Optimized Robin Schwarz waveform relaxation methods for optimal control problems

How to use Fourier analysis properly

Laurence HALPERN

LAGA - Université Sorbonne Paris Nord

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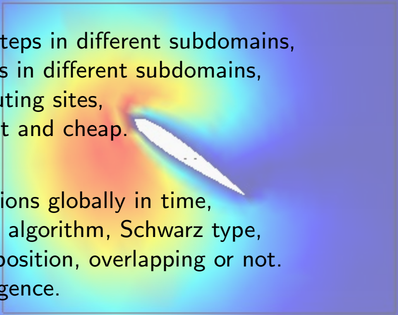
Schwarz waveform relaxation and parabolic problems

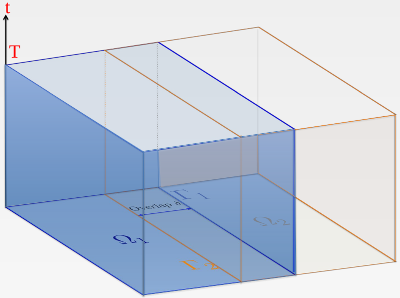
$$(\partial_t + \mathcal{L})u = f \text{ in } \Omega \times (0, T)$$

- The goals
  - Different time-steps in different subdomains,
  - Different models in different subdomains,
  - Different computing sites,
  - Easy to use, fast and cheap.
- The means
  - Treat the equations globally in time,
  - Use an iterative algorithm, Schwarz type,
  - Domain decomposition, overlapping or not.
  - Optimal convergence.



The Schwarz waveform relaxation algorithm

$$(\partial_t + \mathcal{L})u = f \text{ in } \Omega \times (0, T)$$



$$\left\{\begin{array}{rcll} (\partial_t + \mathcal{L})u_1^{n+1} & = & f & \text{in } \Omega_1 \times (0, T) \\ u_1^{n+1}(\cdot, 0) & = & u_0 & \text{in } \Omega_1 \\ \mathcal{B}u_1^{n+1} & = & \mathcal{B}u_2^{n} & \text{on } \Gamma_1 \times (0, T) \end{array}\right.$$

$$\left\{\begin{array}{rcll} (\partial_t + \mathcal{L})u_2^{n+1} & = & f & \text{in } \Omega_2 \times (0, T) \\ u_2^{n+1}(\cdot, 0) & = & u_0 & \text{in } \Omega_2 \\ \mathcal{B}u_2^{n+1} & = & \mathcal{B}u_1^{n} & \text{on } \Gamma_2 \times (0, T) \end{array}\right.$$

$\Omega = (a, b) \times (c, d), \Omega_1 = (a, b_1) \times (c, d),\ \Omega_2 = (b_1 - \delta) \times (c, d)$

$\Gamma_1 = \{b_1\} \times (c, d),\ \Gamma_2 = \{b_1 - \delta\} \times (c, d)$

Dirichlet boundary conditions elsewhere

Optimized Schwarz Waveform relaxation

$$\partial_t + \mathcal{L}, \quad \mathcal{L} = a \cdot \nabla - \nu \Delta + cI$$

$$\mathcal{B}_j u_j^{n+1} = \mathcal{B}_j u_i^n \text{ on } \Gamma_j$$

$$\begin{array}{ll} \mathcal{B} = I & \text{Dirichlet} \\ \mathcal{B} = \nu \partial_n + pI & \text{Robin} \\ \mathcal{B} = \nu \partial_n + pI - q(a_{tan}\partial_{tan} - \nu \partial_{tan}^2) & \text{Ventcel} \end{array}$$

Fourier analysis to optimize the convergence factor
Attention ! $[0, T] \to \mathbb{R}$

$$\inf_{(p,q)>0} \sup_{|k|\in K, |\eta|\in H} \left| \frac{f(z) - (p+qz)}{f(z) + (p+qz)} e^{-\delta f(z)} \right|$$

$$z = 4\nu(i(k + a_y\eta) + \nu\eta^2), \; f(z) = \sqrt{a^2 + 4\nu c + z}$$

Optimized Schwarz Waveform relaxation

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Fourier analysis to optimize the convergence factor

$$z = 4\nu(i(k + a_y\eta) + \nu\eta^2), \; f(z) = \sqrt{a^2 + 4\nu c + z}$$

$$\inf_{P \in \mathbf{P}_1} \sup_{|k| \in K, |\eta| \in H} \left| \frac{f(z) - P(z)}{f(z) + P(z)} e^{-\delta f(z)} \right|$$

Homographic weighted best approximation problem.
Remark For fixed η, $f(z)$ runs on a branch of hyperbola

1-D Problem, Robin , $\inf_{P \in \mathbf{P}_1} \sup_{Z \in \mathcal{H}} \left| \frac{Z - p}{Z + p} e^{-\delta Z} \right|, \; \mathcal{H} : \; x^2 - y^2 = a^2 + 4\nu c.$

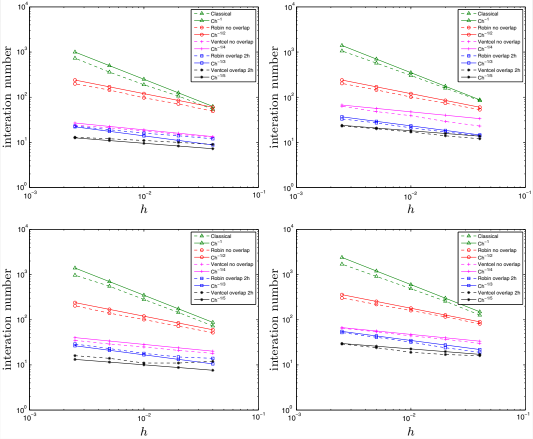
Performance of Optimized Schwarz Waveform relaxation

