Nonoverlapping domain decomposition of nonlinear p-type optimal control problems on metric graphs

by the example of gas networks

Jean-Morlet Chair 2022: Research School - Domain Decomposition for Optimal Control Problems
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TRR 154

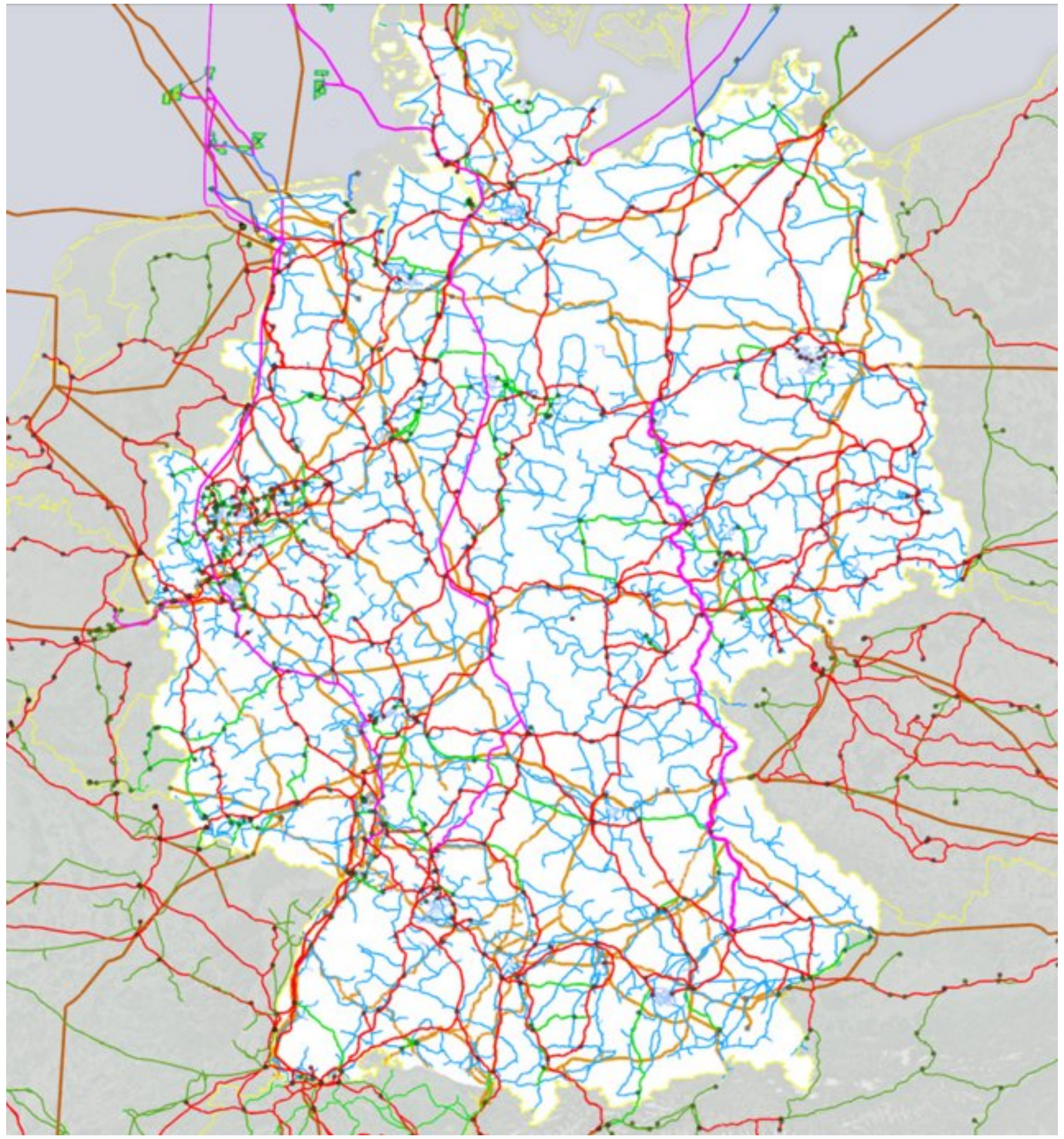
Domain decomposition of flow problems on metric graphs

Why?

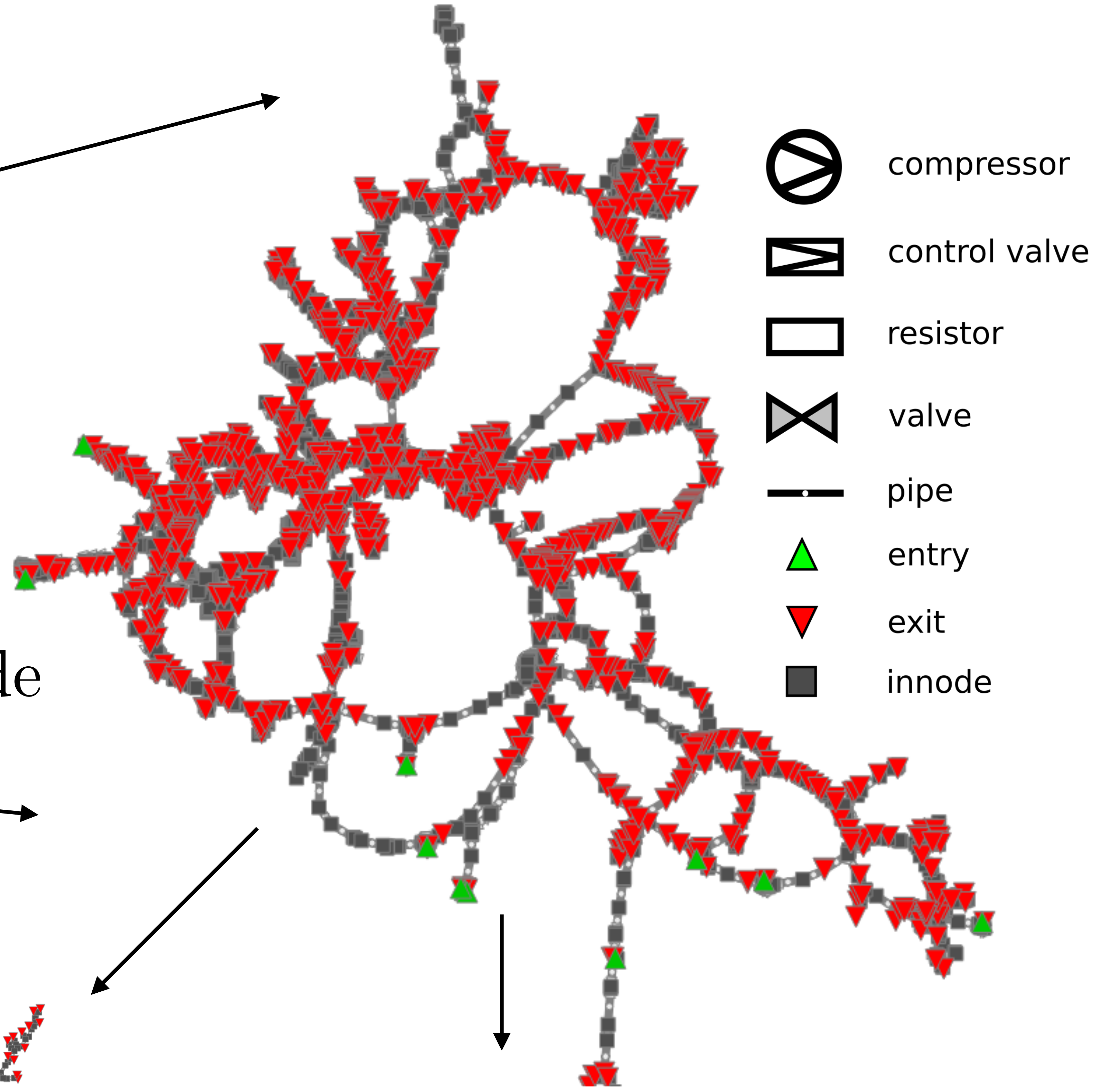
- Large scale networks may contain more than major 20K pipes and many nonlinear elements as compressors, valves etc. See e.g. the German gas network
- For each pipe, one needs space-time discretization for the nonlinear PDEs (e.g. Euler system, shallow water or water-hammer system) and discrete as well as continuous control variables leading to large-scale optimality systems (see Stefan Volkwein's lectures)
- In order to incorporate randomness (of the system data), we need to solve optimality systems repeatedly (see Tommaso Vanzan's talk on Monday)
- Moreover, in the control of gas networks one faces realtime constraints
- Therefore, real-time capable optimal control on large scale flow networks is beyond the current scope of numerical reallizations
- Hence, decomposition is at order at almost every turn (i.e. the optimization level , the network and the time). (See Victorita Doelan's lectures)

