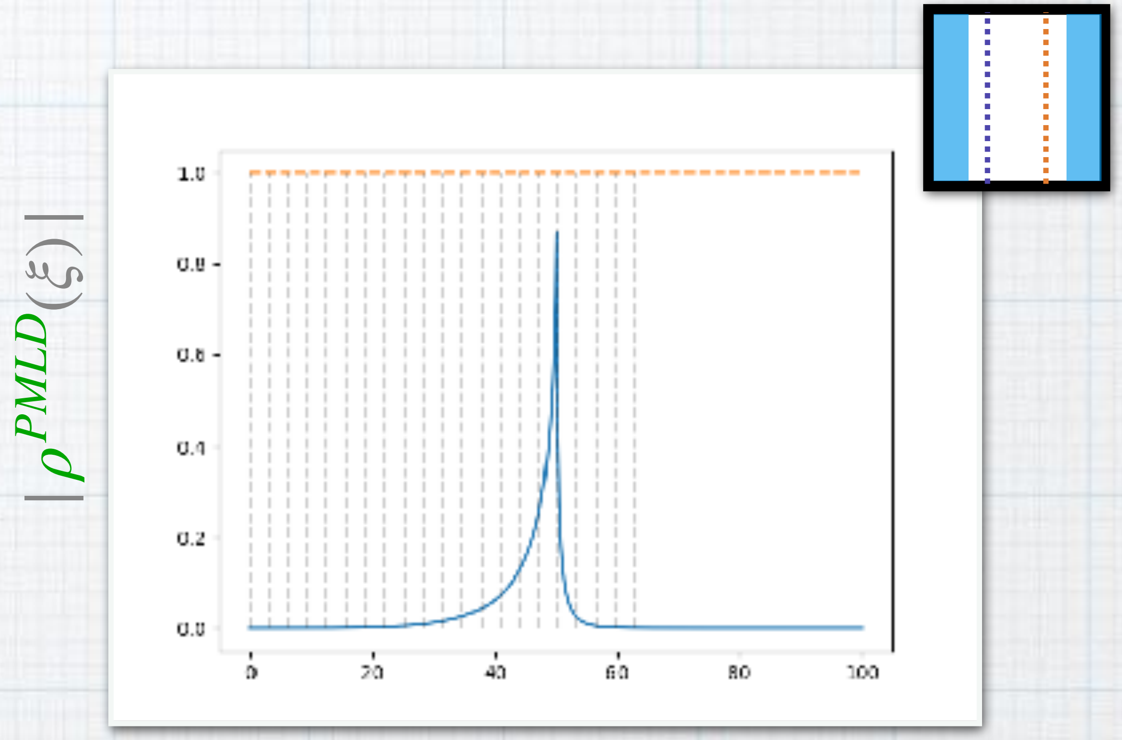
$|\rho^{PMLPML}(\xi)|$

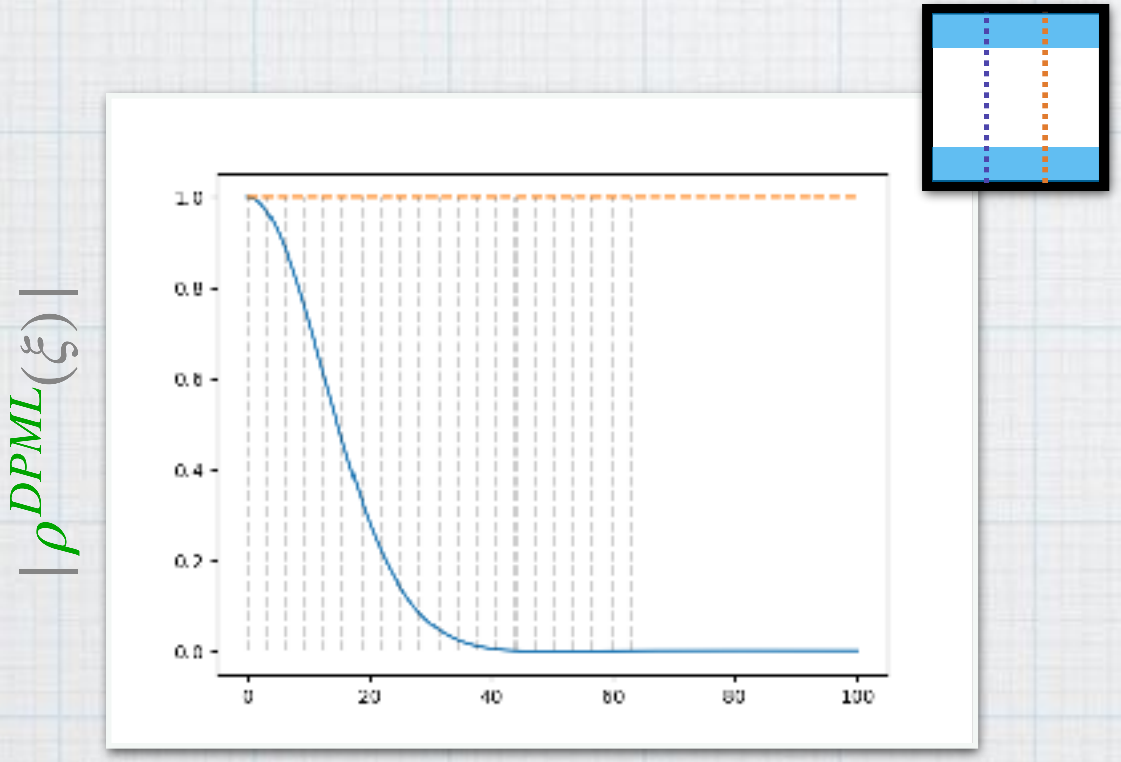


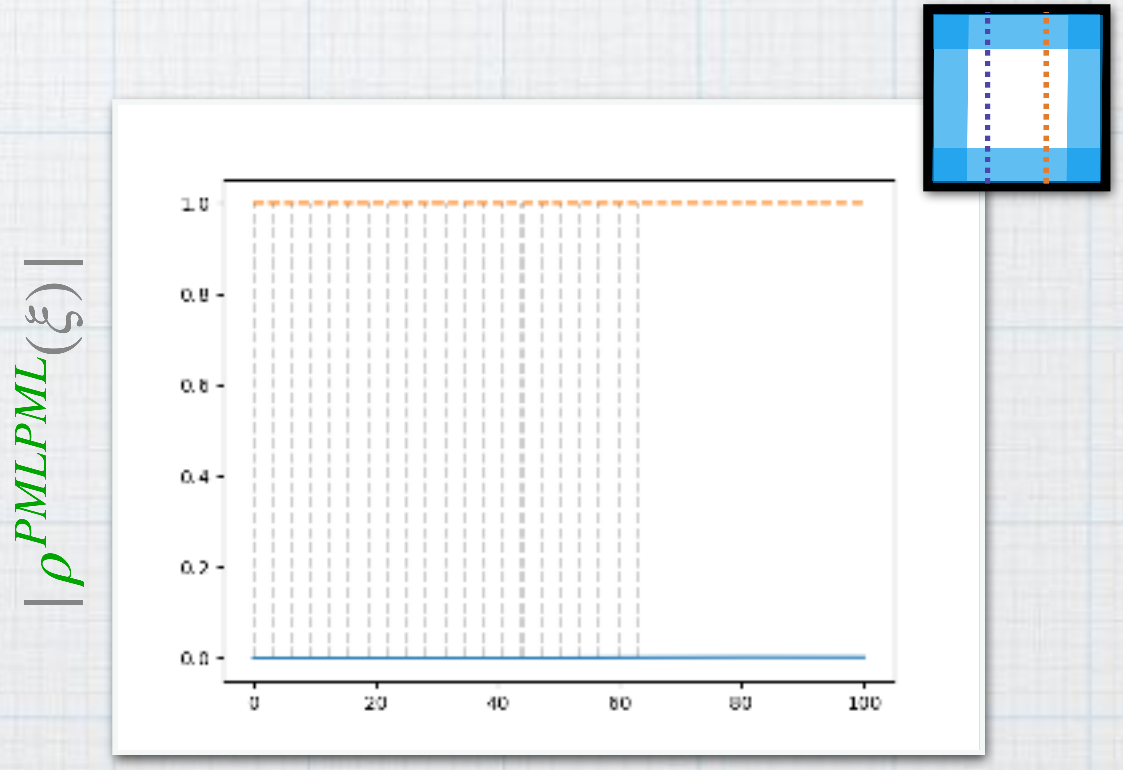
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 50$, $\sigma_{PML} = 10$)