Algorithms with overlap $2h$

		Iterative					*GMRES*				
h		0.04	0.02	0.01	0.005	0.0025	0.04	0.02	0.01	0.005	0.0025
	2x1	54	106	189	360	733	27	40	58	83	117
Dirichlet	2x2	84	159	303	570	1058	37	56	82	118	166
	4x1	73	145	282	553	969	38	60	89	127	179
$1/h$	4x4	127	258	487	912	1706	54	94	143	209	296
	2x1	12	14	16	19	23	8	10	12	14	17
Robin	2x2	14	17	21	27	33	11	14	17	20	24
	4x1	14	15	18	23	29	11	13	16	20	24
$1/\sqrt[3]{h}$	4x4	19	24	32	41	52	14	20	26	32	40
	2x1	9	10	11	12	13	6	7	8	9	10
Ventcel	2x2	12	14	17	20	23	8	10	11	13	16
	4x1	12	11	11	14	16	10	9	9	11	13
$1/\sqrt[5]{h}$	4x4	16	17	19	24	29	13	13	14	18	22

Bennequin-Gander-Gouarin-Halpern, 2016.

Performance of Optimized Schwarz Waveform relaxation



Bennequin-Gander-Gouarin-Halpern, 2016.

Some bibliography

- V. Martin. An optimized Schwarz waveform relaxation method for the unsteady convection diffusion equation in two dimensions. *Applied Numerical Mathematics*, 52(4):401–428, 2005
- M. J. Gander and L. Halpern. Optimized Schwarz waveform relaxation methods for advection reaction diffusion problems. *SIAM J. Numer. Anal.*, 45(2):666–697, 2007
- D. Bennequin, M. J. Gander, and L. Halpern. A homographic best approximation problem with application to optimized Schwarz waveform relaxation. *Math. Comput.*, 78:185–223, 2009
- D. Bennequin, M. J. Gander, L. Gouarin, and L. Halpern. Optimized Schwarz waveform relaxation for advection reaction diffusion equations in two dimensions. *Numer. Math.*, 134(3):513–567, 2016

1-D Heat equation, periodic in time case

The substructured algorithm

$$\Omega = \mathbb{R}, \quad \Omega_1 = (-\infty, \delta), \quad \Omega_2 = (0, +\infty)$$

$$\begin{cases} (\partial_t - \partial_{xx} + c)u = f & \text{in } \Omega \times (0, T) \\ u(\cdot, 0) = u(\cdot, T) & \text{in } \Omega \end{cases}$$

$$\begin{cases} (\partial_t - \partial_{xx} + c)u_1^{n+1} = f & \text{in } \Omega_1 \times (0, T) \\ u_1^{n+1}(\cdot, 0) = u_1^{n+1}(\cdot, T) & \text{in } \Omega_1 \\ (\partial_x + p)u_1^{n+1}(\delta, \cdot) = g_1^n & \text{in } (0, T). \end{cases}$$

$$\begin{cases} (\partial_t - \partial_{xx} + c)u_2^{n+1} = f & \text{in } \Omega_2 \times (0, T) \\ u_2^{n+1}(\cdot, 0) = u_2^{n+1}(\cdot, T) & \text{in } \Omega_2 \\ (-\partial_x + p)u_2^{n+1}(0, \cdot) = g_2^n & \text{in } (0, T) \end{cases}$$

$$h_1^{n+1} = (\partial_x + p)u_2^{n+1}(\delta, \cdot), \quad h_2^{n+1} = (-\partial_x + p)u_1^{n+1}(0, \cdot)$$

$$\text{Error } u_j^n \to u_j^n - u, \; g_j^n \to g_j^n - g_j$$

The substructured algorithm for the error

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Fourier series in time

$$u_j^n(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_j^n(x,k) e^{\frac{2i\pi kt}{T}}$$

$$\left\{\begin{array}{ll} (\frac{2i\pi k}{T} + c - \partial_{xx})\hat{u}_1^{n+1} = 0 & \text{on } (-\infty,\delta) \\ (\partial_x + p)\hat{u}_1^{n+1}(\delta,\cdot) = \hat{g}_1^n & \text{in } (0,T) \end{array}\right.$$

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$$\hat{g}_1^{n+1} = (\partial_x + p)\hat{u}_2^{n+1}(\delta), \quad \hat{g}_2^{n+1} = (-\partial_x + p)\hat{u}_1^{n+1}(0).$$

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Explicit computation for fixed $k \in \mathbb{Z}$,

$$r = \sqrt{\frac{2i\pi k}{T} + c}, \quad e^{\varepsilon r x} \notin \mathcal{S}'(\varepsilon\mathbb{R}_+).$$

$$\hat{u}_1^{n+1} = \frac{\hat{g}_1^n}{p+r} e^{r(x-\delta)}, \quad \hat{u}_2^{n+1} = \frac{\hat{g}_2^n}{p+r} e^{-rx}$$

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Convergence factor

$$r = \sqrt{\frac{2i\pi k}{T} + c}, \quad \rho(k,p,\delta) = \frac{p-r}{p+r} e^{-r\delta}, \quad k \in \mathbb{Z},\ p \in \mathbb{R}_+,\ \delta \geq 0.$$

Fourier series in time

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1. Convergence of the algorithm:

$$\operatorname{Re} r \geq \sqrt{c} \implies |\rho(k,p,\delta)| \leq e^{-\sqrt{c}\delta}$$

   Lebesgue+Parseval theorem.
   Attention ! For $\delta = 0$, $\lim_{k\to\infty} |\rho(k,p,\delta)| = 1$.