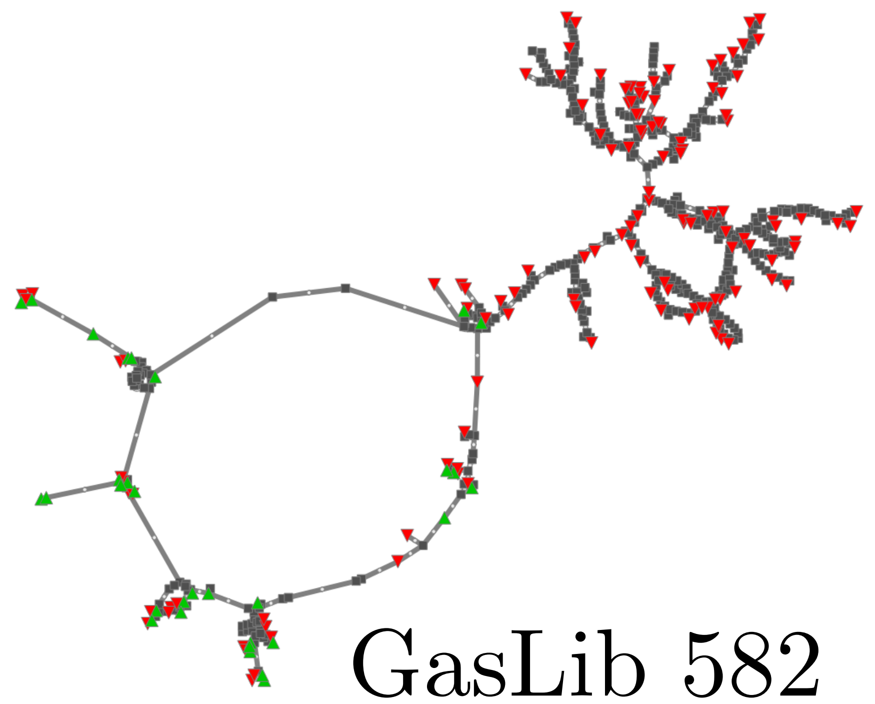
Domain decomposition of optimal control problems on metric graphs

The scope

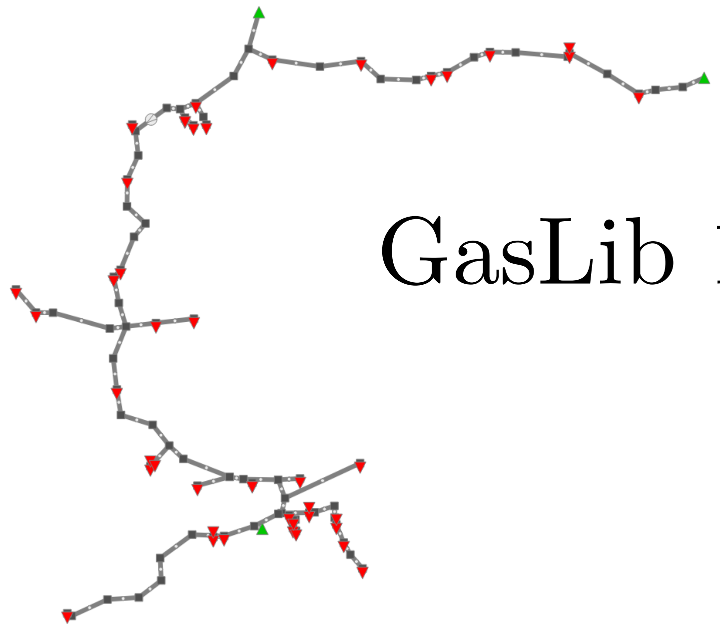


technical zoom
out of a subnetwork
GasLib 4197
see: http://gaslib.zib.de





GasLib 582



GasLib 134

Gas flow in pipe networks

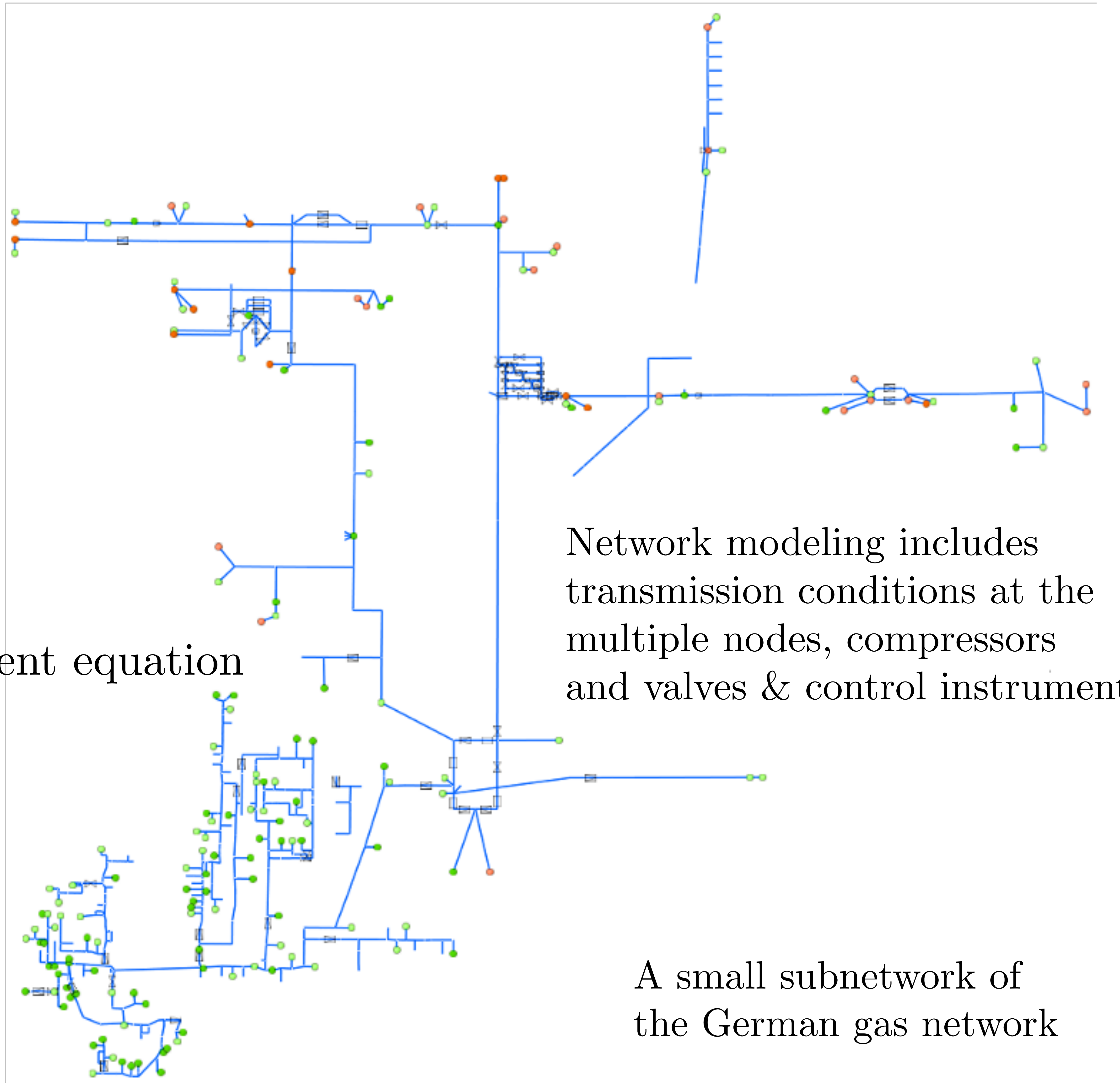
Derivation of the model equations

We start with the Euler system

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0,$$
$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) = -\frac{\lambda}{2D}\rho v|v|.$$

We use $q = \rho v a$ and neglect the inertia in the moment equation

$$\frac{\partial p}{\partial t} + \frac{c^2}{a}\frac{\partial}{\partial x}q = 0,$$
$$\frac{\partial p^2}{\partial x} = -\frac{\lambda c^2}{Da^2}q|q| =: -\gamma^2 q|q|.$$



Network modeling includes transmission conditions at the multiple nodes, compressors and valves & control instruments

A small subnetwork of the German gas network

Network modeling for friction dominated flow

The doubly nonlinear parabolic PDE

We now set $y := p^2$ and obtain from the second equation

$$q = -\frac{1}{\gamma}\frac{\frac{\partial y}{\partial x}}{\sqrt{|\frac{\partial y}{\partial x}|}}.$$

M. A. Stoner 1969
P.J. Wong, R.E. Larson 1968

With $\alpha_0 := \frac{\gamma a}{c}$, we obtain

A.Bamberger, M. Sorin, J.P. Yvon'79

$$\alpha_0 \frac{\partial}{\partial t}\frac{y}{\sqrt{|y|}} - \frac{\partial}{\partial x}\frac{\frac{\partial y}{\partial x}}{\sqrt{|\frac{\partial y}{\partial x}|}} = 0.$$

More generally, with $\beta_p(s) := |s|^{p-2}$ (above $p = \frac{3}{2}$) we obtain

$$\alpha_0 \frac{\partial}{\partial t}\beta_\alpha(y) - \frac{\partial}{\partial x}\beta_p(\frac{\partial y}{\partial x}) = 0.$$

P. A. Raviart'70

It is also possible to write this down in the p-Laplace format:

Wellposedness

Theorem (Raviart 1970): Let $\alpha, p > 1$, $\alpha < p+1$ be given. Let $p' = \frac{p}{p-1}$. Let f, u_0 be functions such that

$$f, \frac{d}{dt} f \in L^{p'}(0,T; W^{-1,p'}(\Omega)); u_0 \in W^{1,q}(\Omega) \cap L^{\alpha}(\Omega).$$

Then there exists a function u such that

$$u \in L^{\infty}(0,T; W^{1,p}(\Omega)) \cap L^{\infty}(0,T; L^{\alpha}(\Omega))$$

$$\frac{d}{dt}(|u|^{\frac{\alpha-2}{2}} u) \in L^2(0,T; L^2(\Omega))$$

$$\frac{d}{dt}(|u|^{\alpha-2} u) \in L^{\infty}(0,T; W^{1,p'}(\Omega))$$

$$\frac{d}{dt}(|u|^{\alpha-2} u) - \sum_{1}^{m} \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p-2} \frac{\partial u}{\partial x_i}\right) = f$$

$$u(0) = u_0.$$

homogeneous boundary conditions

Graph notation

• Graph $G = (V, E)$, with vertices $V = \{n_1, n_2, \ldots, n_{|V|}\} = \{n_j | j \in \mathcal{J}\}$ and edges $E = \{e_1, e_2, \ldots, e_{|E|}\} = \{e_i | i \in \mathcal{I}\}$.
• Edge-to-node incidence matrix

$$d_{ij} = \begin{cases} -1, & \text{if the edge } e_i \text{ starts at node } n_j, \\ +1, & \text{if the edges } e_i \text{ end at node } n_j, \\ 0, & \text{else.} \end{cases}$$

• Each edge e_i is given in general by a line segment $[0, \ell_i]$
• $e_i = [n_j, n_k]$ such that $d_{ij} = -1, d_{ik} = 1$, then $x = 0$, $x = \ell_i$ correspond to the nodes n_j, n_k respectively.
• More precisely, we introduce the notion x_{ij}, where $x_{ij} = 0$ if $d_{ij} = -1$, $x_{ij} = \ell_i$ if $d_{ij} = 1$.