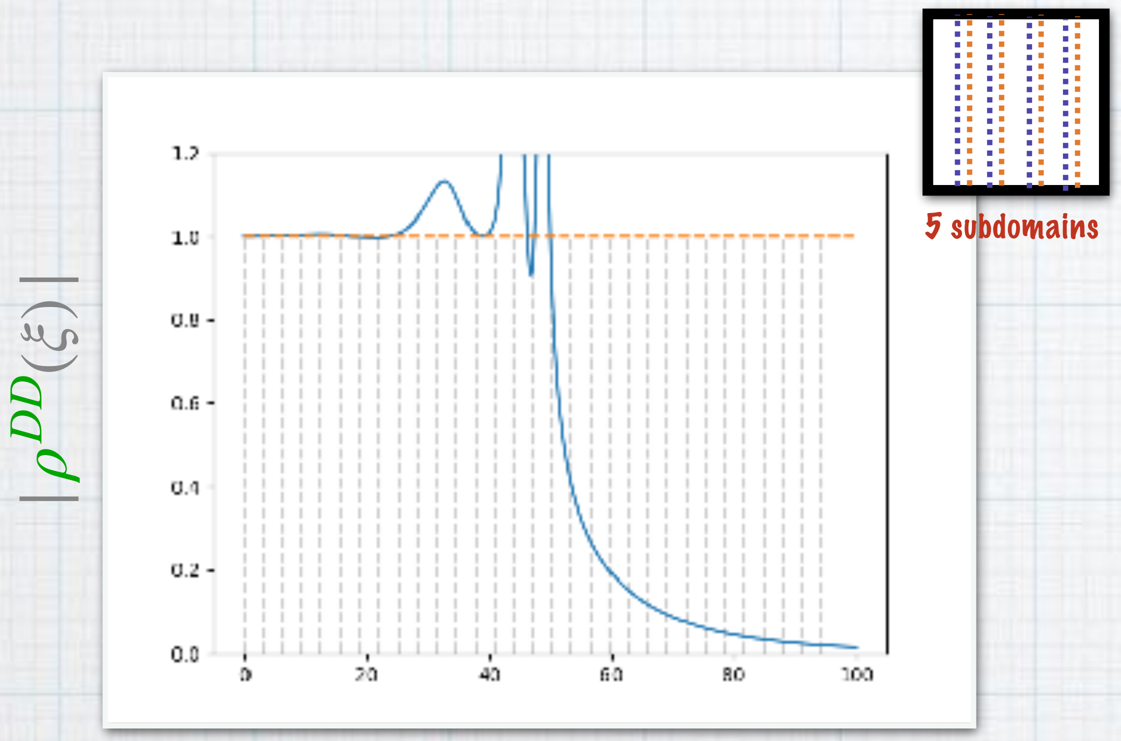


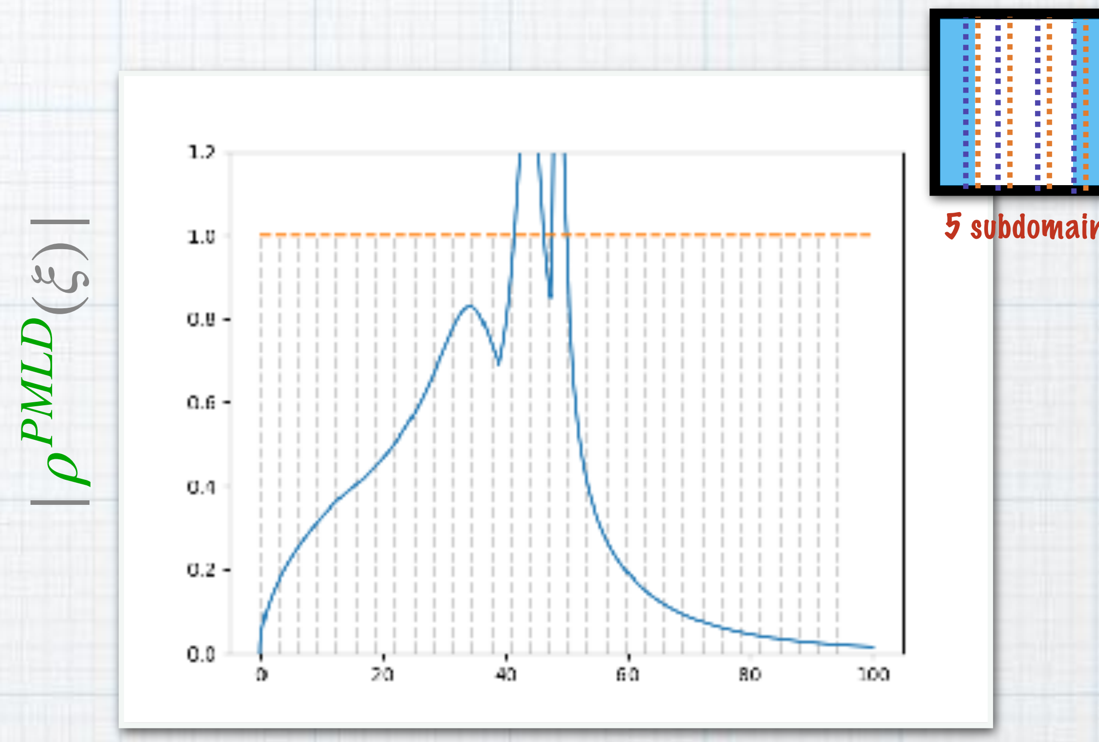


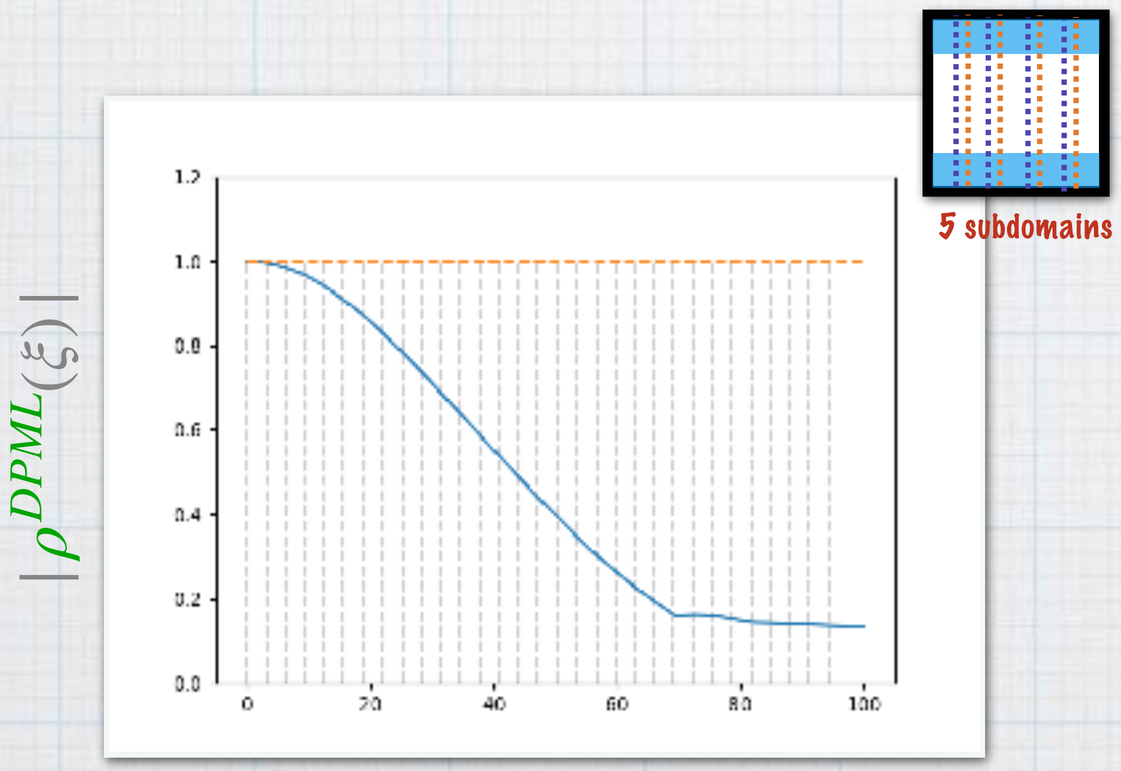


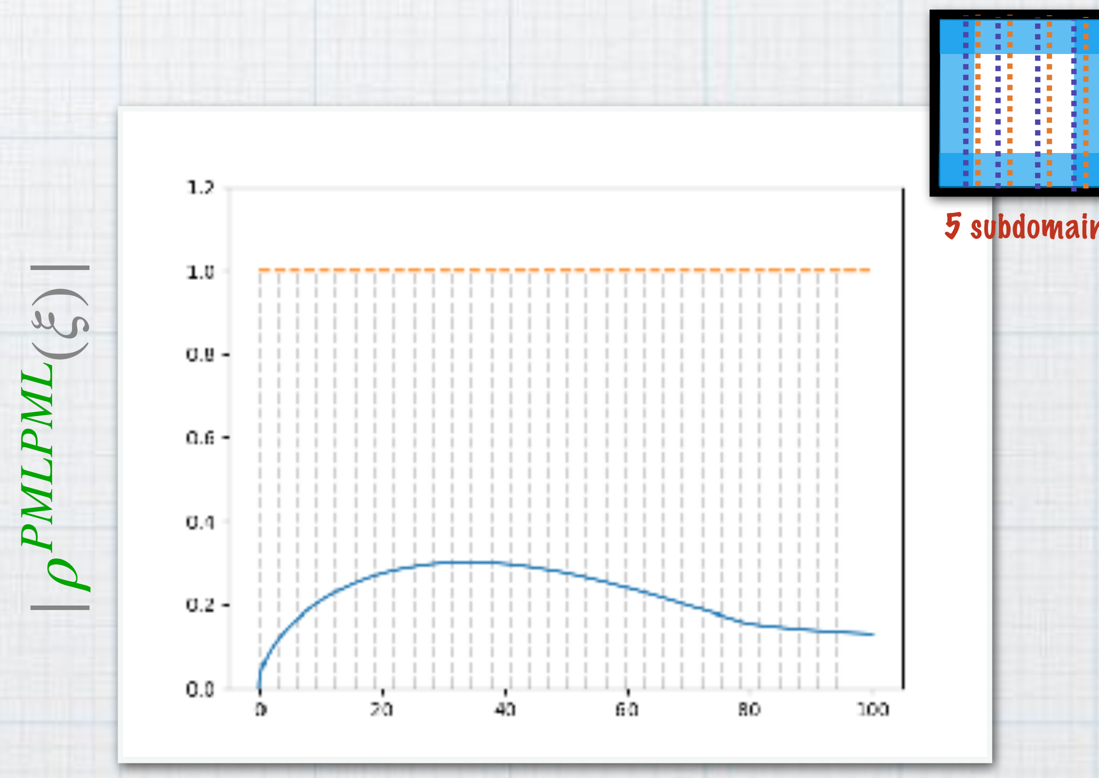
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 50$, $\sigma_{PML} = 10$)









Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 50$, $\sigma_{PML} = 10$)

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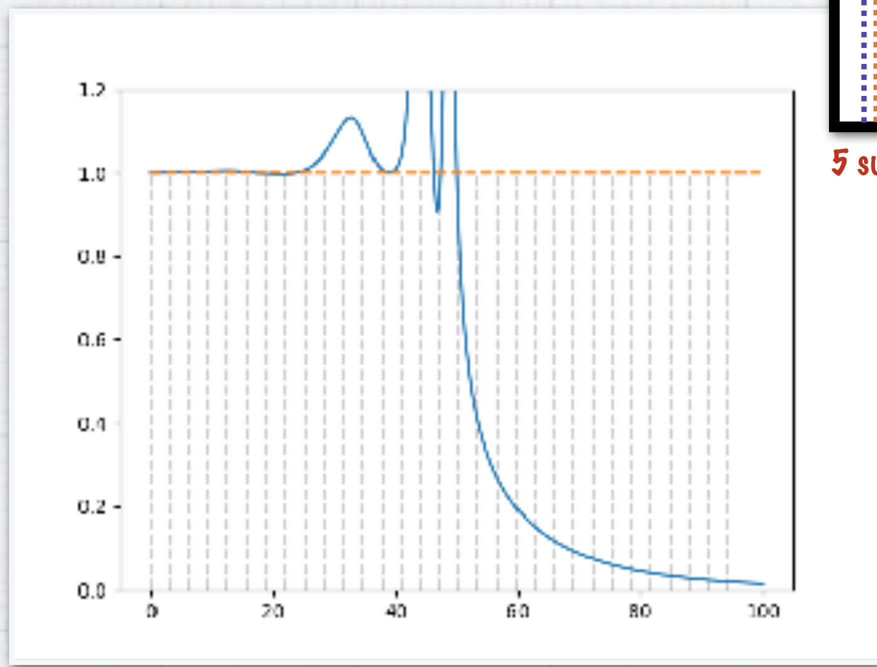
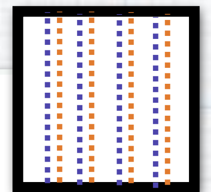
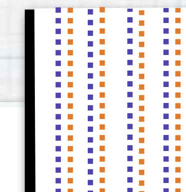


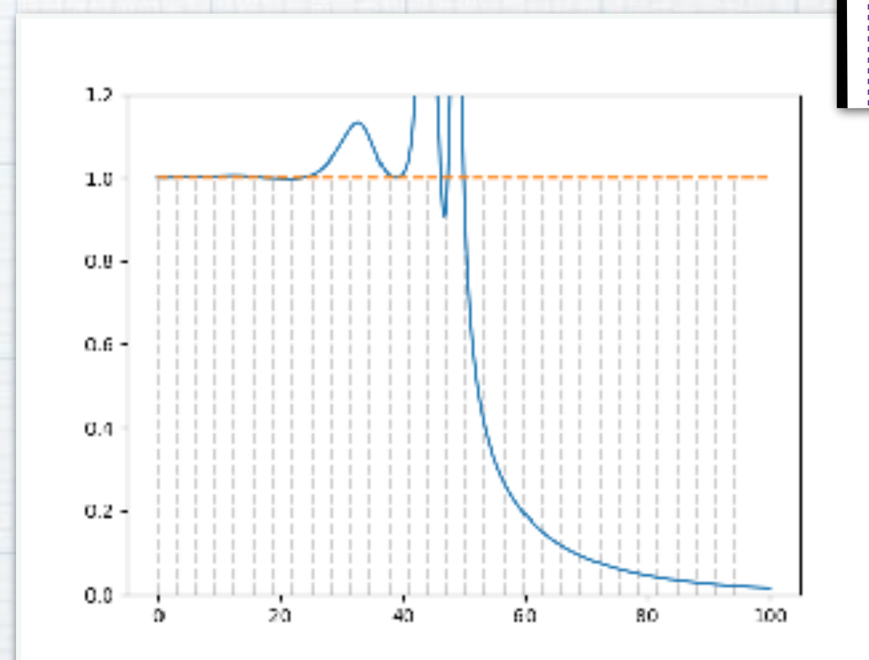


5 subdomains

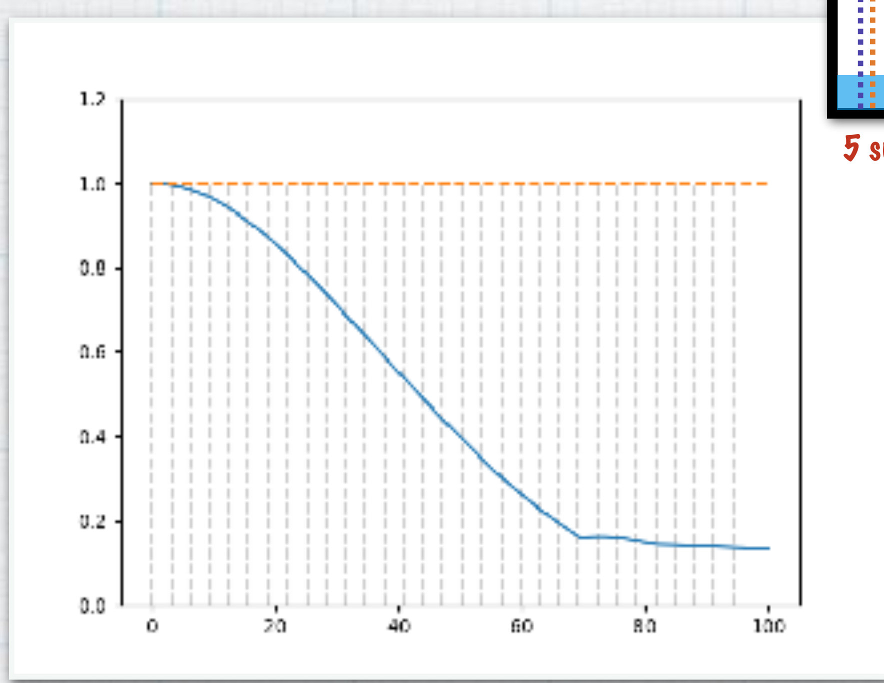
Same convergence factor
but $\neq$ solution !

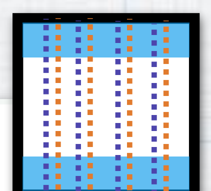
$|\rho^{DInf}(\xi)|$





$|\rho^{DPML}(\xi)|$





5 subdomains

« Same solution »
but $\neq$ convergence factor

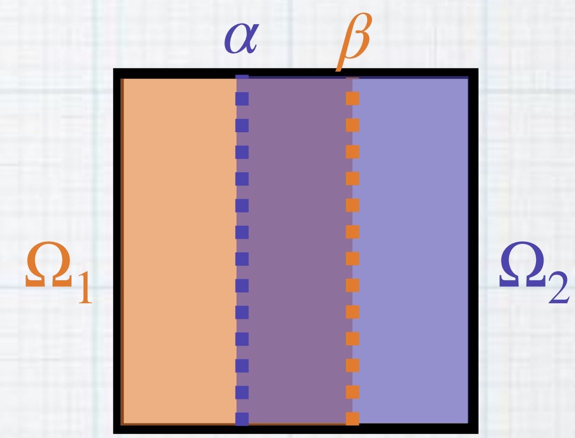
In the next...

1. Motivation
2. Link with Helmholtz equation
3. Convergence analysis on a toy problem
4. An alternated iterative algorithm
5. Conclusion

An alternated iterative algorithm

Step 1 $i \in \{1,2\}$

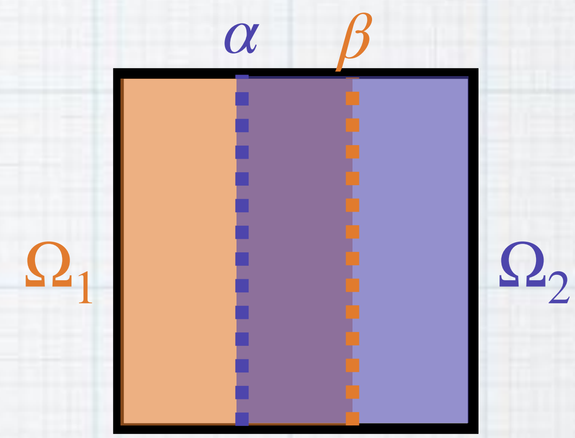
$$
\begin{cases}
\mathscr{L}_{CH}\mathsf{u}^{i,n+1/2} = \mathsf{f}_i & \text{in} \quad \Omega_i \\
\mathsf{u}^{i,n+1/2} = 0 & \text{on} \quad \partial\Omega \\
(\partial_x + p_{i,i\pm1})\mathsf{u}^{i,n+1/2} = (\partial_x + p_{i,i\pm1})\mathsf{u}^{i\pm1,n} & \text{on} \quad \Gamma_{ii\pm1}
\end{cases}
$$



An alternated iterative algorithm

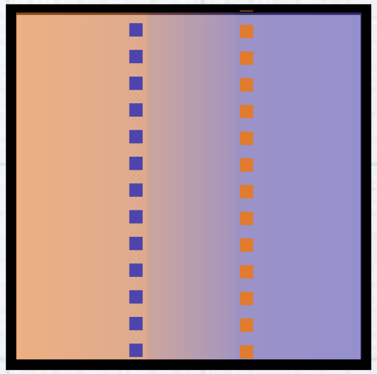
Step 1 $i \in \{1,2\}$

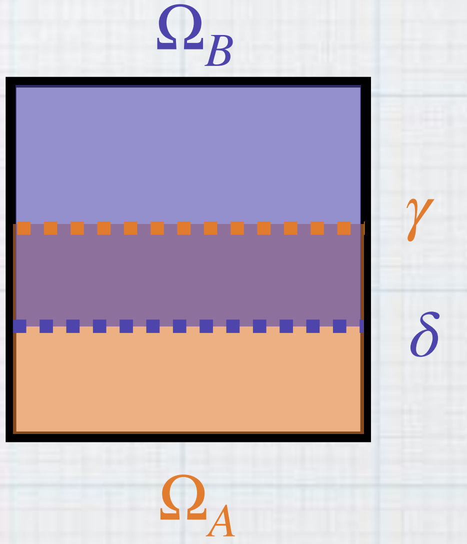
$$\begin{cases} \mathscr{L}_{CH}\mathbf{u}^{i,n+1/2} = \mathsf{f}_i & \text{in } \Omega_i \\ \mathbf{u}^{i,n+1/2} = 0 & \text{on } \partial\Omega \\ (\partial_x + p_{i,i\pm1})\mathbf{u}^{i,n+1/2} = (\partial_x + p_{i,i\pm1})\mathbf{u}^{i\pm1,n} & \text{on } \Gamma_{ii\pm1} \end{cases}$$