2. Optimization of the algorithm

Fourier series in time

Convergence factor

$$r = \sqrt{\frac{2i\pi k}{T} + c}, \quad \rho(k,p,\delta) = \frac{p-r}{p+r} e^{-r\delta}, \quad k \in \mathbb{Z},\ p \in \mathbb{R}_+,\ \delta \geq 0.$$

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2. Optimization of the algorithm

Optimization of the convergence factor

$$g_j^n(t) = \sum_{k\in\mathbb{Z}} \hat{g}_j^n(k) e^{\frac{2i\pi kt}{T}}$$

Convergence factor

$$r = \sqrt{\frac{2i\pi k}{T} + c}, \quad \rho(k,p,\delta) = \frac{p-r}{p+r} e^{-r\delta}, \quad k \in \mathbb{Z},\ p \in \mathbb{R}_+,\ \delta \geq 0.$$

Optimization problem

$$\inf_{p\geq 0} \sup_{k\in K} |\rho(k,p,\delta)|, \quad K = (1, \frac{T}{\Delta t}).$$

Heat equation, general case

The substructured algorithm

$$\Omega = \mathbb{R}, \quad \Omega_1 = (-\infty, \delta), \quad \Omega_2 = (0, +\infty)$$

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The substructured algorithm for the error

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Optimized Robin. 1D case

$$(\partial_t - \partial_{xx} + c)u = f \text{ in } \Omega \times (0, T), \quad u(\cdot, 0) = u^0$$

Algorithm for the error. Initial guesses g_1, g_2 in ${}_0H^1(0, T) \subset \mathcal{C}([0, T])$

$$\begin{cases} (\partial_t - \partial_{xx} + c)u_1 = 0 \text{ in } \Omega_1 \times (0, T) \\ u_1(\cdot, 0) = 0 \text{ in } \Omega_1 \\ (\partial_x + p)u_1(\delta, \cdot) = g_1 \text{ in } (0, T) \end{cases} \begin{cases} (\partial_t - \partial_{xx} + c)u_2 0 \text{ in } \Omega_2 \times (0, T) \\ u_2(\cdot, 0) = 0 \text{ in } \Omega_2 \\ (-\partial_x + p)u_2(0, \cdot) = g_2 \text{ in } (0, T) \end{cases}$$

$$g_1' = (\partial_x + p)u_2(\delta, \cdot), \quad g_2' = (-\partial_x + p)u_1(0, \cdot).$$

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$$g_1' = (\partial_x + p)u_2(\delta, \cdot), \quad g_2' = (-\partial_x + p)u_1(0, \cdot).$$

Fourier transform in time?

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-ikt}\, dt$$

Optimized Robin. 1D case

$$(\partial_t - \partial_{xx} + c)u = f \text{ in } \Omega \times (0, T), \quad u(\cdot, 0) = u^0$$

Algorithm for the error. Initial guesses g_1, g_2 in ${}_0H^1(0, T) \subset \mathcal{C}([0, T])$

$$\begin{cases} (\partial_t - \partial_{xx} + c)u_1 = 0 \text{ in } \Omega_1 \times (0, T) \\ u_1(\cdot, 0) = 0 \text{ in } \Omega_1 \\ (\partial_x + p)u_1(\delta, \cdot) = g_1 \text{ in } (0, T) \end{cases} \begin{cases} (\partial_t - \partial_{xx} + c)u_2 0 \text{ in } \Omega_2 \times (0, T) \\ u_2(\cdot, 0) = 0 \text{ in } \Omega_2 \\ (-\partial_x + p)u_2(0, \cdot) = g_2 \text{ in } (0, T) \end{cases}$$

$$g_1' = (\partial_x + p)u_2(\delta, \cdot), \quad g_2' = (-\partial_x + p)u_1(0, \cdot).$$

But g_j are defined on $[0, T]$ only.

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$$\left\{ \begin{array}{l} (\partial_t - \partial_{xx} + c)u_1 = 0 \text{ in } \Omega_1 \times (0, T) \\ u_1(\cdot, 0) = 0 \text{ in } \Omega_1 \\ (\partial_x + p)u_1(\delta, \cdot) = g_1 \text{ in } (0, T) \end{array} \right. \left\{ \begin{array}{l} (\partial_t - \partial_{xx} + c)u_2 0 \text{ in } \Omega_2 \times (0, T) \\ u_2(\cdot, 0) = 0 \text{ in } \Omega_2 \\ (-\partial_x + p)u_2(0, \cdot) = g_2 \text{ in } (0, T) \end{array} \right.$$

$$g_1' = (\partial_x + p)u_2(\delta, \cdot), \quad g_2' = (-\partial_x + p)u_1(0, \cdot).$$

But g_j are defined on $[0, T]$ only.
Extend g_j to $\mathbb{R}$ properly.