• The edge degree is $d_j := |\mathcal{I}_j|$.

• $\mathcal{J} = \mathcal{J}^M \cup \mathcal{J}^S$, where $\mathcal{J}^M = \{j \in \mathcal{J} | d_j > 1\}$ represents the multiple nodes and $\mathcal{J}^S = \{j \in \mathcal{J} | d_j = 1\}$ the simple nodes. According to Dirichlet or Neumann boundary conditions a the simple nodes, we further decompose $\mathcal{J}^S = \mathcal{J}^S_D \cup \mathcal{J}^S_N$.

The network model

$$
\begin{aligned}
&\alpha_i \partial_t \beta(y_i(x,t)) - \partial_x \left(\beta(\partial_x y_i(x,t))\right) = u_i(x,t), && i \in \mathcal{I},\ x \in (0,\ell_i),\ t \in (0,T),\\
&y_i(n_j,t) = y_k(n_j,t), && \forall i,k \in \mathcal{I}_j,\ j \in \mathcal{J}^M,\ t \in (0,T),\\
&\sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i(n_j,t)) = 0, && j \in \mathcal{J}^M,\ t \in (0,T)\\
&y_i(n_j,t) = 0, && i \in \mathcal{I}_j, j \in \mathcal{J}^S_D,\ t \in (0,T),\\
&d_{ij} \beta(\partial_x y_i)(n_j,t) = u_j(t), && i \in \mathcal{I}_j, j \in \mathcal{J}^S_N,\ t \in (0,T),\\
&y_i(x,0) = y_i^0(x), && x \in (0,\ell_i),
\end{aligned}
\tag{NET}
$$

where the functions $u_i, i \in \mathcal{I}$, $u_j, j \in \mathcal{I}_j, j \in \mathcal{J}^S_N$ serve as distributed and boundary controls, respectively.

The optimal control problem

$$I_y(y) := \sum_{i\in\mathcal{I}} \int_0^T \int_0^{\ell_i} \frac{\kappa_i}{2} |y_i(x,t) - y_i^d(x,t)|^2 dx dt, \; I_T(y(x,T)) := \sum_{i\in\mathcal{I}} \int_0^{\ell_i} \frac{\kappa_{i,T}}{2} |y_i(x,T) - y_{i,T}|^2 dx$$

for the state, while the norms of the controls are penalized as follows

$$I_u(u) := \sum_{i\in\mathcal{I}} \frac{\nu_{i,d}}{2} \int_0^T \int_0^{\ell_i} |u_i(x,t)|^2 dx dt + \sum_{j\in\mathcal{J}_N^S} \frac{\nu_{i,b}}{2} \int_0^T |u_j(t)|^2 dt,$$

where $\kappa_i, \kappa_{i,T} \geq 0, \nu_{i,d}, \nu_{i,b} \geq 0$ serve as penalty parameters. We pose the following optimal control problem for (1)

$$\begin{aligned} &\min_{(y,u)} I(y,u) := I_y(y) + I_T(y(\cdot,T)) + I_u(u) \\ &\qquad\qquad s.t. \\ &(y,u) \text{ satisfies } (NET). \end{aligned} \tag{OCP}$$

The corresponding optimality system

$$
\begin{aligned}
&\alpha_i \partial_t \beta(y_i(x,t)) - \partial_x \left(\beta(\partial_x y_i(x,t))\right) = \frac{1}{\nu_{i,d}} p_i(x,t), \\
&\alpha_i \beta'(y_i(x,t)) \partial_t p_i(x,t) + \partial_x \left(\beta'(\partial_x y_i(x,t)) \partial_x p_i(x,t)\right) = \kappa_i (y_i - y_i^d), && i \in \mathcal{I},\ x \in (0,\ell_i),\ t \in (0,T), \\
&\quad y_i(n_j,t) = y_k(n_j,t),\ p_i(n_j,t) = p_k(n_j,t), && \forall i,k \in \mathcal{I}_j,\ j \in \mathcal{J}^M,\ t \in (0,T), \\
&\quad \sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i(n_j,t)) = 0,\ \sum_{i \in \mathcal{I}_j} d_{ij} \beta'(\partial_x y_i(n_j,t)) \partial_x p_i(n_j,t) = 0, && j \in \mathcal{J}^M,\ t \in (0,T), \\
&\quad y_i(n_j,t) = 0,\ p_i(n_j,t) = 0, && i \in \mathcal{I}_j, j \in \mathcal{J}_D^S,\ t \in (0,T), \\
&\quad d_{ij} \beta(\partial_x y_i(n_j,t)) = \frac{1}{\nu_{i,b}} p_j(n_j,t),\ d_{ij} \beta'(\partial_x y_i(n_j,t)) \partial_x p_i(n_j,t) = 0, && i \in \mathcal{I}_j, j \in \mathcal{J}_N^S, t \in (0,T), \\
&\quad y_i(x,0) = y_{i,0}(x),\ p_i(x,T) = -\kappa_{i,T}(y_i(x,T) - y_{iT}^d(x)), && x \in (0,\ell_i),
\end{aligned}
\tag{GOS}
$$

where p denotes the adjoint variable (Lagrange multiplier).

We need to be careful with possibly 'flat regions'

Time discretization

- We decompose $[0,T]$ into break points $t_0 = 0 < t_1 < \cdots < t_N = T$ with widths $\Delta t_n := t_{n+1} - t_n, n = 0, \ldots, N-1$
- We denote $y_i(x, t_n) := y_{i,n}(x), n = 0, \ldots, N-1$ and similarly for the controls.
- We consider an implicit Euler scheme and a standard quadrature rule for the time integrals represented by weights ω_n.
- We introduce the semi-discrete cost functions

$$I_y^{\Delta t}(y_i) := \sum_{i \in \mathcal{I}} \sum_{n=1}^{N-1} \omega_n \int_0^{\ell_i} \frac{\kappa_i}{2} |y_{i,n} - y_{i,n}^d|^2 dx, \; I_N(y_{i,N}) := \sum_{i \in \mathcal{I}} \int_0^{\ell_i} \frac{\kappa_{i,T}}{2} |y_{i,N} - y_{i,N}^d|^2 dx,$$

$$I_u^{\Delta t}(u) := \sum_{i \in \mathcal{I}} \frac{\nu_{i,d}}{2} \sum_{n=1}^{N-1} \omega_n \int_0^{\ell_i} |u_{i,n}|^2 dx + \sum_{j \in \mathcal{J}_N^S} \omega_i \frac{\nu_{i,b}}{2} \sum_1^N \omega_n |u_{j,n}|^2$$

Time-discrete optimal control problem

$$\min_{(y,u)} I(y,u) := I_y^{\Delta t}(y) + I_T(y(\cdot,N) + I_u^{\Delta t}(u)$$

$$s.t.$$

$$\frac{1}{\Delta t}\beta(y_{i,n+1})(x) - \partial_x\left(\beta(\partial_x y_{i,n+1}(x)\right) = \frac{1}{\Delta t}\beta(y_{i,n})(x) + u_{i,n+1}(x), \qquad x \in (0,\ell_i),$$

$$y_{i,n+1}(n_j) = y_{k,n+1}(n_j), \qquad \forall i,k \in \mathcal{I}_j,\ j \in \mathcal{J}^M,$$

$$\sum_{i\in\mathcal{I}_j} d_{ij}\beta(\partial_x y_{i,n+1})(n_j) = 0, \qquad j \in \mathcal{J}^M,$$

$$\beta(\partial_x y_{i,n+1})(n_j) = u_{j,n+1}, \qquad d_j = 1,\ i \in \mathcal{I}_j,\ j \in \mathcal{J}_N^S,$$

$$y_{i,n+1}(n_j) = 0, \qquad i \in \mathcal{I}_j,\ j \in \mathcal{J}_D^S,$$

$$y_{i,0}(x) = y_i^0(x), \qquad i \in \mathcal{I},\ x \in (0,\ell_i),\ i \in \mathcal{I},$$

with $n = 1, \ldots, N-1$.