Step 2 $i \in \{1,2\},\ j \in \{A,B\}$

$$\begin{cases} \mathbf{u}^n_{Glo} = \mathscr{P}_{12\to Glo}((\mathbf{u}^{i,n+1/2})_i) \\ \mathbf{u}^{j,n} = \mathbf{u}^n_{Glo}|_{\Omega_j} \end{cases}$$





An alternated iterative algorithm

Step 1 ($i \in \{1,2\}$)

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\begin{cases}
\mathscr{L}_{CH} \mathbf{u}^{i,n+1/2} = \mathbf{f}_i & \text{in } \Omega_i \\
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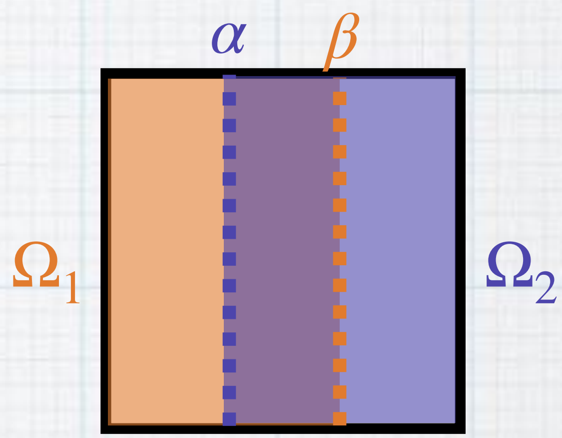
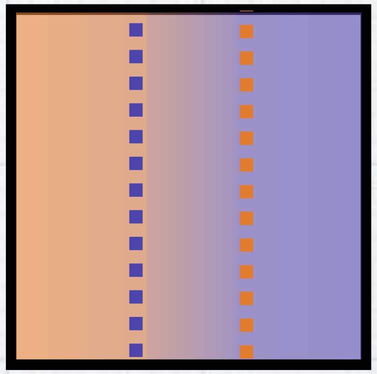
\end{cases}
$$



Step 2 ($i \in \{1,2\}$, $j \in \{A,B\}$)

$$
\begin{cases}
\mathbf{u}^n_{Glo} = \mathscr{P}_{12\to Glo}((\mathbf{u}^{i,n+1/2})_i) \\
\mathbf{u}^{j,n} = \mathbf{u}^n_{Glo}|_{\Omega_j}
\end{cases}
$$



Step 3 ($j \in \{A,B\}$)

$$
\begin{cases}
\mathscr{L}_{CH} \mathbf{u}^{j,n+1/2} = \mathbf{f}_j & \text{in } \Omega_j \\
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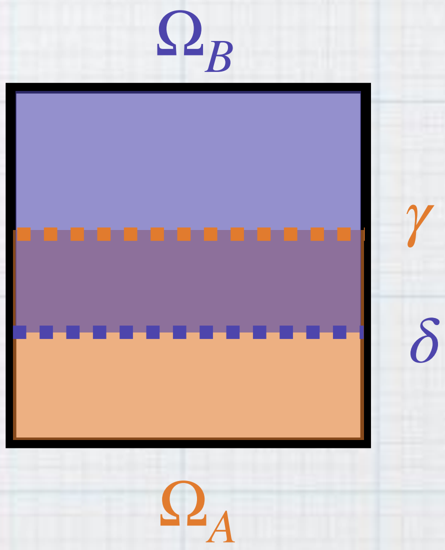
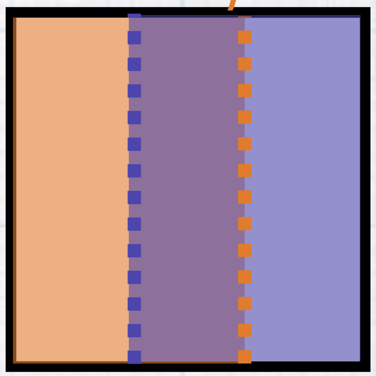
(\partial_x + p_{j,j\pm1}) \mathbf{u}^{j,n+1/2} = (\partial_x + p_{j,j\pm1}) \mathbf{u}^{j\pm1,n} & \text{on } \Gamma_{jj\pm1}
\end{cases}
$$



An alternated iterative algorithm

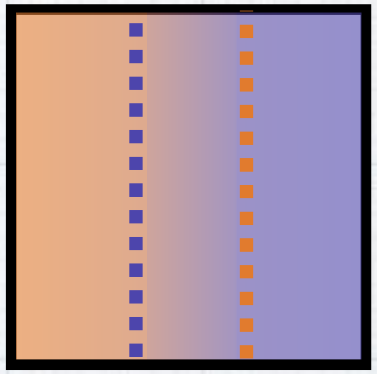
Step 1 ($i \in \{1,2\}$)

$$\begin{cases} \mathscr{L}_{CH} \mathbf{u}^{i,n+1/2} = \mathsf{f}_i & \text{in } \Omega_i \\ \mathbf{u}^{i,n+1/2} = 0 & \text{on } \partial\Omega \\ (\partial_x + p_{i,i\pm1}) \mathbf{u}^{i,n+1/2} = (\partial_x + p_{i,i\pm1}) \mathbf{u}^{i\pm1,n} & \text{on } \Gamma_{ii\pm1} \end{cases}$$



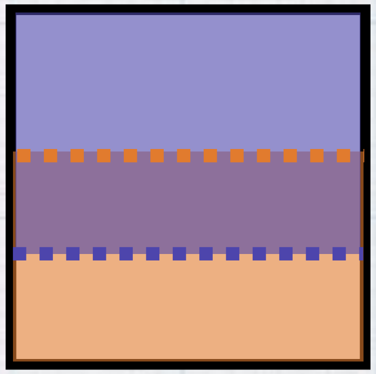
Step 2 ($i \in \{1,2\},\ j \in \{A,B\}$)

$$\begin{cases} \mathbf{u}^n_{Glo} = \mathscr{P}_{12\to Glo}((\mathbf{u}^{i,n+1/2})_i) \\ \mathbf{u}^{j,n} = \mathbf{u}^n_{Glo}|_{\Omega_j} \end{cases}$$



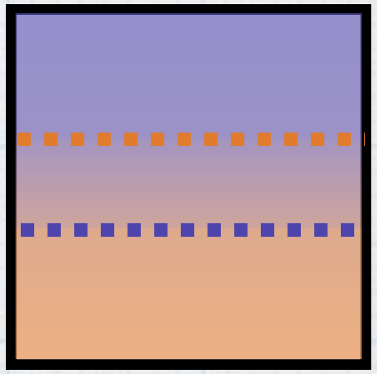
Step 3 ($j \in \{A,B\}$)

$$\begin{cases} \mathscr{L}_{CH} \mathbf{u}^{j,n+1/2} = \mathsf{f}_j & \text{in } \Omega_j \\ \mathbf{u}^{j,n+1/2} = 0 & \text{on } \partial\Omega \\ (\partial_x + p_{j,j\pm1}) \mathbf{u}^{j,n+1/2} = (\partial_x + p_{j,j\pm1}) \mathbf{u}^{j\pm1,n} & \text{on } \Gamma_{jj\pm1} \end{cases}$$



Step 4 ($i \in \{1,2\},\ j \in \{A,B\}$)

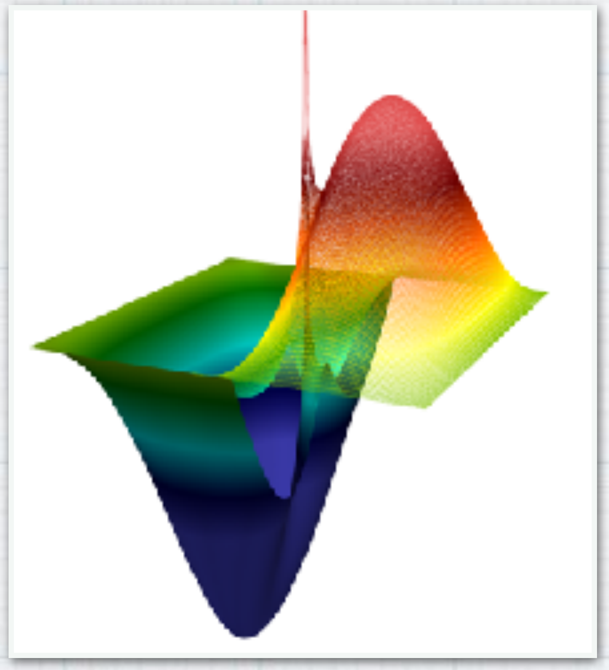
$$\begin{cases} \mathbf{u}^{n+1}_{Glo} = \mathscr{P}_{AB\to Glo}((\mathbf{u}^{j,n+1/2})_j) \\ \mathbf{u}^{i,n+1} = \mathbf{u}^n_{Glo}|_{\Omega_i} \end{cases}$$



An alternated iterative algorithm

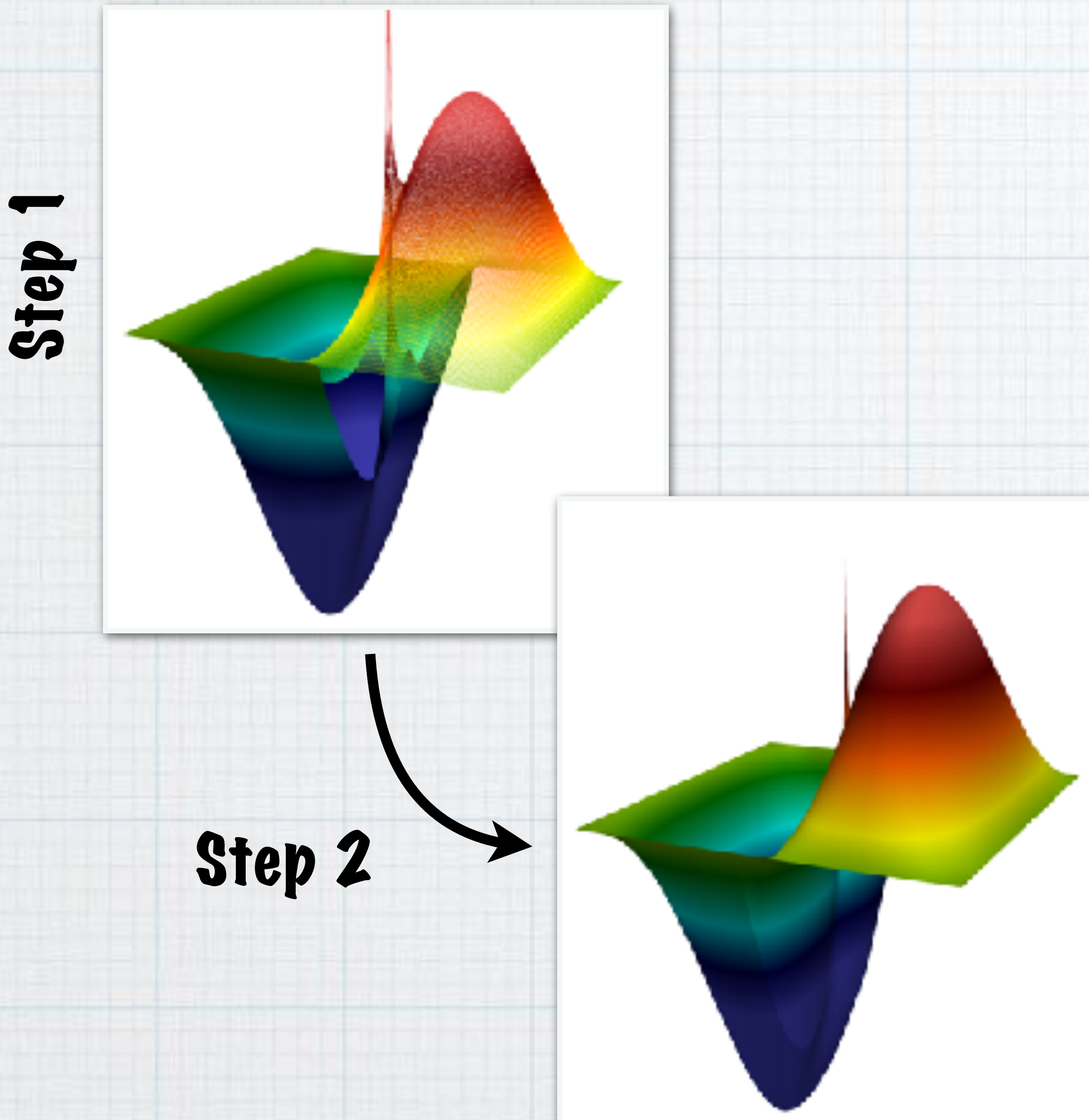
Illustration of the algorithm (Wave Ray: $-\Delta\mathbf{u}+i\mathbf{a}\cdot\nabla\mathbf{u}=\delta,\ \widetilde{\omega}=5$)