Optimized Robin. Fourier analysis

$$\begin{cases} (\partial_t - \partial_{xx} + c)u_1 = 0 \text{ in } \Omega_1 \times (0,T) \\ u_1(\cdot,0) = 0 \text{ in } \Omega_1 \\ (\partial_x + p)u_1(\delta,\cdot) = g_1 \text{ in } (0,T) \end{cases} \begin{cases} (\partial_t - \partial_{xx} + c)u_2 = 0 \text{ in } \Omega_2 \times (0,T) \\ u_2(\cdot,0) = 0 \text{ in } \Omega_2 \\ (-\partial_x + p)u_2(0,\cdot) = g_2 \text{ in } (0,T) \end{cases}$$

$$g_1' = (\partial_x + p)u_2(\delta,\cdot), \quad g_2' = (-\partial_x + p)u_1(0,\cdot).$$

- $g_1 \in {}_0H^1(0,T)$. Extend by $\tilde{g}_1 \in H^1(\mathbb{R})$ (vanishing for $t \le 0$) .
- Extend the equation to $t \in \mathbb{R}$

$$(EE) \quad (\partial_t - \partial_{xx} + c)\tilde{u}_1 = 0 \text{ in } (-\infty,\delta)\times\mathbb{R}, \quad (\partial_x + p)\tilde{u}_1(\delta,\cdot) = \tilde{g}_1 \text{ in } \mathbb{R}$$

- Fourier transform in time the equation

$$\forall k \in \mathbb{R},\ (\partial_{xx} - (ik + c))S = 0 \text{ in } (-\infty,\delta),\ (\partial_x + p)S(\delta,\cdot) = \mathcal{F}(\tilde{g}_1) \text{ in } \mathbb{R}$$

- write explicitly the solution and compute $(-\partial_x + p)S(0,\cdot)$

$$(-\partial_x + p)S(0,\cdot) = \frac{p-r}{p+r}e^{r(x-\delta)}\mathcal{F}(\tilde{g}_1), \quad r = \sqrt{ik + c}.$$

- By Paley-Wiener theorem, the inverse Fourier transform of S is vanishing for negative t and satisfies (EE) .
- Uniqueness comes from energy estimates) .
- Conclude by the causality principle that $u_1 = \tilde{u}_1|_{[0,T]}$.

Optimized Robin. Fourier analysis

$$\begin{cases} (\partial_t - \partial_{xx} + c)u_1 = 0 \text{ in } \Omega_1 \times (0,T) \\ u_1(\cdot,0) = 0 \text{ in } \Omega_1 \\ (\partial_x + p)u_1(\delta,\cdot) = g_1 \text{ in } (0,T) \end{cases} \begin{cases} (\partial_t - \partial_{xx} + c)u_2 = 0 \text{ in } \Omega_2 \times (0,T) \\ u_2(\cdot,0) = 0 \text{ in } \Omega_2 \\ (-\partial_x + p)u_2(0,\cdot) = g_2 \text{ in } (0,T) \end{cases}$$

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Convergence factor

$$(g_1, g_2) \in ({}_0H^1(0,T))^2 \;\to\; (\tilde{g}_1', \tilde{g}_2') \in ({}_0H^1(0,T))^2,$$

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Convergence of the algorithm: Lebesgue+Parseval theorem.

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2. Take the infimum over all extensions

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- L. Halpern and J. Szeftel. Optimized and quasi-optimal Schwarz waveform relaxation for the one-dimensional Schrödinger equation. *Mathematical Models and Methods in Applied Sciences*, 20(12):2167–2199, 2010
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Earlier experiments

How to choose the transmission operators ?

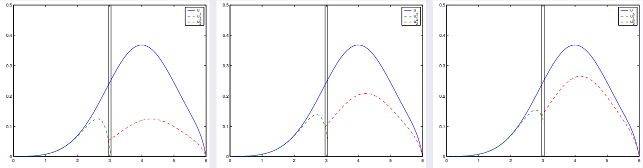
Classical Schwarz

$\mathcal{B}_j \equiv I$ AND overlap.

1D Numerical experiment

$$a = 1, \nu = 0.2, \Omega = (0, 6), T = 2.5, L = 0.08.$$

$u_1^1(\cdot, T), u_2^2(\cdot, T)$ $\quad$ $u_1^3(\cdot, T), u_2^4(\cdot, T)$ $\quad$ $u_1^5(\cdot, T), u_2^6(\cdot, T)$



Earlier experiments

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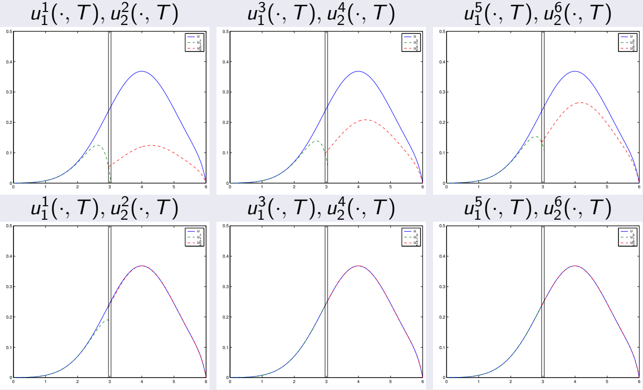
Comparison



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First attempt : Robin transmission condition