Instantaneous control

- We replace $\alpha_i := \frac{1}{\Delta t}$, $f_i^1 := \alpha_i \beta(y_{i,n})$ and omit the weights ω_n.
- For each $n = 1, \ldots, N-1$ and given $y_{i,n}$, we consider the cost functions at each time t_n:

$$\tilde{I}_y^{\Delta t}(y_i) := \sum_{i \in \mathcal{I}} \int_0^{\ell_i} \frac{\kappa_i}{2} |y_i - y_{i,n}^d|^2 dx,$$

$$\tilde{I}_u^{\Delta t}(u) := \sum_{i \in \mathcal{I}} \frac{\nu_{i,d}}{2} \int_0^{\ell_i} |u_i|^2 dx + \sum_{j \in \mathcal{J}_N^S} \frac{\nu_{i,b}}{2} |u_j|^2.$$

$$I(y,u) := \tilde{I}_y^{\Delta}(y) + \tilde{I}_u^{\Delta t}(u) =: \sum_{i \in \mathcal{I}} J_i(y_i, u_i).$$

Instantaneous control

$$\min_{(y,u)} I(y,u)$$

$$s.t.$$

$$\begin{aligned}
&\alpha_i\beta(y_i)(x) - \partial_x\left(\beta(\partial_x y_i(x))\right) = u_i(x) + f_i^1(x), && i \in \mathcal{I},\ x \in (0,\ell_i),\\
&y_i(n_j) = y_k(n_j), && \forall i,k \in \mathcal{I}_j,\ j \in \mathcal{J}^M,\\
&\sum_{i\in\mathcal{I}_j} d_{ij}\beta(\partial_x y_i)(n_j) = 0, && j \in \mathcal{J}^M,\\
&\beta(\partial_x y_i)(n_j) = u_j, && i \in \mathcal{I}_j,\ j \in \mathcal{J}_N^S,\\
&y_i(n_j) = 0, && i \in \mathcal{I}_j,\ j \in \mathcal{J}_D^S.
\end{aligned}$$

Decomposition

Principal remarks

- We want to iteratively decompose the optimality system (GOS) on the ‚global‘ network G into subnetworks (*Network tearing and Interconnection* NETI), in fact here, to each individual edge. Analysis in the continuous setting!

- The decomposed optimality system (DOS) should itself be an optimality system for an optimal control problem on the subnetwork (i.e. edge) including *virtual controls* at the multiple nodes (interfaces), in the sense of J.L. Lions and O. Pironneau 1999.

- The decomposition should be *non-overlapping* (in the sense of P.L. Lions 1989) overlapping domains are not intuitive at multiple nodes. Overlapping Schwarz-type methods at serial connections (‚cutting out stars') are also under consideration (not here, however), see Gon, Kwok, Tan 2022

- Space-time domain decomposition

Previous work

- **General domains (manifolds, continuous level, no controls; very selective list)**: Early work by P. L. Lions’1989 and O. Pironneau & J.L. Lions’1999 pursued later by J.-D. Benamou’1992-99 for elliptic and parabolic problems, A. Quarteroni’1988-16, F. Nataf’ 91-’, M. Gander’00-, G. Ciaramella’17-,L. Halpern’00-, J. Haslinger’00-14, J. Kucera,T. Sassi (Signorini-type contact problems), E. Engström, E. Hansen’22 (Robin-type p-Laplace)…M. Dryia, W. Hackbusch’97 (general finite dimensional(!) nonlinear problems)

- **Time domain decomposition (continuous level; again very selective list):** J.L. Lions, Y. Maday, G. Turinici’01, J. Salomon’07-, M. Gander’07-, F. Kwok’18-,G. Ciaramella’21 (semi-linear elliptic) ….(parareal/ multiple shooting)…space-time…

- **Optimal control problems:** M. Heinkenschloss’00-11, M. Herty’07, S. Ulbrich’07, M. Gander,’00- F. Kwok’17-, V. Agoshkov’85-, P. Gervasio’04-16, A. Quarteroni’05/06, B. Delourme, L. Halpern, B. Nguyen’06, W. Gong, F. Kwok, Z. Tan’22 (overlapping domains) many others, for linear elliptic and parabolic problems (in almost all cases).

- **Networked domains and optimal control** (non-manifolds; multiple nodes in 1-D and interfaces in 2- or 3-D): J. E. Lagnese & G.L. 2003, G.L. (et al.) 2018-2022.

Example: diamond graph

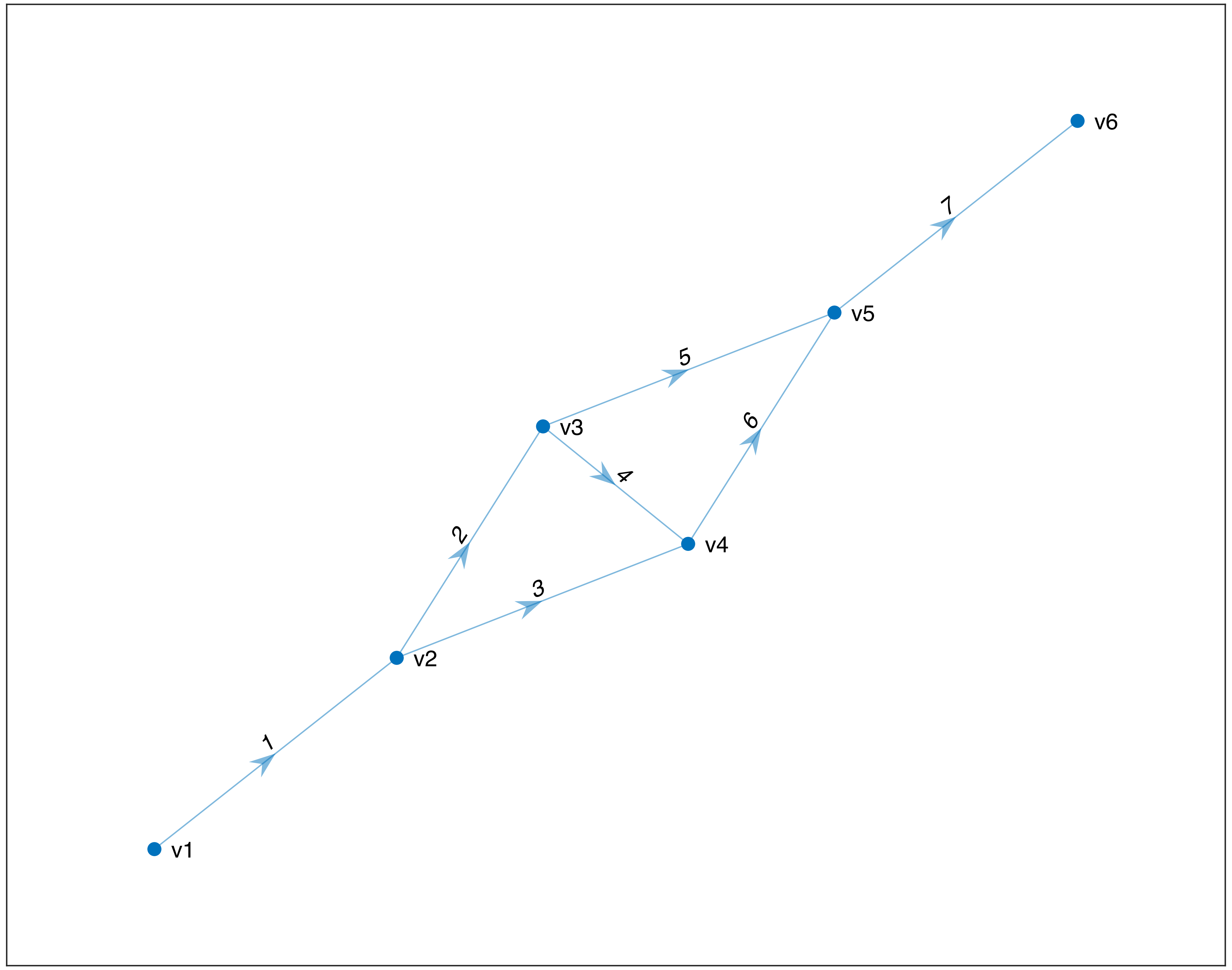
- We consider the so-called diamond graph,

- We apply a Neumann condition at n_6 and a boundary control at n_1.

- We want to steer y_4 to the constant value 1, applying running costs and terminal costs, individually.