Step 1



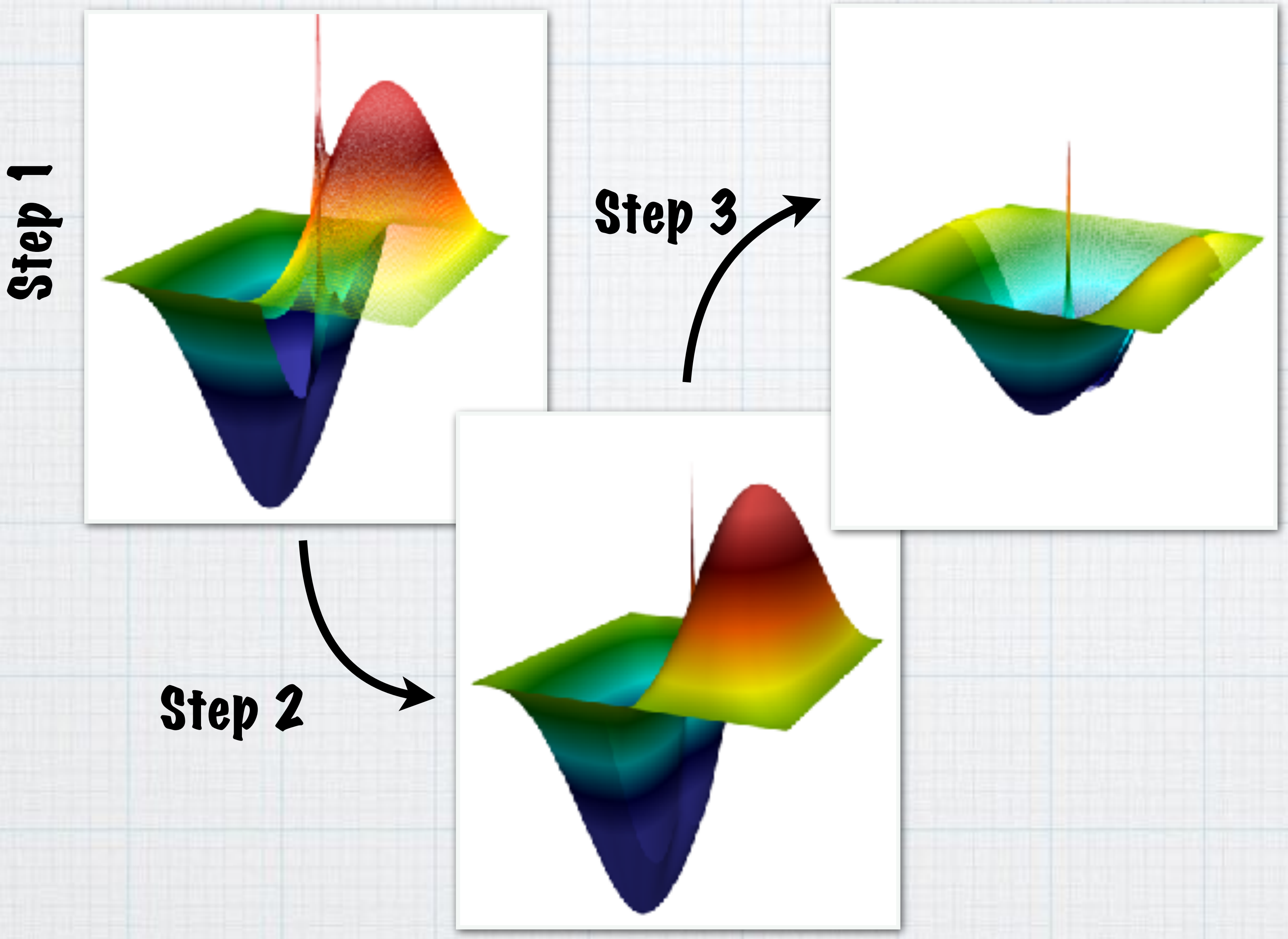
An alternated iterative algorithm

Illustration of the algorithm (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = \delta$, $\widetilde{\omega} = 5$)



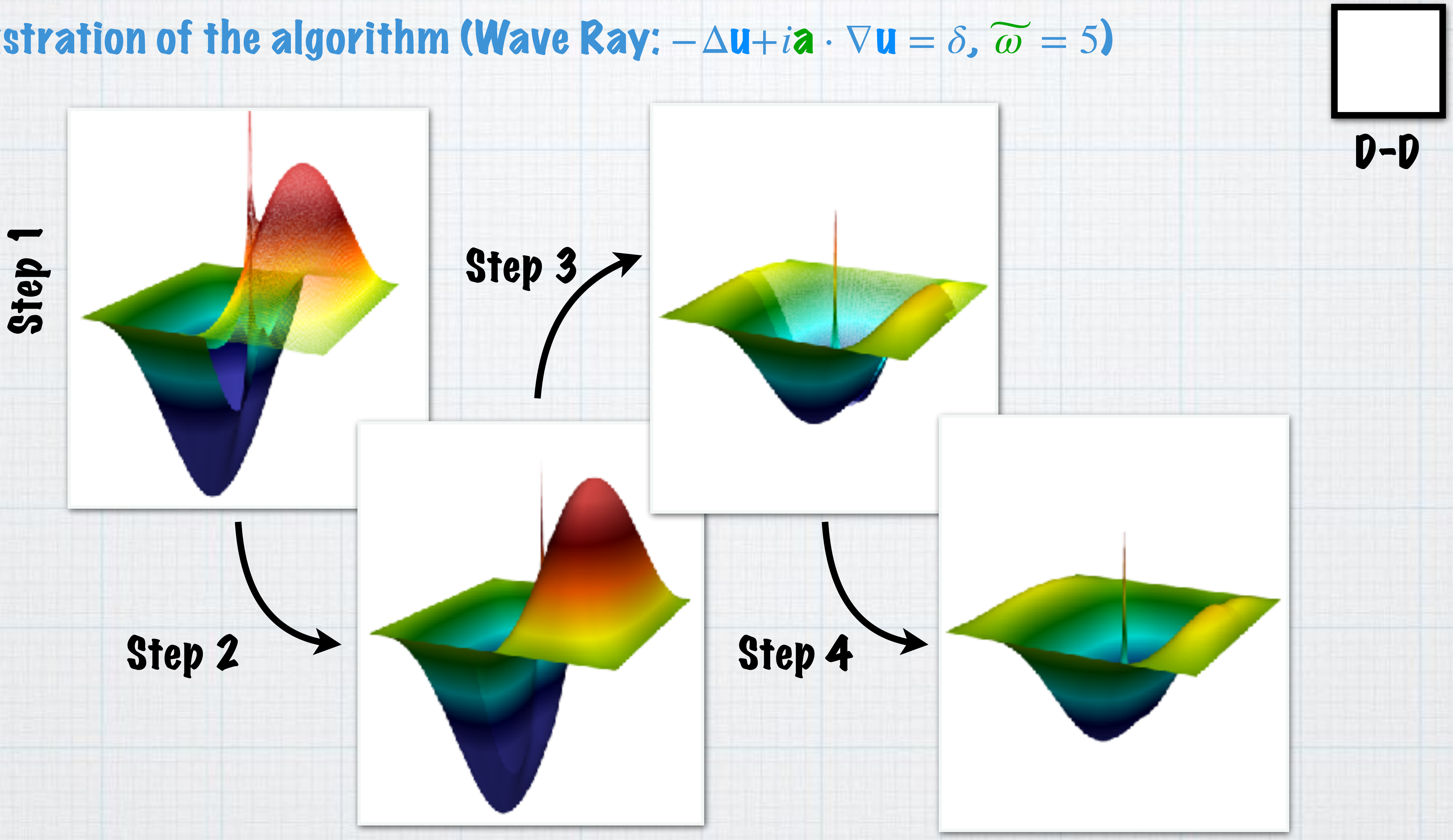
An alternated iterative algorithm

Illustration of the algorithm (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = \delta$, $\widetilde{\omega} = 5$)



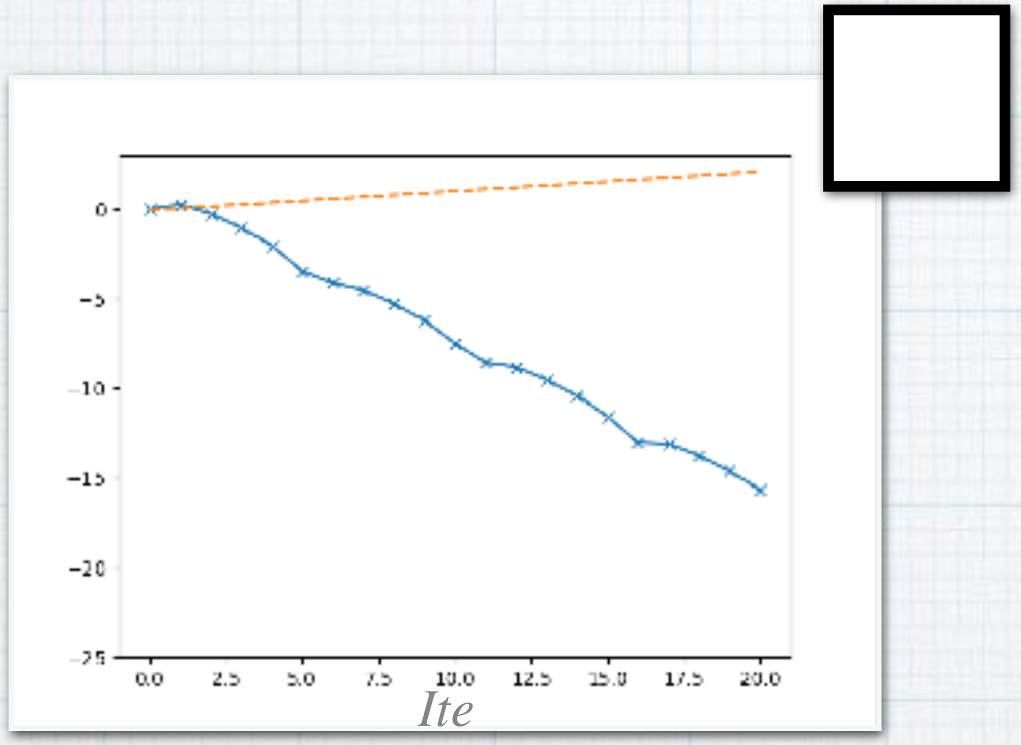
An alternated iterative algorithm

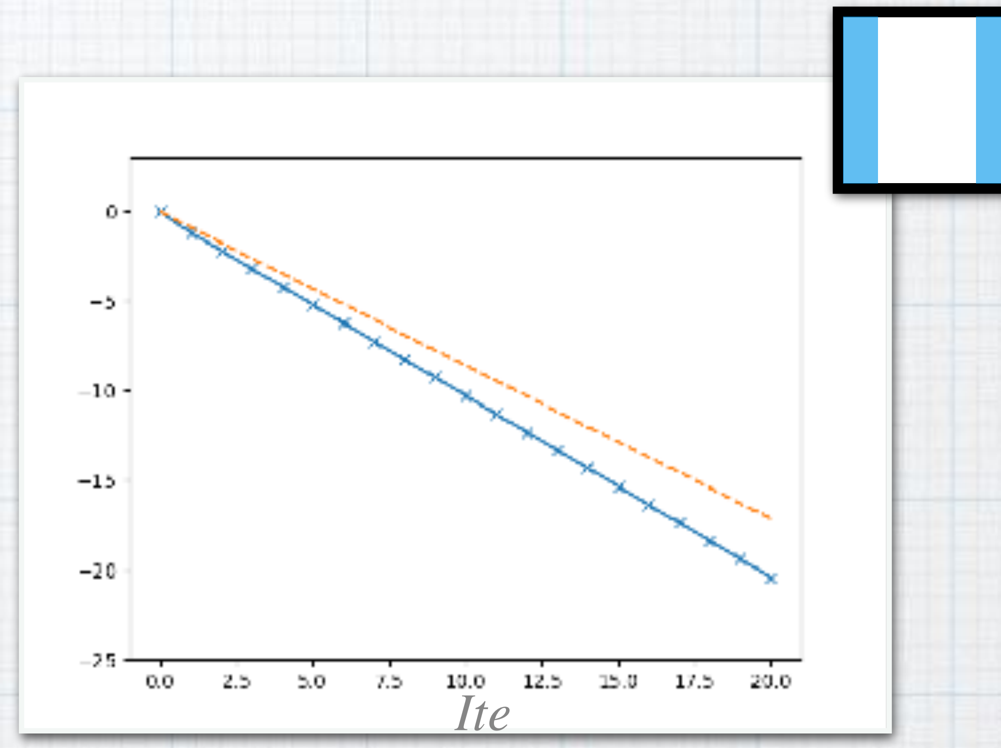
Illustration of the algorithm (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = \delta, \widetilde{\omega} = 5$)

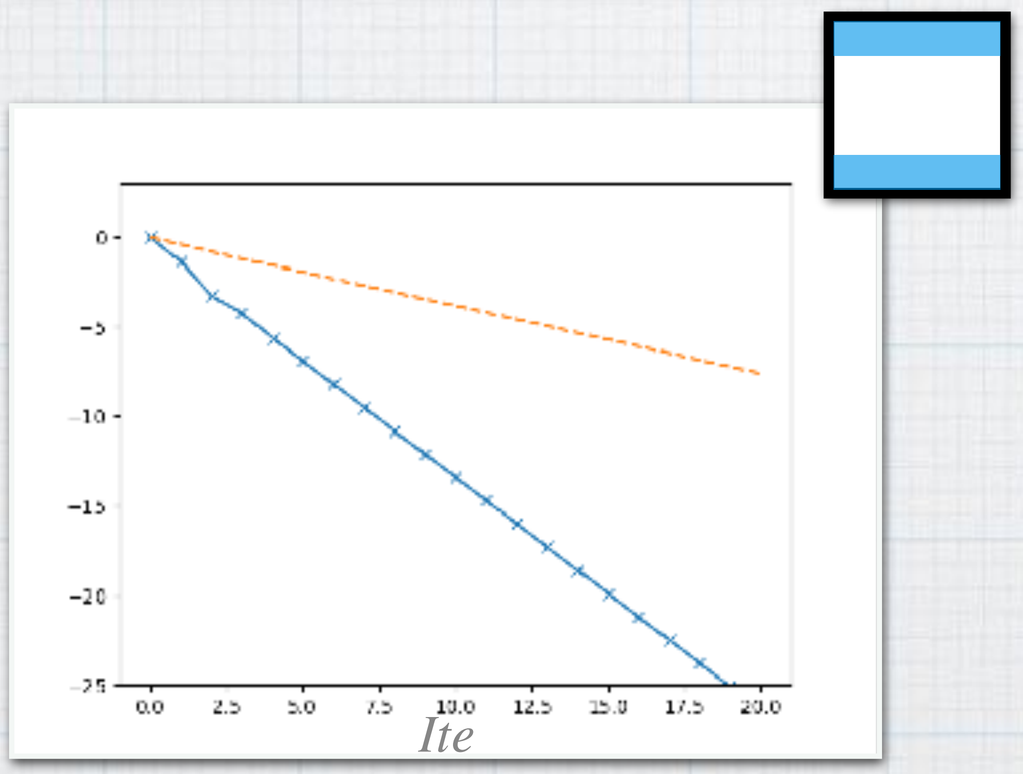


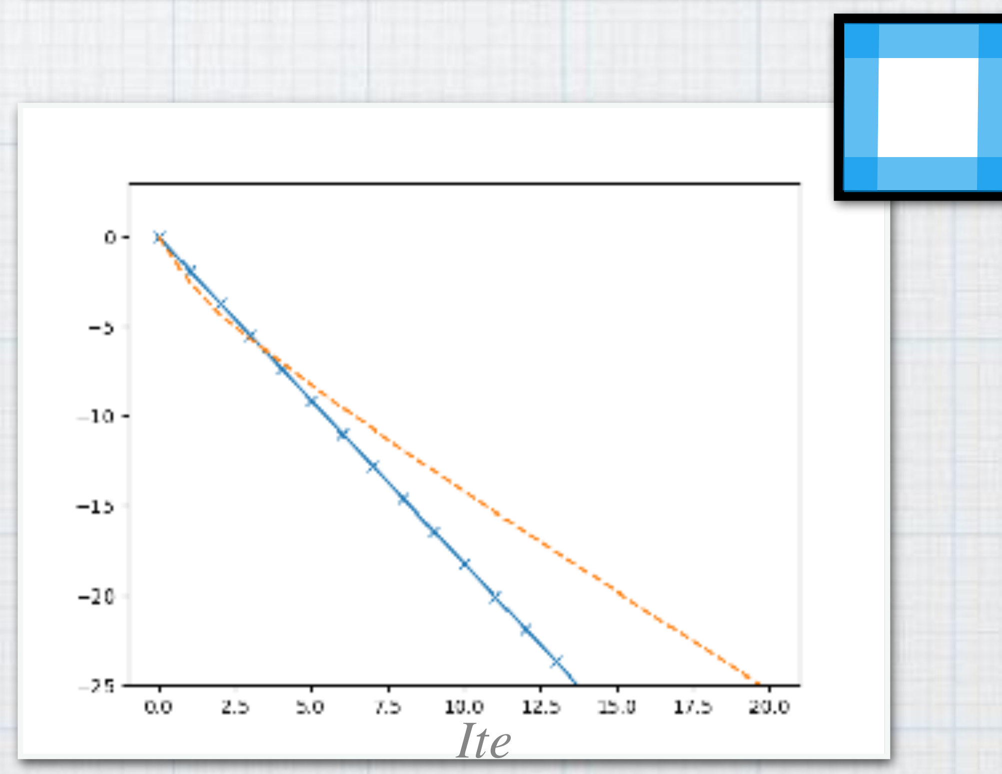
An alternated iterative algorithm

Convergence in the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\sigma_{PML} = 10$)









Legend

$\log_{10}(\sqrt{err})$ (alternate)

$\log_{10}(err)$ (only vert.)