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k=1 ; k=2 ; k=3 ;

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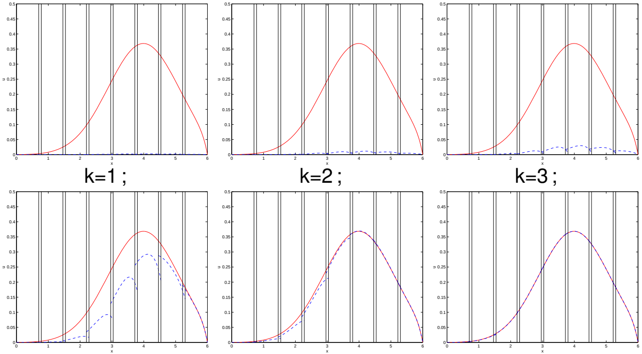
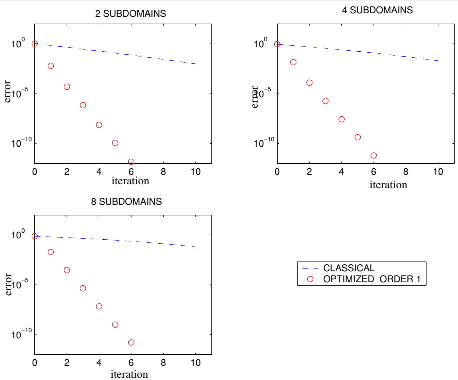


With 2 subdomains : Gander, L.H, Nataf, DD 11, 1998.

Earlier experiments

One dimension : comparison



Optimal control problem in time

The optimal control problem periodic in time

$$J(y,u) = \frac{1}{2}\|y - y_Q\|^2_{L^2(\Omega\times(0,T))} + \frac{\sigma}{2}\|u\|^2_{L^2(\Omega\times(0,T))}$$

subject to the linear parabolic constraint

$$\begin{array}{l} \partial_t y - \lambda\partial_{xx} y + dy = u \text{ in } (0,T)\times\Omega \\ y(\cdot,0) = y(\cdot,T) \text{ in } \Omega \end{array}$$

with $\sigma, \lambda, d > 0$.

J $\alpha-$convex, so well-posed problem in adapted spaces $H^{2r,r}$. P. G. Ciarlet. *Introduction to Numerical Linear Algebra and Optimisation.* Cambridge Texts in Applied Mathematics. Cambridge University Press, 1989

The optimality system

$$\sigma u - q = 0$$

Forward heat equation

$$\partial_t y - \lambda \partial_{xx} y + dy = u \text{ in } \Omega \times (0, T)$$
$$y(\cdot, 0) = y(\cdot, T) \text{ in } \Omega$$

Backward heat equation

$$-\partial_t q - \lambda \partial_{xx} q + dq = y_Q - y \text{ in } \Omega \times (0, T)$$
$$q(\cdot, 0) = q(\cdot, T) \text{ in } \Omega$$

J. L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations.*
Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer-Verlag, 1971

The parallel Robin-Schwarz waveform relaxation algorithm

$$\mathcal{T}(\underline{g}_1, \underline{g}_2) = (\underline{g}_1', \underline{g}_2') :$$

$$\begin{aligned}
&q_j = \sigma u_j \text{ in } \Omega_j \times (0, T) \\
&(\partial_t - \lambda\partial_{xx} + d)y_j = u_j \text{ in } \Omega_j \times (0, T) \\
&(\partial_{n_j} + p)y_j = g_j \text{ on } \Gamma_j \times (0, T) \\
&y_j(\cdot, 0) = y_j(\cdot, T) \text{ in } \Omega_j \\[1ex]
&(-\partial_t - \lambda\partial_{xx} + d)q_j = y_Q - y_j \text{ in } \Omega_j \times (0, T) \\
&(\partial_{n_j} + p)q_j = \tilde{g}_j \text{ on } \Gamma_j \times (0, T) \\
&q_j(\cdot, 0) = q_j(\cdot, T) \text{ in } \Omega_j \\
&(g', \tilde{g}')_i = (\partial_{n_i} + p)(y, q)_j.
\end{aligned}$$

Well-posedness in $(H_{\#}^{3/4}(0, T))^4$.

Theorem The coupled system is the Euler system for the minimisation of the functionals

$$J_j(y_j, u_j) = \frac{1}{2}\|y_j - y_Q\|^2_{L^2(\Omega_j \times (0,T))} + \frac{\sigma}{2}\|u_j\|^2_{L^2(\Omega_j \times (0,T))} - (\tilde{g}_j, y_j)_{L^2(\Gamma_j \times (0,T))}$$

Some bibliography

1. R. Bartlett, M. Heinkenschloss, D. Ridzal, and B. van

   Bloemen Waanders. Domain decomposition methods for advection dominated linear quadratic elliptic optimal control problem.
   Comput. Methods. Appl. Mech. Eng., 195, 2006

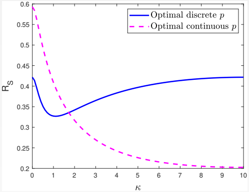
2. J. Lagnese and G. Leugering. *Domain in Decomposition Methods in Optimal Control of Partial Differential Equations.*
   Springer Verlag NY, 2004

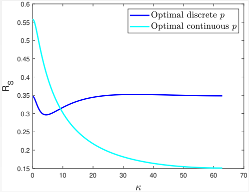
3. J.-D. Benamou. A domain decomposition method for control problems.
   In *Proceedings of the 9th International Conference on Domain Decomposition Methods*, pages 266–273, 1998

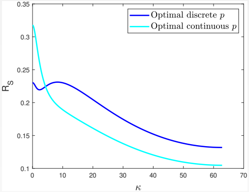
4. B. Delourme and L. Halpern. A complex homographic best approximation problem. application to optimized Robin–Schwarz algorithms, and optimal control problems.
   SIAM Journal on Numerical Analysis, 59(3):1769–1810, 2021

Semi-discretization in time

Semi-discretization in time







Nooverlap

Overlap Case 1

Overlap Case 2.