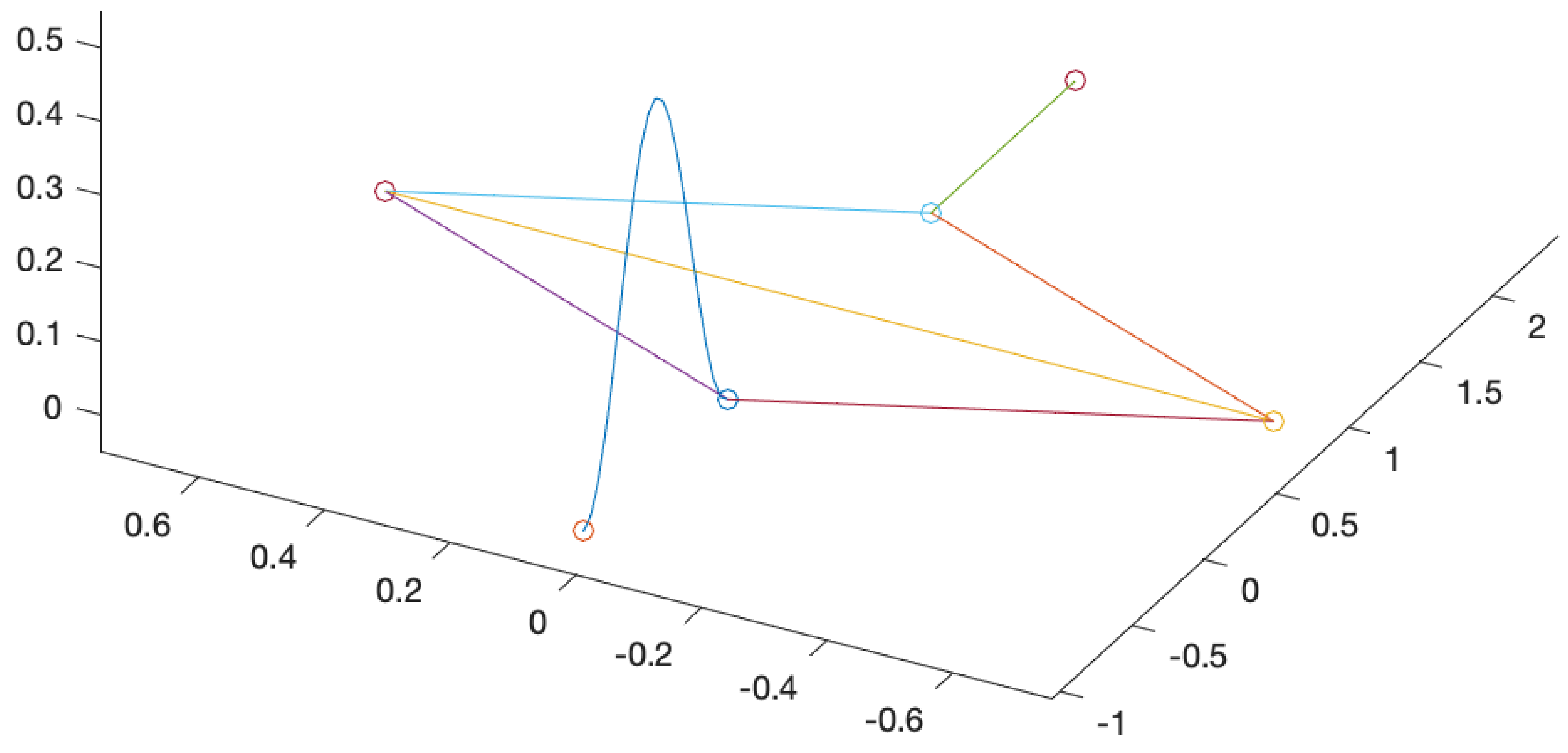
- For the penalty data, we take $\kappa = 1.e4, \nu = 1$

- We use standard discretization in space and time, as already proposed by Bamberger'77 and Raviart'70