An alternated iterative algorithm

Some ideas on the convergence analysis of the alternated algorithm:

$\mathbf{A}^{i,n}(\xi_k)$

An alternated iterative algorithm

Some ideas on the convergence analysis of the alternated algorithm:

$$\mathbf{A}^{i,n}(\xi_k) \longrightarrow \rho(\xi_k)\, \mathbf{A}^{i,n}(\xi_k)$$
$$= \mathbf{A}^{i,n+1/2}(\xi_k)$$

Step 1

An alternated iterative algorithm

Some ideas on the convergence analysis of the alternated algorithm:

$$\mathbf{A}^{j,n}(\zeta_0)$$

$$\vdots$$

$$\mathbf{A}^{i,n}(\xi_k) \longrightarrow \rho(\xi_k)\, \mathbf{A}^{i,n}(\xi_k) = \mathbf{A}^{i,n+1/2}(\xi_k) \longrightarrow \mathbf{A}^{j,n}(\zeta_k)$$

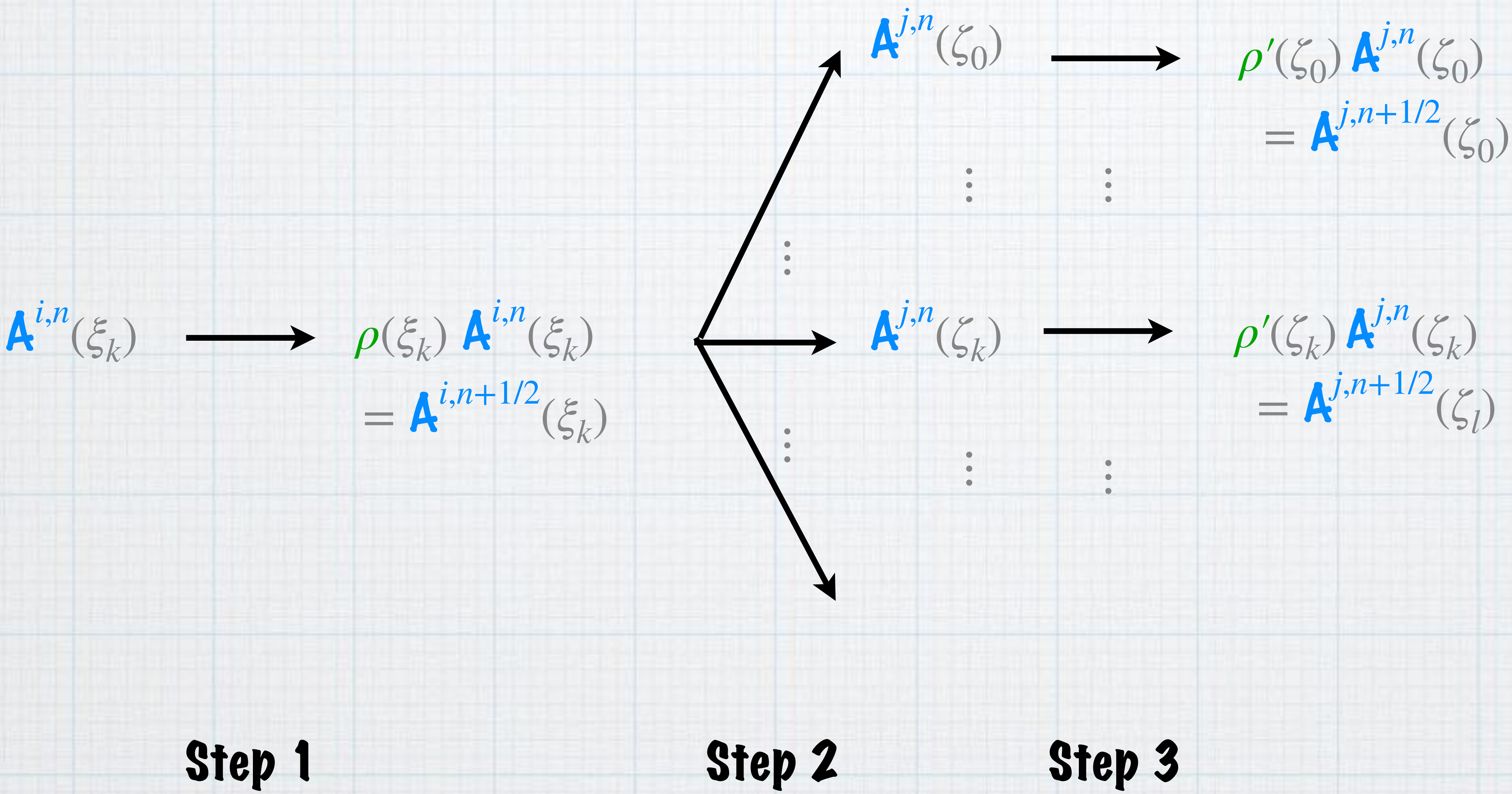
$$\vdots$$

Step 1

Step 2

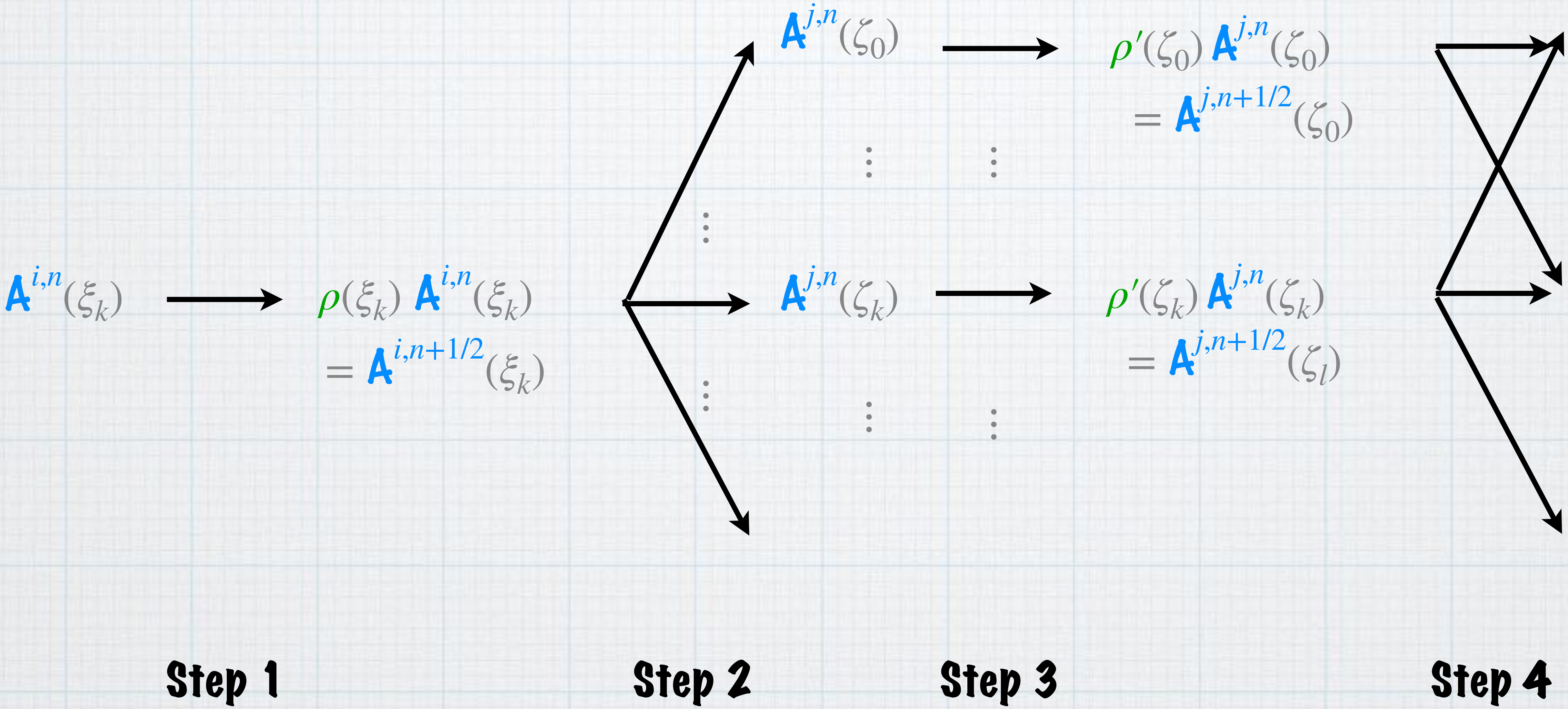
An alternated iterative algorithm

Some ideas on the convergence analysis of the alternated algorithm:



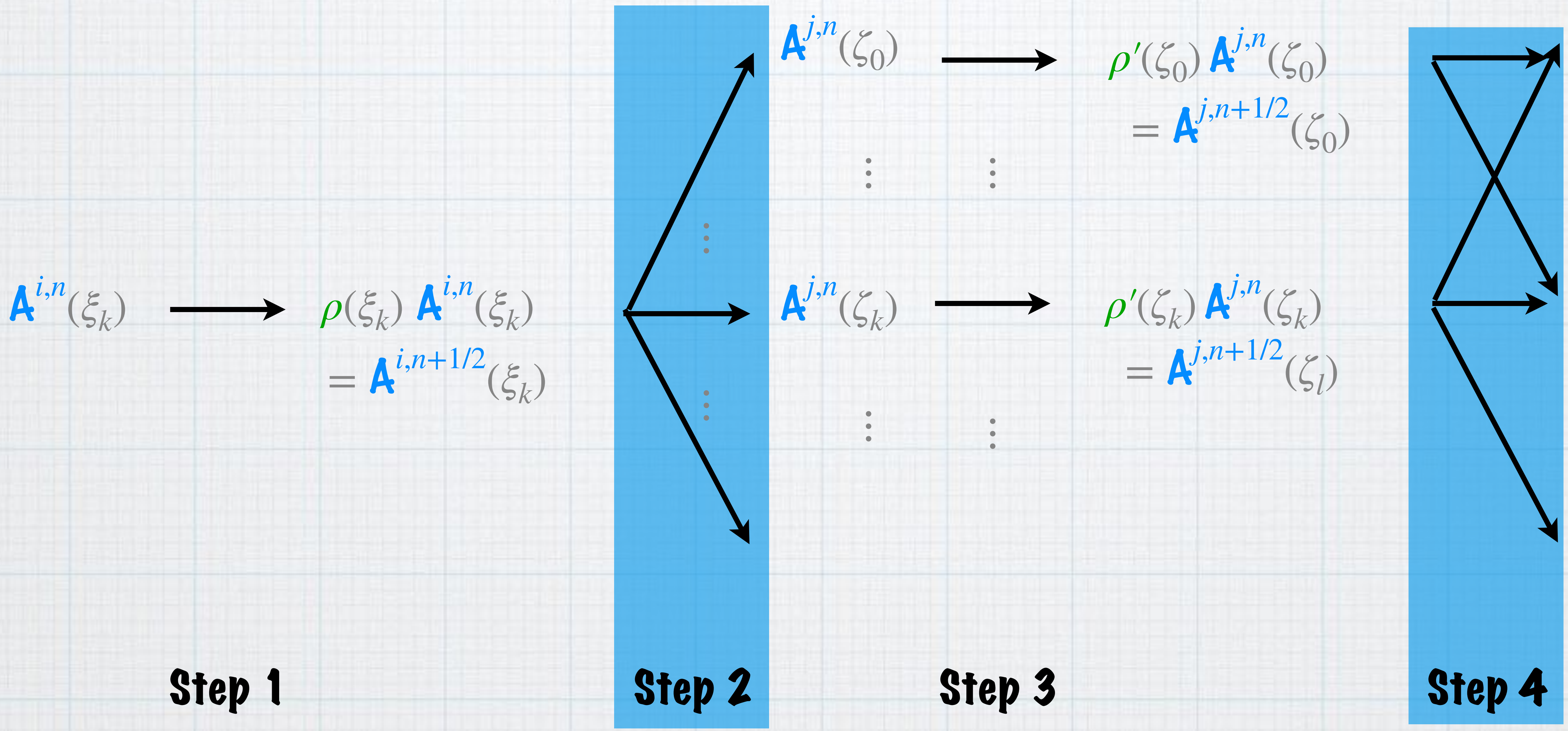
An alternated iterative algorithm

Some ideas on the convergence analysis of the alternated algorithm:



An alternated iterative algorithm

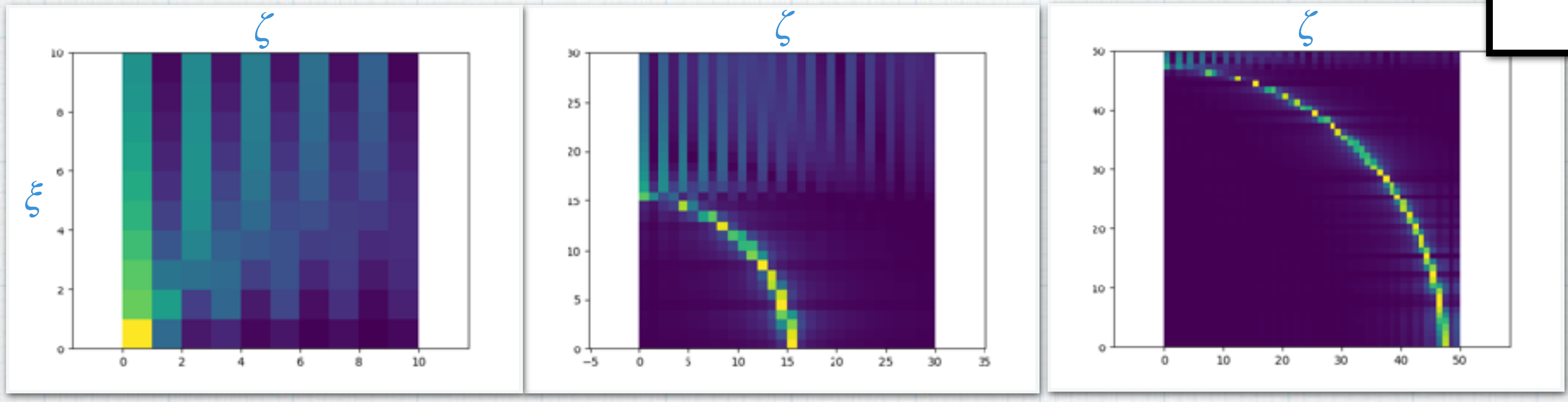
Some ideas on the convergence analysis of the alternated algorithm:



Let us denote by $\left[\mathscr{P}_{12\to AB}\right]_{kl}$ the coefficients that maps $\mathbf{A}^{i,n+1/2}(\xi_k)$ to $\mathbf{A}^{j,n}(\zeta_l)$.