Semi-discretisation in time

$\Delta t = T/S$, $t_s = s\Delta t$ for $s = 0, \dots, S$. The functions y and q of t and x are approximated by vectors Y and Q in $\mathbb{R}^{S+1}$, functions of x, with components indexed by s. Y_Q is the vector defined by $(Y_Q)_s = y_Q(t_s)$. Implicit Euler scheme:

$$\left| \begin{array}{l} \frac{1}{\Delta t}(Y_s - Y_{s-1}) - \lambda\partial_{xx}Y_s + dY_s = U_s \text{ in } [\![1, S]\!] \times \Omega, \\ Y_0 = Y_S \text{ in } \Omega, \end{array} \right. \tag{1a}$$

$$\sigma U = Q \text{ in } \text{ in } [\![1, S]\!] \times \Omega \tag{1b}$$

$$\left| \begin{array}{l} \frac{1}{\Delta t}(Q_s - Q_{s+1}) - \lambda\partial_{xx}Q_s + dQ_s = (Y_Q)_s - Y_s \text{ in } [\![0, S-1]\!] \times \Omega, \\ Q_S = Q_0 \text{ in } \Omega. \end{array} \right. \tag{1c}$$

Theorem This is the optimality system for the minimization of

$$J(U, Y) = \frac{1}{2}\|Y - Y_Q\|^2_{\mathbb{L}^2(\Omega)} + \frac{\sigma}{2}\|U\|^2_{\mathbb{L}^2(\Omega)} \tag{2}$$

subject to (1a).

Semi-discretized Robin SWR

$$\mathcal{T}_{\Delta t}(\underline{G}_1, \underline{G}_2) = (\underline{G}_1', \underline{G}_2') :$$

For $j = 1, 2$, given $\underline{G}_j = (G_j, \tilde{G}_j) \in R^2_{\#}$ (periodic in time), solve (3a)

$$Q_j = \sigma U_j \tag{3b}$$

$$\left| \begin{aligned} &\frac{Y_j(s) - Y_j(s-1)}{\Delta t} - \lambda \partial_{xx} Y_j(s) + dY_j(s) = U_j(s) \text{ in } [\![1, S]\!] \times \Omega_j, \\ &\partial_{n_j} Y_j(\cdot, x_j) + pY_j(\cdot, x_j) = G_j \text{ in } [\![0, S]\!], \\ &Y_j(0, \cdot) = Y_j(S, \cdot) \text{ in } \Omega_j, \end{aligned} \right. \tag{3c}$$

$$\left| \begin{aligned} &\frac{Q_j(s) - Q_j(s+1)}{\Delta t} - \lambda \partial_{xx} Q_j(s) + dQ_j(s) = Y_Q(s) - Y_j(s) \text{ in } [\![0, S-1]\!] \times \Omega_j, \\ &\partial_{n_j} Q_j(\cdot, x_j) + pQ_j(\cdot, x_j) = \tilde{G}_j \text{ in } [\![0, S]\!], \\ &Q_j(0, \cdot) = Q_j(S, \cdot) \text{ in } \Omega_j, \end{aligned} \right. \tag{3d}$$

Compute for $i \neq j$, in $[\![0, S]\!]$,

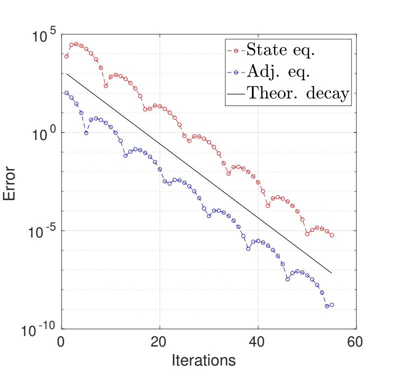
$$\left| \begin{aligned} &G_i' = \partial_{n_i} Y_j(\cdot, x_i) + pY_j(\cdot, x_i) \text{ in } [\![0, S]\!], \\ &\tilde{G}_i' = \partial_{n_i} Q_j(\cdot, x_i) + pQ_j(\cdot, x_i) \text{ in } [\![0, S]\!], \\ &\underline{G}_i' = (G_i', \tilde{G}_i') \in R^2_{\#}. \end{aligned} \right. \tag{3e}$$

Convergence analysis: Fourier series in time + diagonalisation

Theorem Let $\lambda > 0$ and $d > 0$. There is a constant $C > 0$ such that, for any initial guess $\underline{G}^0 \in R^4_\#$, and for any $p > 0$ and $\sigma > 0$,

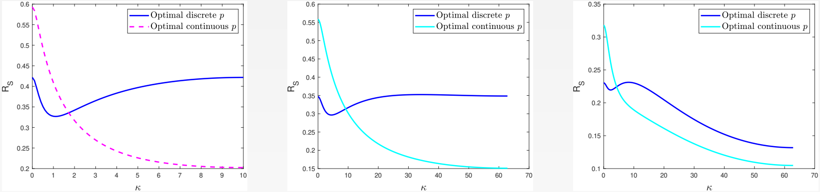
$$\|\underline{\mathcal{G}}^n\| \leq C \sup_{\kappa \in [\![0, S-1]\!]} |\rho_S(\kappa, p, \delta)|^n \|\underline{\mathcal{G}}^0\|.$$

with $\mathcal{G}_j = G_j - Y(x_j)$. Furthermore, $\sup_{\kappa \in [\![0,S-1]\!]} |\rho_S(\kappa, p, \delta)| < 1$, therefore the sequence is convergent.