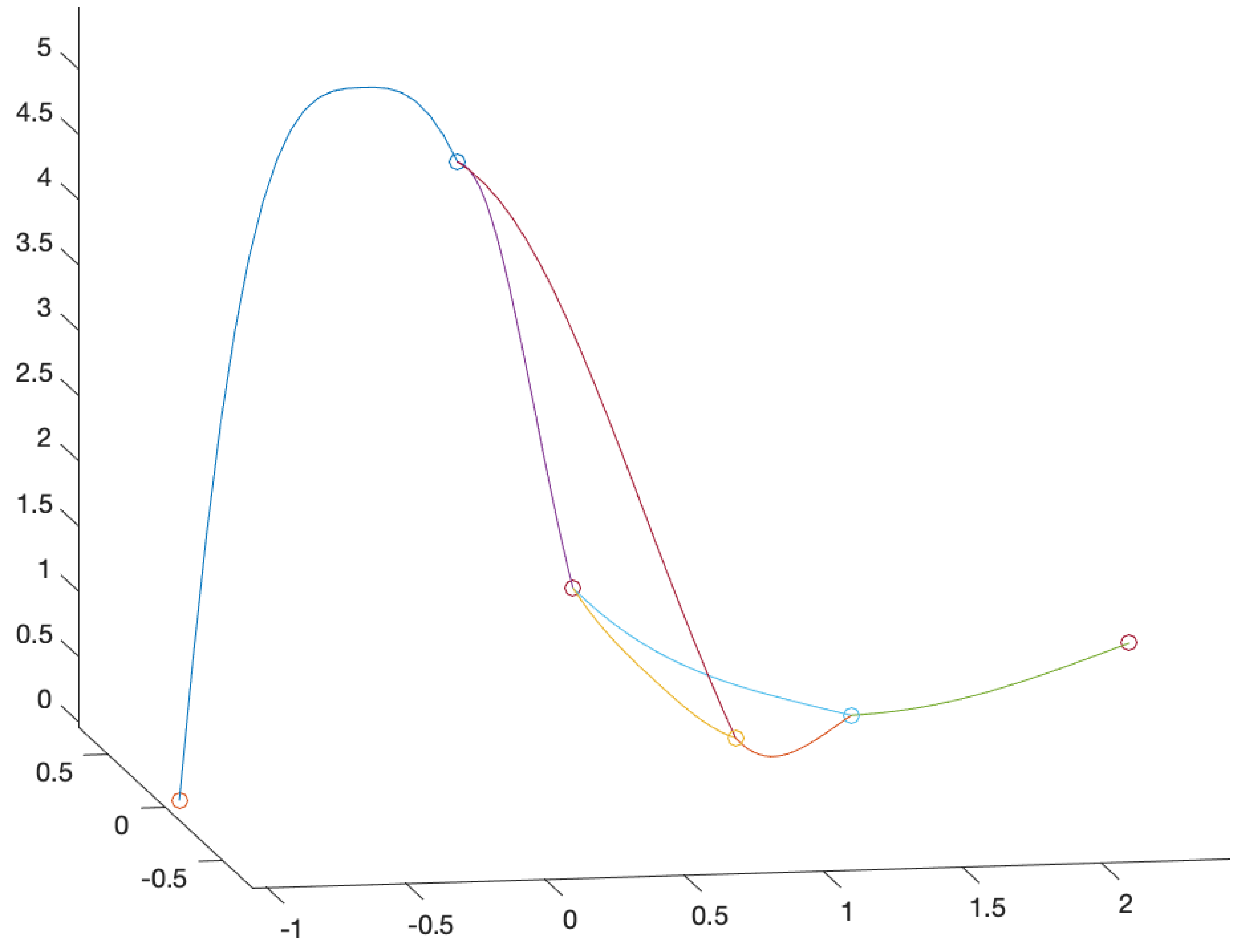


The diamond graph

Example full network

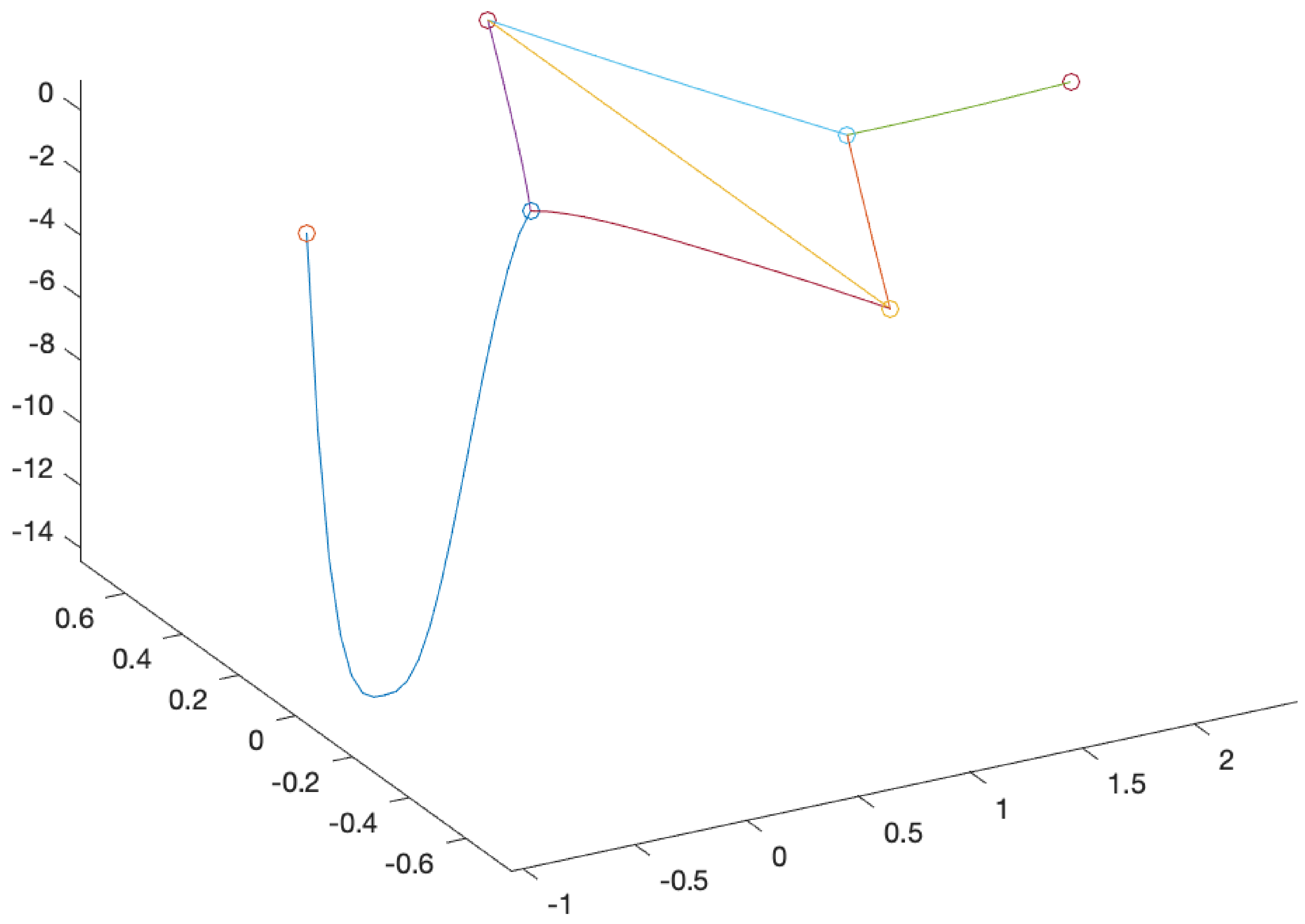


Initial condition

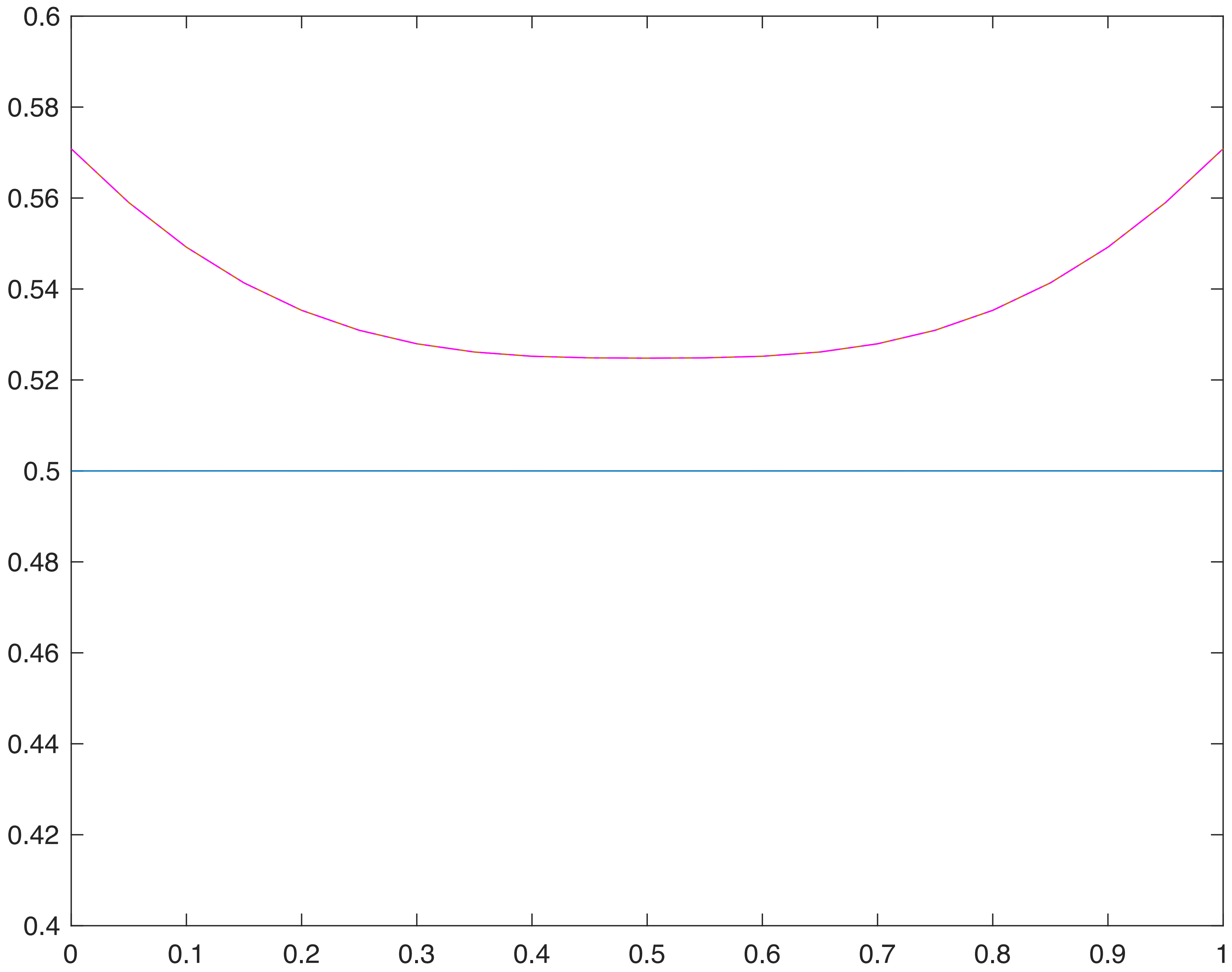


At final time with running costs

Example



Final value control



Comparison of final states

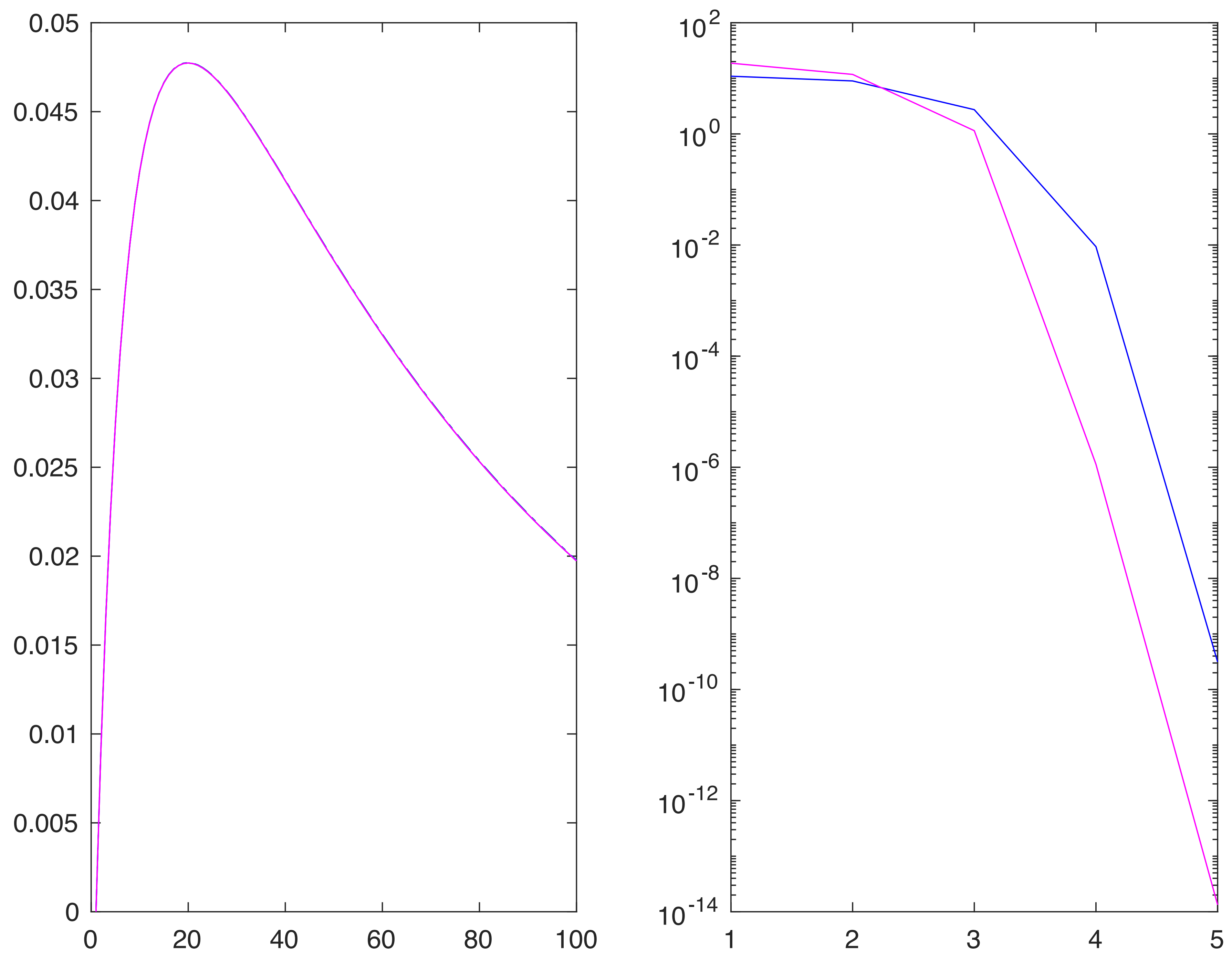
Domain decomposition in space

The P.L. Lions algorithm extended to p-parabolic equations

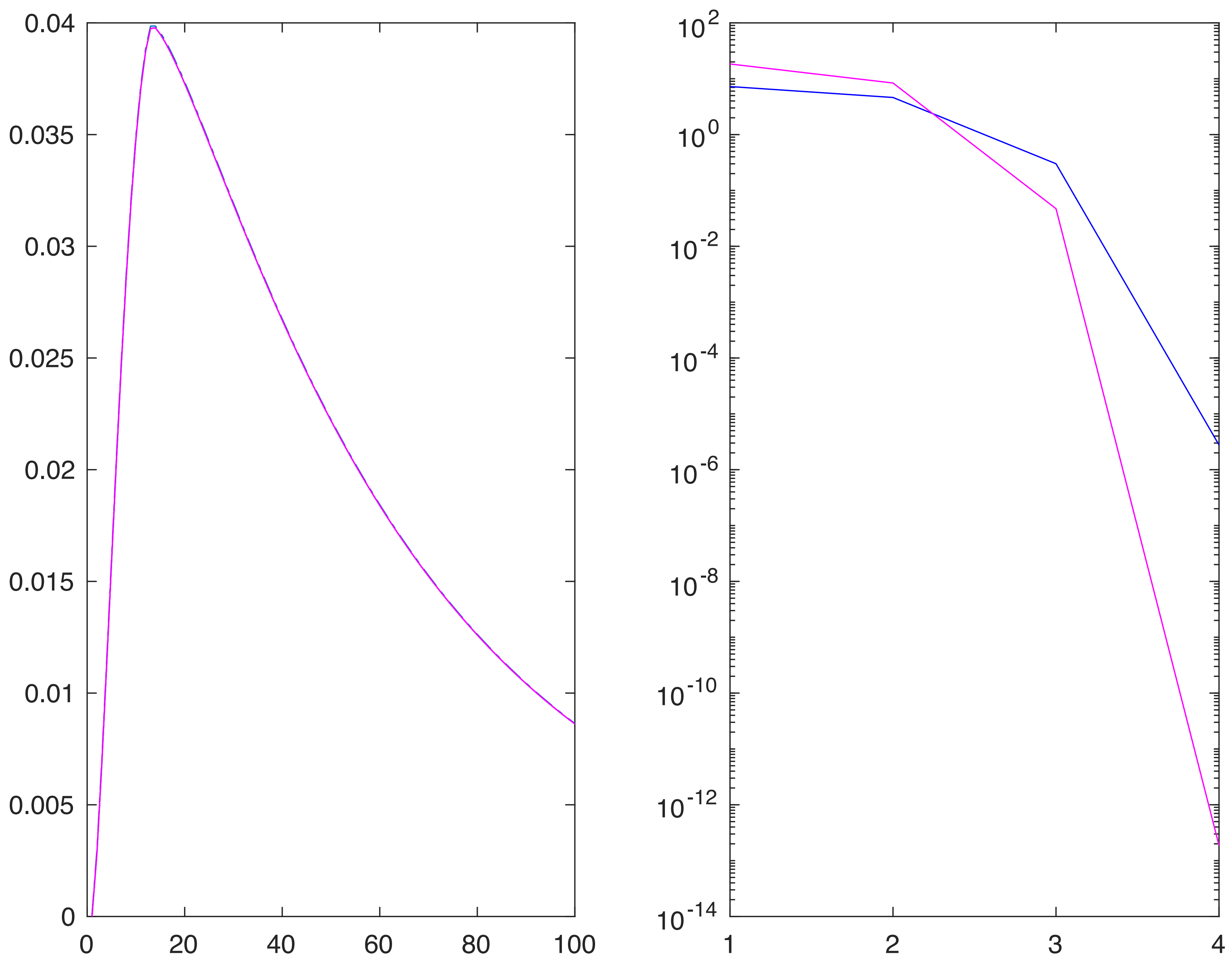
$$
\begin{aligned}
&\partial_t \beta_i(y^{k+1})(x,t) - \partial_x \left(\beta_i(\partial_x y_i^{k+1}(x,t)\right) = f_i(x,t), && i \in \mathcal{I},\ x \in (0,\ell_i),\ t \in (0,T),\\
&\quad y_i^{k+1}(n_j,t) = 0, && i \in \mathcal{I}_j, j \in \mathcal{J}_D^S,\ t \in (0,T),\\
&\quad d_{ij}\beta_i(\partial_x y_i^{k+1}(n_j,t) = 0, && i \in \mathcal{I}_j, j \in \mathcal{J}_N^S,\ t \in (0,T),\\
&\quad d_{ij}\beta_i(\partial_x y_i^{k+1})(x_{ij},t) + \rho y_i^{k+1}(x_{ij},t) = \rho \left(\frac{2}{d_j} \sum_{l \in \mathcal{I}_j} y_l^k(x_{lj},t) - y_i^k(x_{ij},t) \right), &&\\
&\qquad\qquad - \left(\frac{2}{d_j} \sum_{l \in \mathcal{I}_j} d_{lj}\beta_l(\partial_x y_l^k)(x_{lj},t) - d_{ij}\beta_i(\partial_x y_i^k)(x_{ij},t) \right), && j \in \mathcal{J}^M, i \in \mathcal{I}_j,\\
&\quad y_i^{k+1}(x,0) = y_i(x);\ x \in (0,\ell_i), && i \in \mathcal{I}.
\end{aligned}
$$

Example: two-link p-parabolic problem

We take the interval $(0, 2)$ and introduce the interface at $x = 1$. At $x = 0$, we have Dirichlet boundary conditions and at $x = 2$ Neumann conditions, as well as initial conditions $\sin(\pi x)^2$ in each domain. The load is equal to 1 everywhere. We apply the algorithm above with $\rho = .5$ and use the **pdpe** code from Matlab.



p=2; Left: plot of the solutions in domains 1 and 2 on top of ‚true‘ solution
Right: the errors of state and fluxed at the interface



Same as on left figure, but now for p=3/2

Relevance for optimal control problems

We can approach the decomposition of the optimality system by the following fixed point procedure:

1. Choose controls (distributed and boundary controls)
2. Solve the state equation in parallel using the DDM above
3. Input the state in the (linear!) adjoint equation and solve using the classical (still extended) DDM (se e.g. Benamou)