An alternated iterative algorithm

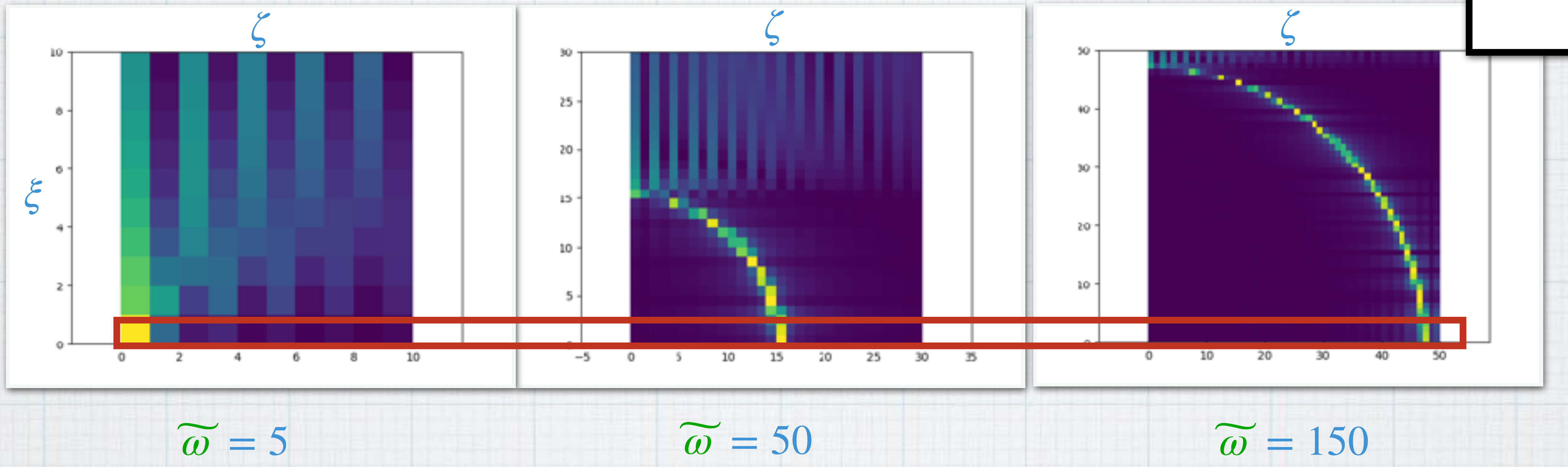
Illustration of the matrix $\mathscr{P}_{12\to AB}$ (Wave Ray: $-\Delta\mathbf{u}+i\mathbf{a}\cdot\nabla\mathbf{u}=0$)



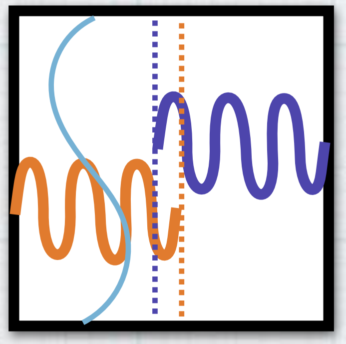
$\widetilde{\omega}=5$ $\widetilde{\omega}=50$ $\widetilde{\omega}=150$

An alternated iterative algorithm

Illustration of the matrix $\mathscr{P}_{12\to AB}$ (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$)

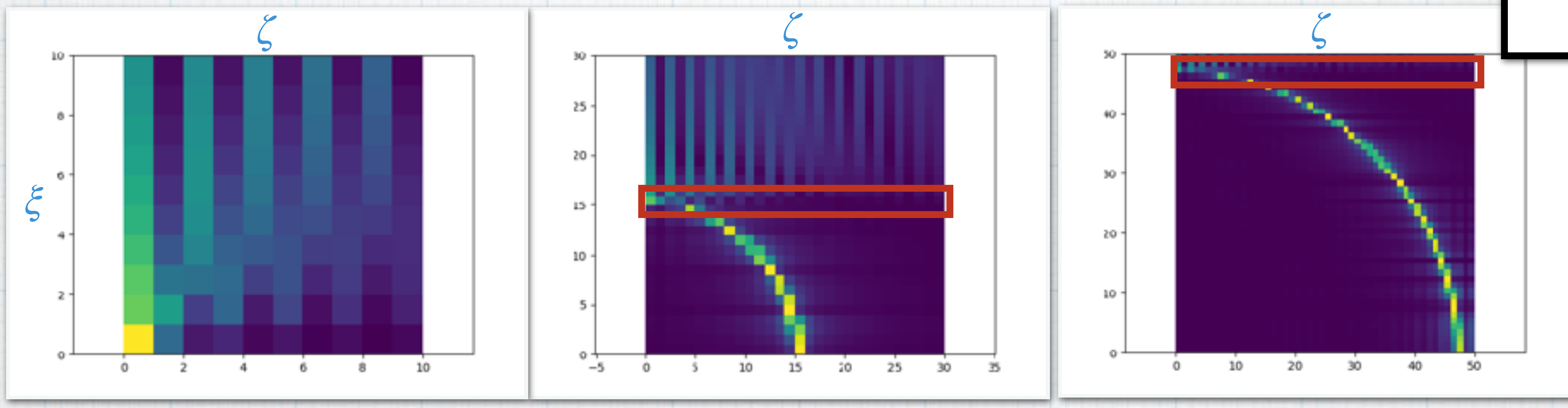


Interpretation: $\mathbf{v}^{i,n}(\xi) = \sin(\xi\pi y)\left(\mathbf{A}^{i,n}(\xi)e^{i\mathcal{S}(\xi)x} + \mathbf{B}^{i,n}(\xi)e^{-i\mathcal{S}(\xi)x}\right)$ **where** $\mathcal{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}$.



An alternated iterative algorithm

Illustration of the matrix $\mathscr{P}_{12\to AB}$ (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a}\cdot\nabla \mathbf{u} = 0$)



$\widetilde{\omega} = 5$ $\widetilde{\omega} = 50$ $\widetilde{\omega} = 150$

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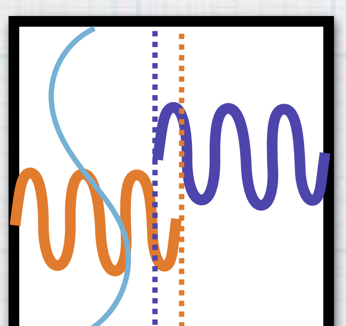
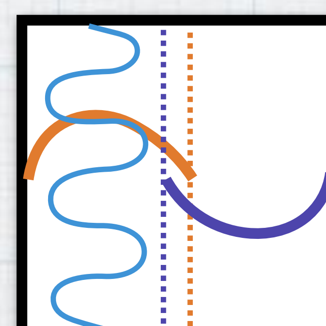
Small ξ



High oscillation

Large ξ



Slow oscillation

An alternated iterative algorithm

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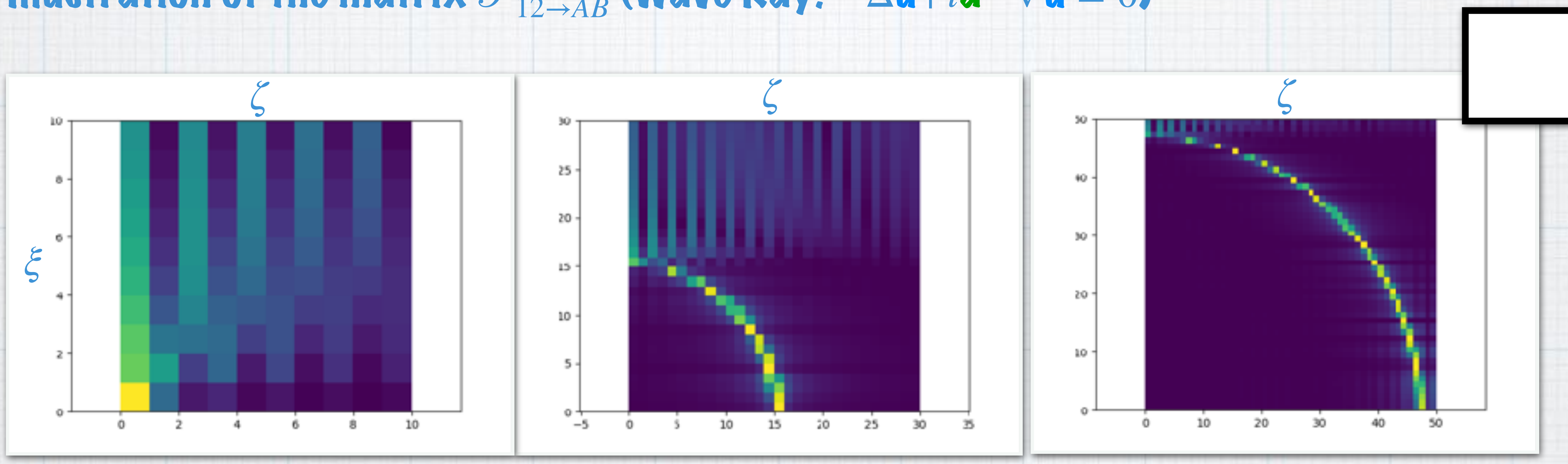

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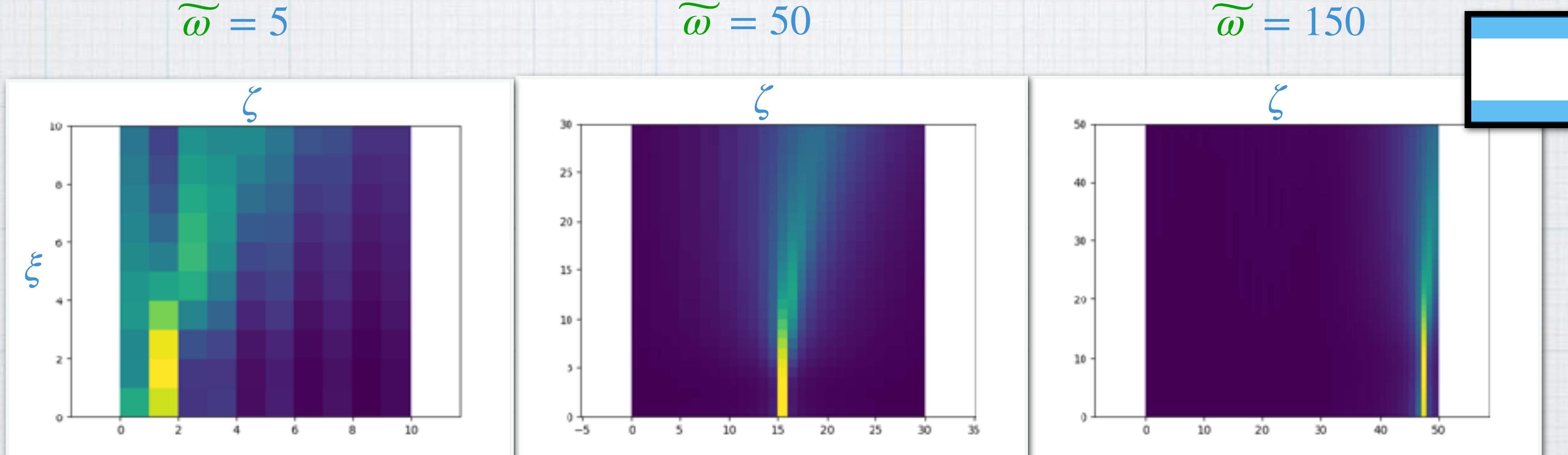
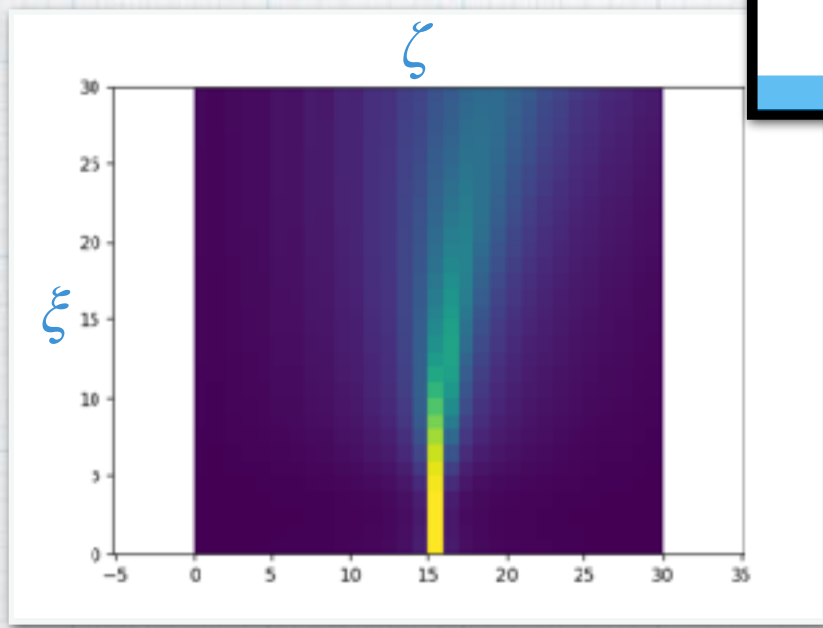


An alternated iterative algorithm

Illustration of the convergence optimization (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 50$)