Optimization

G. Ciaramella, L. Halpern, and L. Mechelli. Convergence analysis and optimization of a robin schwarz waveform relaxation method for periodic parabolic optimal control problems.
submitted, 2023
See DD27 talk, Prague, July 2022.



Nooverlap Overlap Case 1 Overlap Case 2.

Optimal control problem in time nonperiodic case

Optimal control problem in time nonperiodic case

Ongoing work with Gabriele Ciaramella and Luca Mechelli

The optimal control problem

$$J(y,u) = \frac{1}{2}\|y - y_Q\|^2_{L^2(\Omega\times(0,T))} + \frac{\sigma}{2}\|u\|^2_{L^2(\Omega\times(0,T))}$$

subject to the PDE-constraint

$$\begin{aligned} &\partial_t y - \lambda \partial_{xx} y + dy = u \text{ in } (0,T)\times\Omega \\ &y(\cdot,0) = y_0 \text{ in } \Omega \end{aligned}$$

with $\sigma, \lambda, d > 0$.

See R. Glowinski and J.L. Lions.

The optimality system

Adjoint state

$$q = \sigma u$$

Forward heat equation

$$\partial_t y - \lambda \partial_{xx} y + dy = \frac{1}{\sigma} q \text{ in } \Omega \times (0, T)$$

Initial value $y(\cdot, 0) = y_0$ in Ω

Backward heat equation

$$-\partial_t q - \lambda \partial_{xx} q + dq = y_Q - y \text{ in } \Omega \times (0, T)$$

Final value $q(\cdot, T) = 0$ in Ω

The parallel Robin-Schwarz waveform relaxation algorithm

$$q_j = \sigma u_j \text{ in } \Omega_j \times (0,T)$$
$$(\partial_t - \lambda\partial_{xx} + d)y_j = \frac{1}{\sigma} q_j \text{ in } \Omega_j \times (0,T)$$
$$(\partial_{n_j} + p)y_j = Y_j \text{ on } \Gamma_j \times (0,T)$$
$$\text{Initial value } y_j(\cdot,0) = y_0 \text{ in } \Omega_j$$

$$(-\partial_t - \lambda\partial_{xx} + d)q_j = y_Q - y_j \text{ in } \Omega_j \times (0,T)$$
$$(\partial_{n_j} + p)q_j = Q_j \text{ on } \Gamma_j \times (0,T)$$
$$\text{Final value } q_j(\cdot,T) = 0 \text{ in } \Omega_j$$
$$(Y', Q')_j = (\partial_{n_j} + p)(y,q)_i,$$

Lemma The coupled system is the Euler system for the minimisation of the functionals

$$J_j(y_j,u_j) = \frac{1}{2}\|y_j - y_Q\|^2_{L^2(\Omega_j\times(0,T))} + \frac{\sigma}{2}\|u_j\|^2_{L^2(\Omega_j\times(0,T))} - (Q_j, y_j)_{L^2(\Gamma_j\times(0,T))}$$

see Desprès-Benamou, Lagnese, Leugering,..

1. Fourier series in time ? If periodicity
2. Fourier transform in time ? No causality
3. Wanted: diagonalize the operator. Help Mr d'Alembert

The parallel Robin-Schwarz waveform relaxation algorithm

Equations on the error:

$$(\partial_t - \lambda\partial_{xx} + d)y_j = \frac{1}{\sigma}q_j \text{ in } \Omega_j \times (0, T)$$
$$(\partial_{n_j} + p)y_j = Y_j \text{ on } \Gamma_j \times (0, T)$$
Initial value $y_j(\cdot, 0) = \cancel{y_0}\ 0$ in Ω_j

$$(-\partial_t - \lambda\partial_{xx} + d)q_j = \cancel{y_Q} - y_j \text{ in } \Omega_j \times (0, T)$$
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A backward temporal journey in the eighteen century

Joseph Fourier (1768-1830)

Jean le Rond d'Alembert (1717-1783)

The system in the subdomain

$$
\begin{aligned}
&\partial_t y - \lambda \partial_{xx} y + dy = \frac{1}{\sigma} q \text{ in } (0, T) \times \Omega, \\
&y(\cdot, 0) = 0 \text{ in } \Omega, \\
&-\partial_t q - \lambda \partial_{xx} q + dq = -y \text{ in } (0, T) \times \Omega, \\
&q(\cdot, T) = 0 \text{ in } \Omega
\end{aligned}
$$

Separation of variables, d'Alembert [1]

$$y = \varphi(x)\psi(t)$$

[1] D'Alembert, Addition au mémoire sur la courbe que forme une corde tendue, mise en vibration, Hist. Ac. Se. Berlin, 1750.