4. Retrieve the controls using the optimality condition and go back to the first step until done.

Notice, however, that this procedure is not a DDM for the optimality system as a whole and, consequently, does not lead to a substitute optimal control problem on the subnetworks.

DDM algorithm for the optimality system

1. Given $\lambda_{ij}^n, \rho_{ij}^n$,

2. solve for y_i^{n+1}, p_i^{n+1}

$$\partial_t \beta_i(y_i^{n+1}) - \partial_x(\beta_i(\partial_x y_i^{n+1})) = \frac{1}{\nu_{i,d}} p_i^{n+1},$$
$$\beta_i'(y_i^{n+1})\partial_t p_i^{n+1} + \partial_x(\beta_i'(\partial_x y_i^{n+1})\partial_x p_i^{n+1}) = \kappa_i(y_i^{n+1} - y_i^d),$$
$$y_i^{n+1}(x_{ij},t) = 0,\ p_i^{k+1}(x_{ij},t) = 0,$$
$$d_{ij}\beta_i(\partial_x y_i^{n+1})(x_{ij},t) = \frac{1}{\nu_{i,b}} p_i(x_{ij},t),\ d_{ij}\beta_i'(\partial_x y_i^{n+1})\partial_x p_i(x_{ij}^{n+1})(x_{ij},t) = 0,$$

$$d_{ij}\beta_i(\partial_x y_i^{n+1})(x_{ij}) + \sigma y_i^{n+1}(x_{ij}) + \mu p_i^{n+1}(x_{ij}) = -\left(\frac{2}{d_j}\sum_{l\in\mathcal{I}_j} d_{lj}\beta_l(\partial_x y_l^n)(x_{lj}) - d_{ij}\beta_i(\partial_x y_i^n)(x_{ij})\right)$$
$$+\sigma\left(\frac{2}{d_j}\sum_{l\in\mathcal{I}_j} y_l^n(x_{lj},t) - y_i(x_{ij},t)\right) + \mu\left(\frac{2}{d_j}\sum_{l\in\mathcal{I}_j} p_l^n(x_{lj},t) - p_i(x_{ij},t)\right) =: \lambda_{ij}(t)^n,$$

P-parabolic problem: Algorithm

$$d_{ij}\beta_i'(\partial_x y_i^{n+1}(x_{ij},t))\partial_x p_i^{n+1}(x_{ij},t) + \sigma p_i^{n+1}(x_{ij},t) - \mu y_i^{n+1}(x_{ij},t)$$

$$= -\left(\frac{2}{d_j}\sum_{l\in\mathcal{I}_j} d_{lj}\beta_l'(\partial_x y_l^n(x_{lj},t))\beta_l(\partial_x p_l^n)(x_{lj},t) - d_{ij}\beta_i'(\partial_x y_i^n(x_{ij},t))(\beta_i(\partial_x p_i^n)(x_{ij},t)\right)$$

$$+\sigma\left(\frac{2}{d_j}\sum_{l\in\mathcal{I}_j} p_l^n(x_{lj},t) - p_i(x_{ij},t)\right) - \mu\left(\frac{2}{d_j}\sum_{l\in\mathcal{I}_j} y_l^n(x_{lj},t) - y_i(x_{ij})\right) =: \rho_{ij}^{(}t).$$

3. Update $\lambda_{ij}^{n+1}, \rho_{ij}^{n+1}$ for $n \to n+1$.

Equivalent virtual control problem

1. Given $\lambda_{ij}^n, \rho_{ij}^n$,
2. solve for $y_i^{n+1}, u_i^{n+1}, u_j^{n+1}, j \in \mathcal{J}_i$

$$\min_{u,g,y} \left\{ J_i(y_i, u_i) + \frac{1}{2\mu} \sum_{j \in \mathcal{J}_i} \int_0^T \left[|g_{ij}|^2 + |\mu y_i - \rho_{ij}^n|^2 \right] dt \right\}$$