An alternated iterative algorithm

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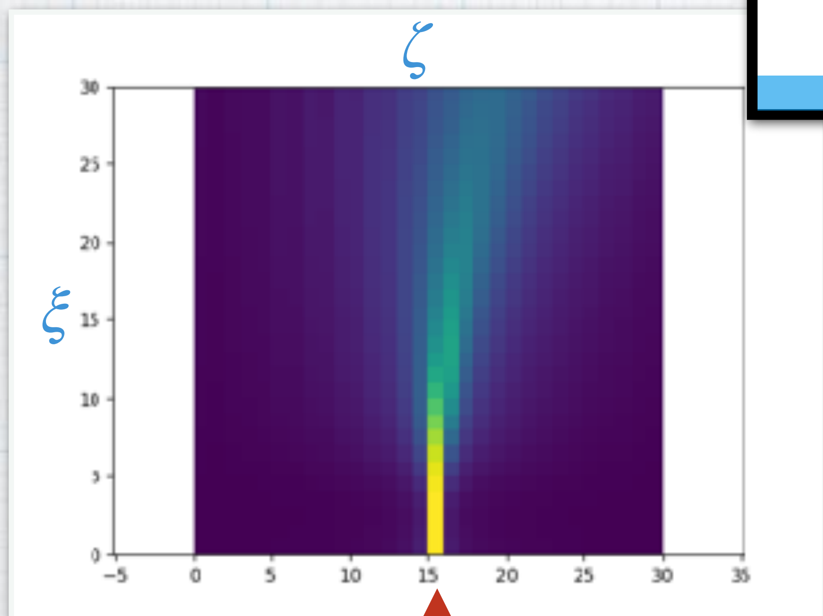
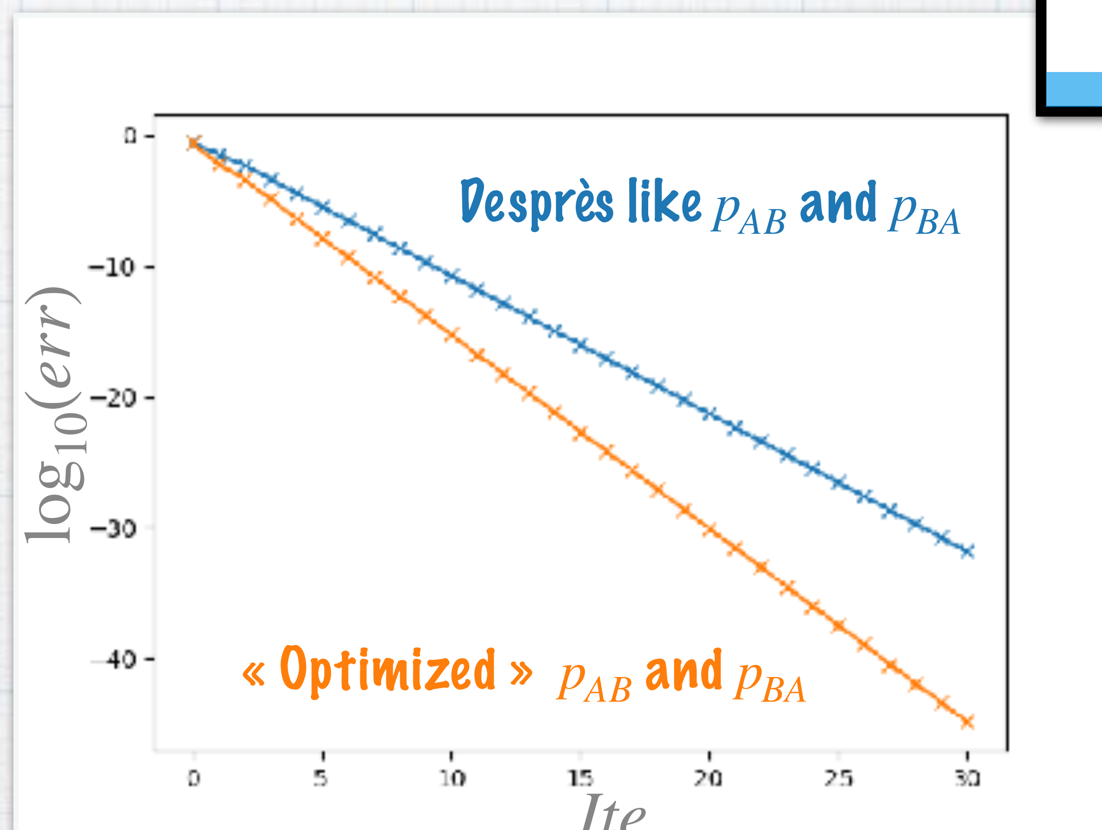


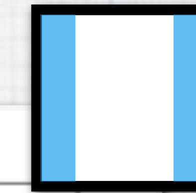
l^*

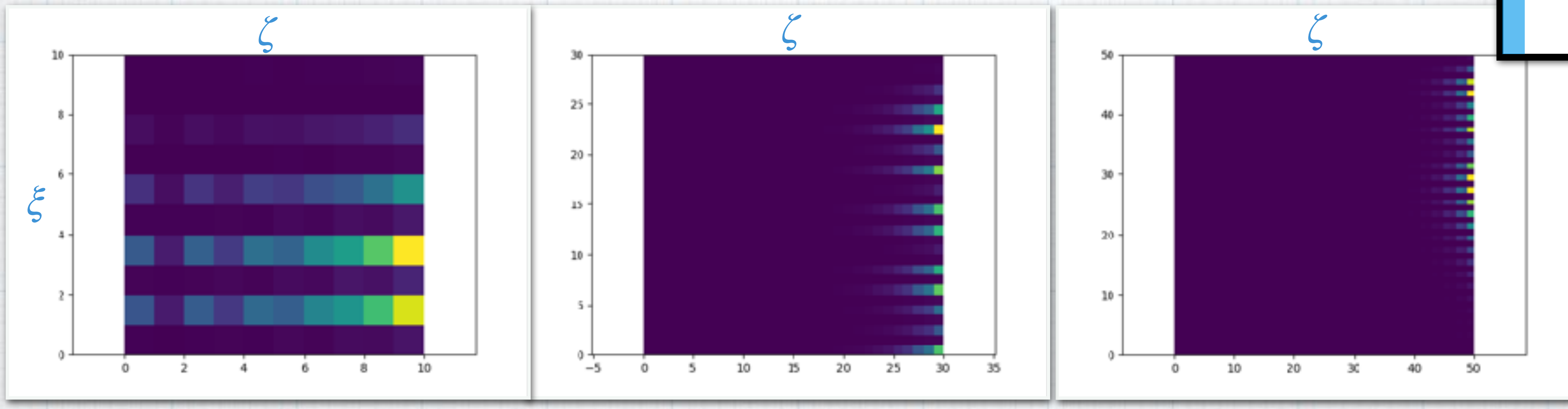
Idea: Choose p_{AB} and p_{BA} s.t. $\rho^{AB}(\zeta_{l^*}) = 0.$



An alternated iterative algorithm

Illustration of the matrix $\mathscr{P}_{12\to AB}$ (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$) (reliable…??)





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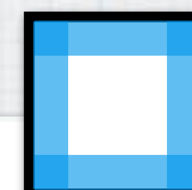

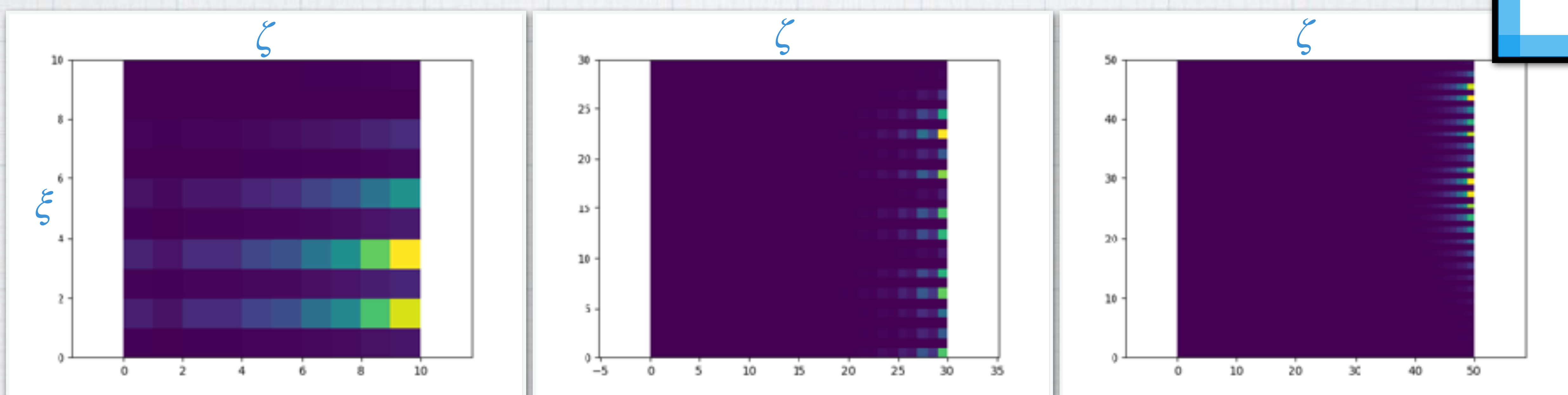

An alternated iterative algorithm

The matrix $\mathscr{P}_{12\to AB}$ for the PML-D and PML PML cases

In the two previous cases, D-D and D-PML cases, the transverse eigenfunctions were

$$\psi_\zeta(x) \quad \propto \quad \sin(\zeta \pi x), \quad \zeta \in \mathbb{N}\,.$$

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In the PML-D or PML-PML cases, we have :

$$\psi_\zeta(x) \quad \propto \quad \sin\left(\frac{\zeta\pi(\widetilde{x}(x) - \widetilde{x}(0))}{1 + 2i\sigma\ell}\right), \quad \zeta \in \mathbb{N}$$

where $\widetilde{x}(x)$ is the stretched variables :

$$\widetilde{x}(x) = \begin{vmatrix} x + i\sigma(x - \ell) & \text{if} & x \in [0, \ell], \\ x & \text{if} & x \in [\ell, 1-\ell], \\ x + i\sigma(x - (1-\ell)) & \text{if} & x \in [1-\ell, \ell]\,. \end{vmatrix}$$

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Although this family of function is a **complete basis**, it is **no more an orthogonal basis !!**

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⟶ **Requires to invert the Gramian** matrix $\mathscr{G}_{lk} = (\psi_l, \psi_k)$ to decompose on this basis (if it is a Riesz basis...)

An alternated iterative algorithm

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Illustration of the (real part of the) eigenfunctions ψ_ζ ($\ell = 0.1$, $\sigma = 10$) :

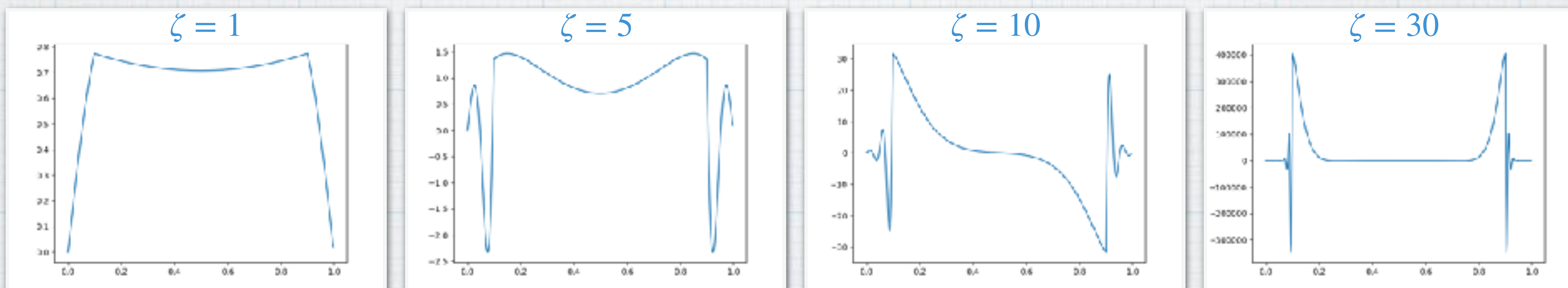

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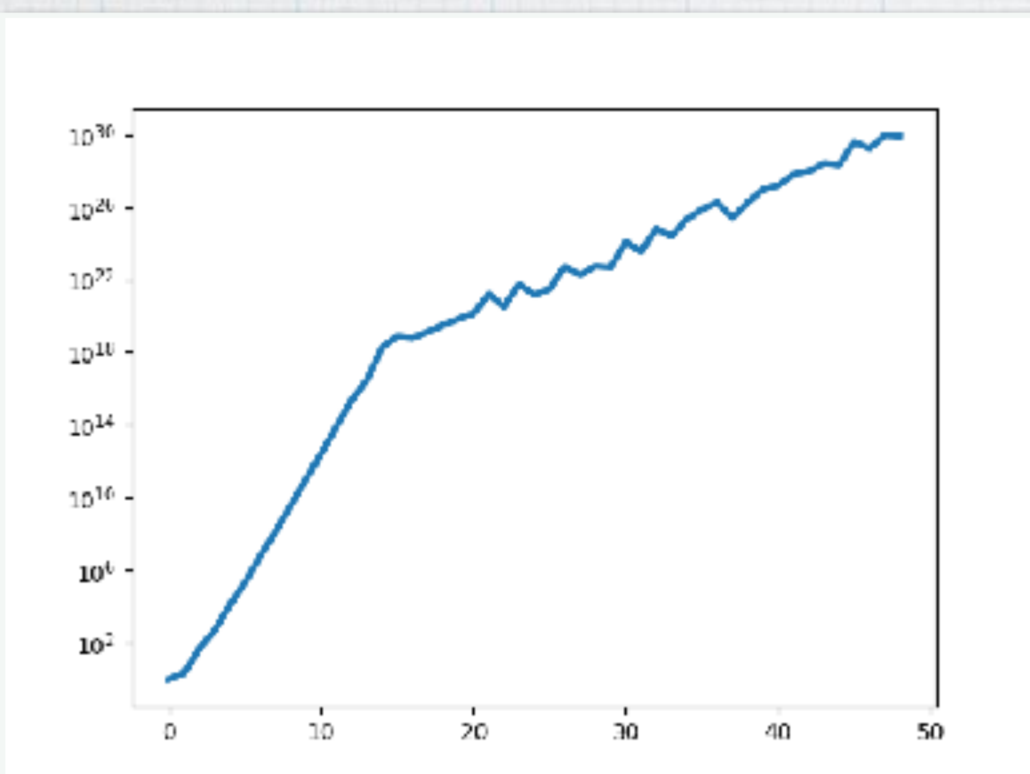
Illustration of the condition number of the Gramian matrix :



Very ill-conditionned Gramian matrix...