Separation of variables, d'Alembert [1]

$$y = e^{-rx}\sin(at)$$
$$\sin^2(aT) + \sigma a^2 = 0, r = \sqrt{\frac{d + a\,\mathrm{cotan}(aT)}{\lambda}}$$

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Theorem 1 Countable family of solutions, $(a_k, \overline{a_k})$.
$a_k \sim \frac{1}{T}(k\pi + i\,\mathrm{sgn}(k)\log(|k|\pi\sigma^{-1})),\ k \in \mathbb{Z}^*$

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$a_k \sim \frac{1}{T}(k\pi + i\,\mathrm{sgn}(k)\log(|k|\pi\sigma^{-1})),\ k \in \mathbb{Z}^*$
Theorem 2 The sequence $\{\sin a_k t\}_{k\geq 1}$ is minimal complete in $\mathcal{C}_{\#}(0,\pi) = \{g \in \mathcal{C}(0,\pi)\ g(0) = 0\}$ and in $L^2(0,\pi)$.
Analysis based on nonharmonic Fourier series theory,

1. R. M. Young. *An introduction to nonharmonic Fourier series.* Academic press, 1981
2. A. Sedleckii. On completeness of the systems {exp (ix (n+ ihn))}. *Analysis Mathematica*, 4(2):125–143, 1978

Related problem: differential-delay equations, $y'(t) = -ay(t-1)$.

[1] D'Alembert, Addition au mémoire sur la courbe que forme une corde tendue, mise en vibration, Hist. Ac. Se. Berlin, 1750.

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&q(\cdot, T) = 0 \text{ in } \Omega
\end{aligned}
$$

Eliminate q

$$\partial_{tt} y - \lambda^2 \partial_x^4 y + 2d\lambda \partial_x^2 y - (d^2 + \sigma^{-1}) y = 0$$

with two boundary conditions in time

$$y(\cdot, 0) = 0, \quad q(\cdot, T) = \sigma(\partial_t y - \lambda \partial_{xx} y + dy)(\cdot, T) = 0.$$

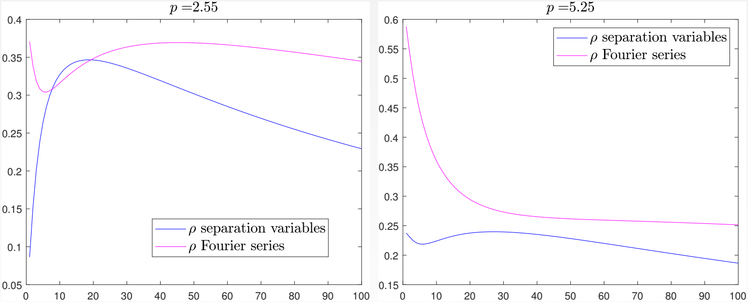
Convergence

$\varphi_k(t) = \sin(a_k t)$ diagonalize the iteration operator.

1. Define the convergence factor

$$\rho(k,p) = \left(\frac{r(k)-p}{r(k)+p} e^{-r(k)L}\right)^2, \; r(k) = \sqrt{\frac{d + a_k \operatorname{cotan}(a_k T)}{\lambda}}$$

2. Convergence for the linear combination of those modes.
3. Optimization of the convergence factor to define the best p.



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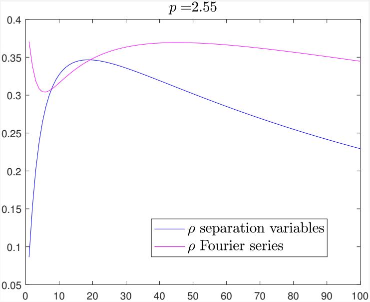

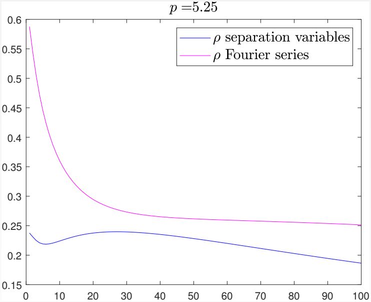

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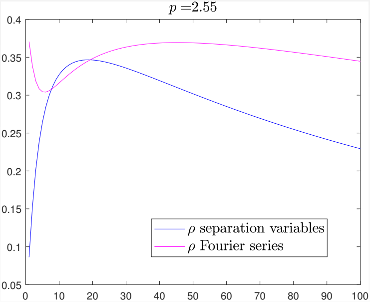

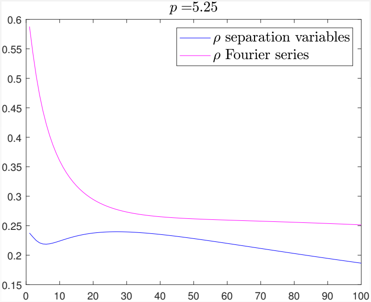

Conclusion

1. Separation of variables powerful tool for control,
2. Extension to the wave equation ongoing,
3. also useful in other DD issues: preconditioner for DD with cross points, Cuvelier, Gander, Halpern DD27 as well.

Reference: *Convergence analysis and optimization of a Robin Schwarz waveform relaxation method for periodic parabolic optimal control problems*, Gabriele Ciaramella, Laurence Halpern and Luca Mechelli, to be submitted soon.

Thank you