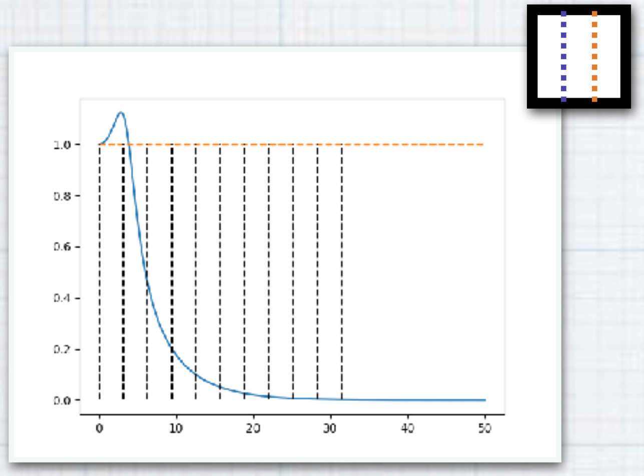
... almost sure it is not a Riesz basis..!

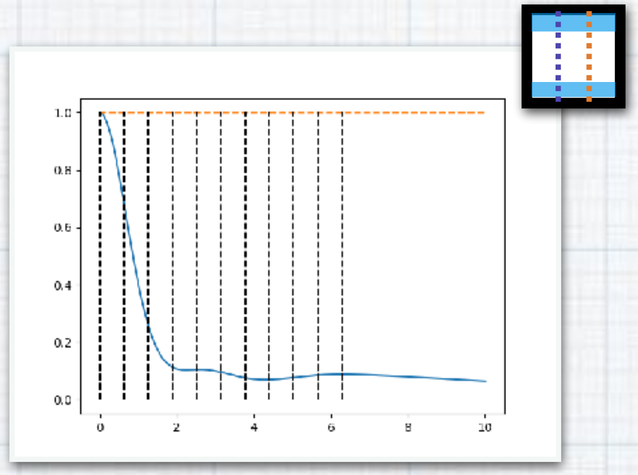
In the next...

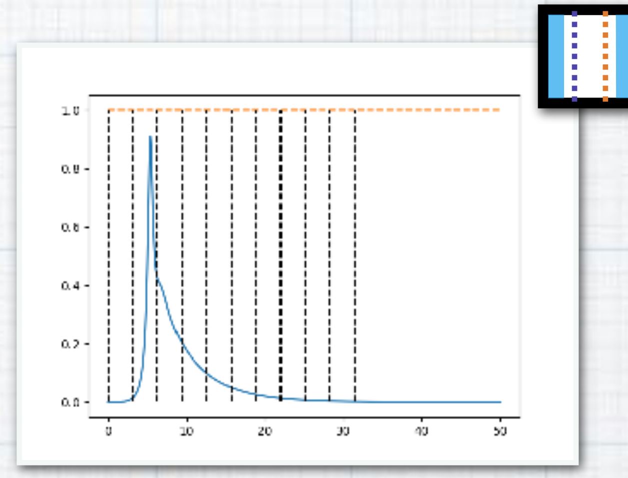
1. Motivation
2. Link with Helmholtz equation
3. Convergence analysis on a toy problem
4. An alternated iterative algorithm
5. Conclusion

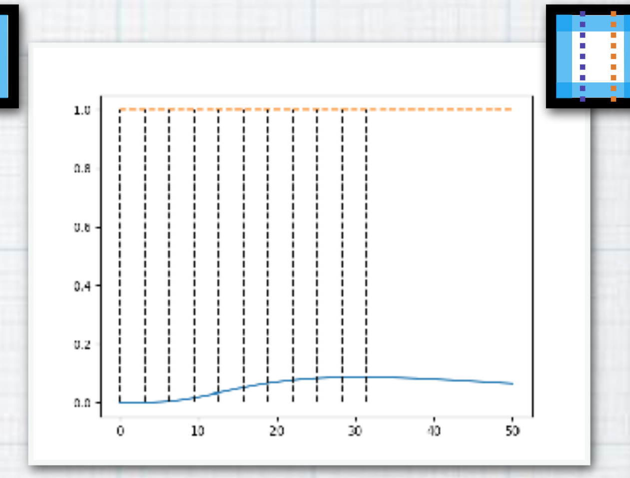
Conclusion

Using **PML has a strong impact on the convergence** of classical iterative algorithm.

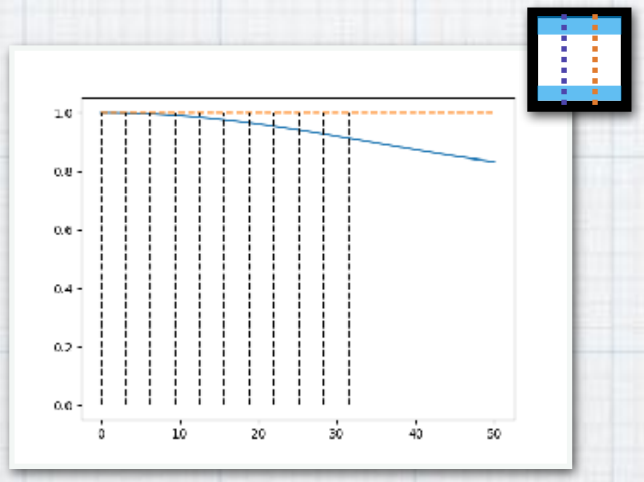


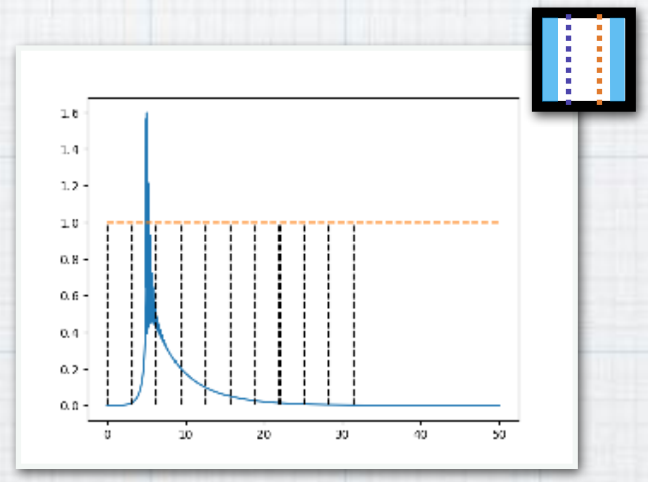






Also, the **PML coefficients must be chosen carefully !**





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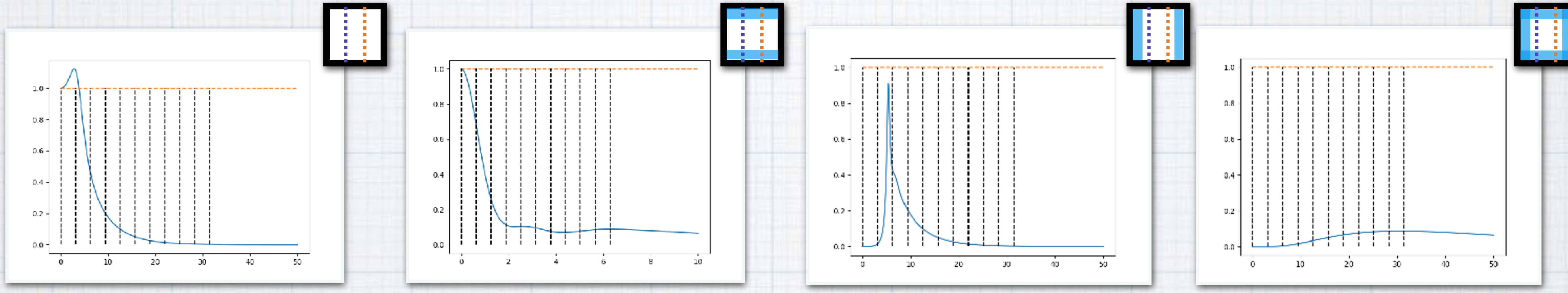
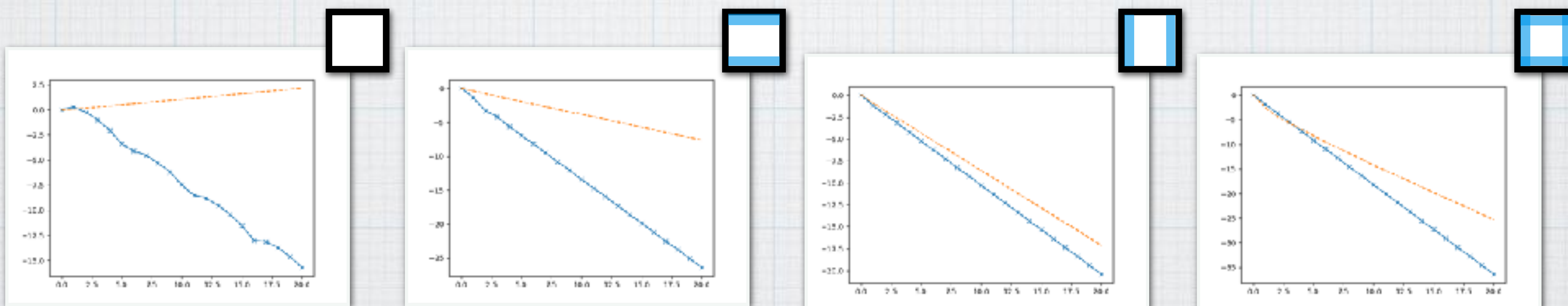


We propose an **alternated algorithm** based on splitting once vertically and once horizontally the domain. This algorithm :

* *improve the convergence factor* in every case



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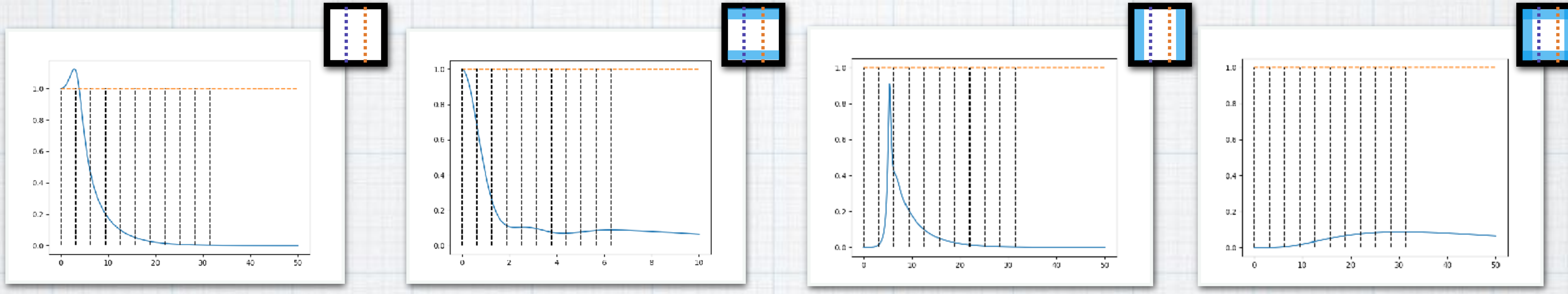
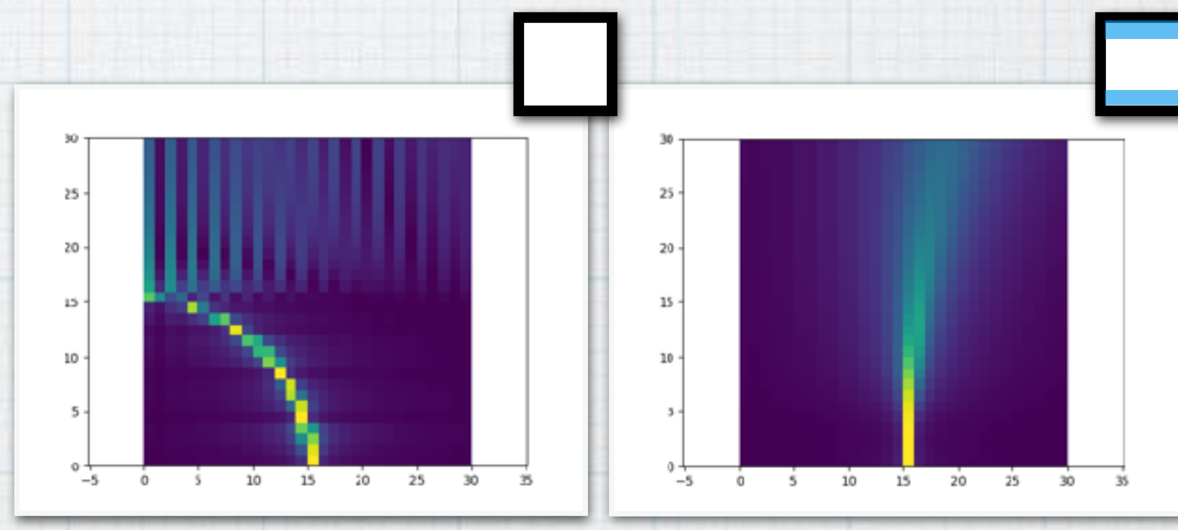


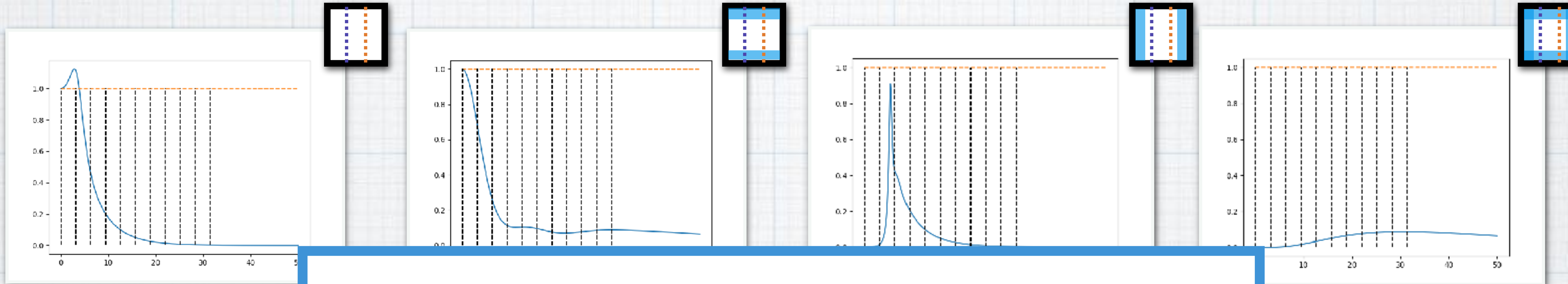
We propose an **alternated algorithm** based on splitting once vertically and once horizontally the domain. This algorithm :

* improve the convergence factor in every case
* and have a different behaviour depending on the PML BCs,



Questions ?

Using **PML has a strong impact on the convergence** of classical iterative algorithm.



Thank you for your attention !!

We propose an ... ically and once horizontally the domain. This algorithm :

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* and have a **different behaviour depending on the PML BCs,**