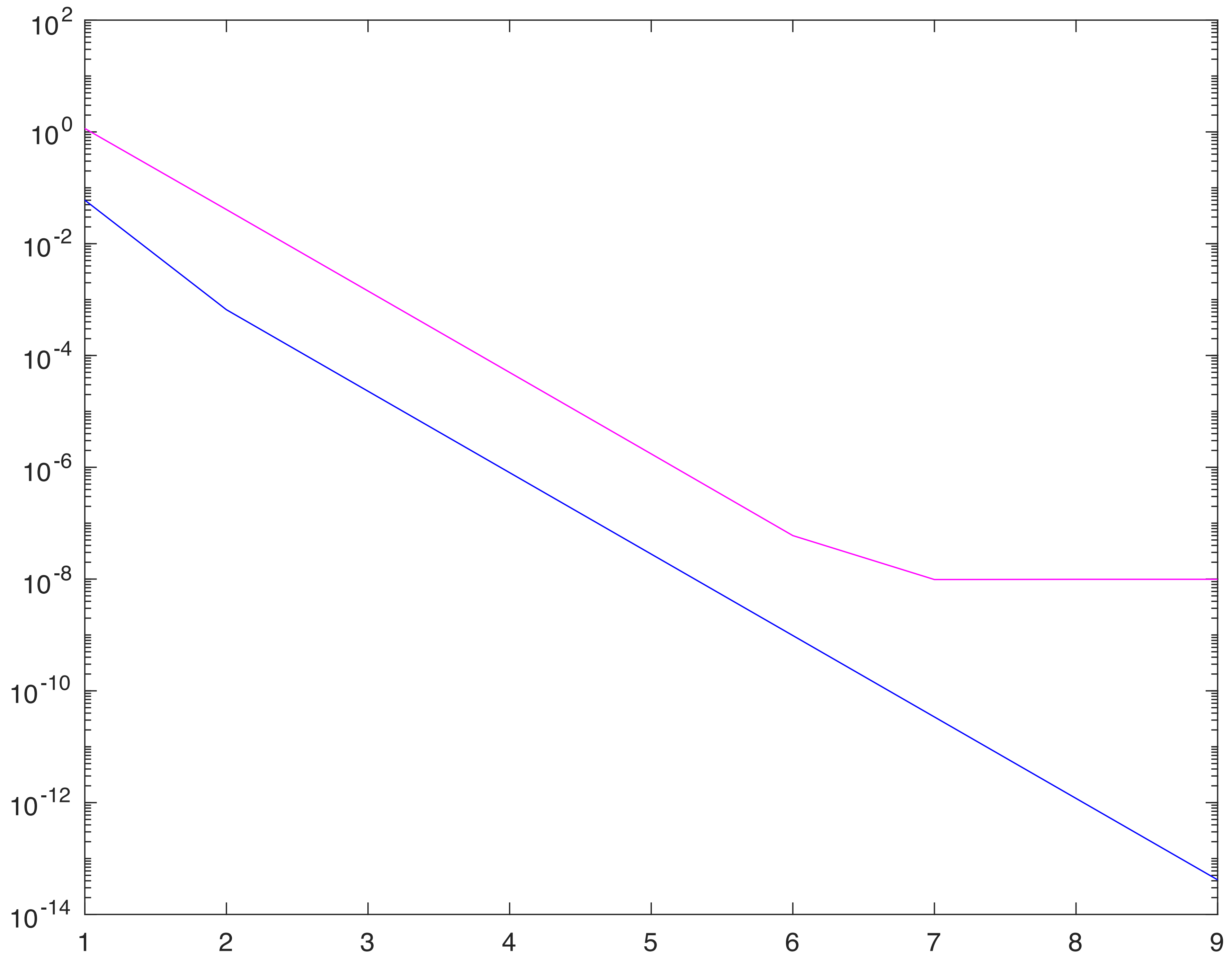
s. t.

$$\begin{aligned}
\partial_t \beta_i(y_i) - \partial_x(\beta_i(\partial_x y_i)) &= u_i, & i \in \mathcal{I}, x \in I_i, t \in (0,T) \\
d_{ij}\beta_i(\partial_x y_i(x_{ij}, t)) + \sigma y_i(x_{ij}, t) &= \lambda_{ij}(t)^n + g_{ij}(t), & j \in \mathcal{J}_i, i \in \mathcal{I}_j, t \in (0,T) \\
y_i, t &= 0, & i \in \mathcal{I}_j, j \in \mathcal{J}_D^S, t \in (0,T) \\
d_{ij}\beta_i(\partial_x y_i(x_{ij}, t)) &= u_j(t), & i \in \mathcal{I}_j, j \in \mathcal{J}_N^S, t \in (0,T).
\end{aligned}$$

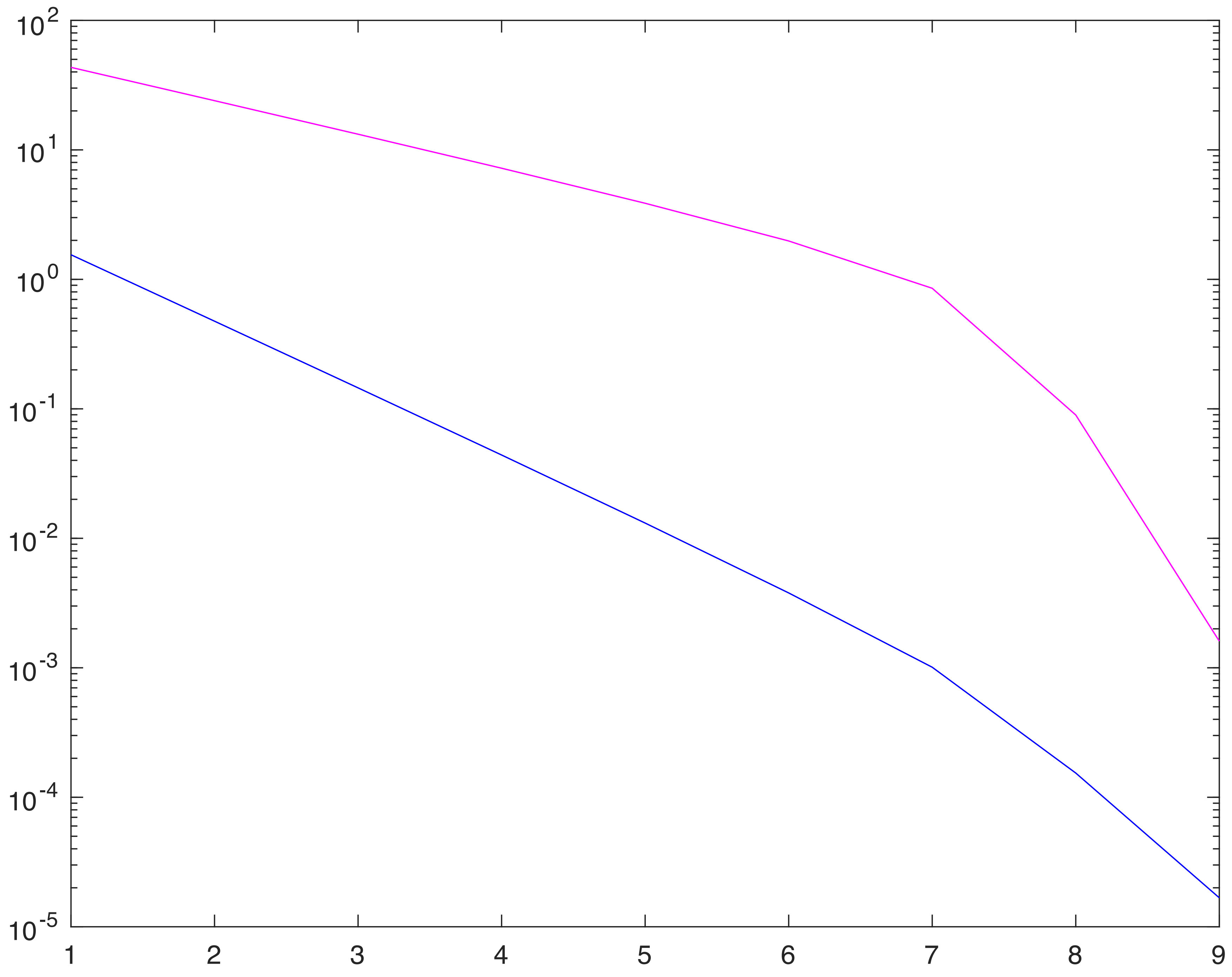
3. Update $\lambda_{ij}^{n+1}, \rho_{ij}^{n+1}$ for $n \to n+1$.

Example

A two-link problem



p=2, sigma=0, mu=10, nu=0, kappa=1000



p=3/2, sigma=50, mu=100, nu=.001, kappa=1000

Convergence

Sketch of proof: two-link case (α=2)

$$\partial_t y_i^{n+1} - \partial_x(\beta_i(\partial_x y_i^{n+1})) = \frac{1}{\nu} p_i^{n+1}, \qquad i = 1, 2,\ x \in I_i$$

$$\partial_t p_i^{n+1} + \partial_x(\beta_i'(\partial_x y_i^{n+1})\partial_x p_i^{n+1}) = \kappa(y_i^{n+1} - y_i^d), \quad i = 1, 2,\ x \in I_i$$

$$y_1^{n+1}(0) = 0,\ y_2^{n+1}(2) = 0,$$

$$p_1^{n+1}(0) = 0,\ p_2^{n+1}(2) = 0,$$

We omit t in the following where all equations are taken at time t^*

$$d_{ij}\beta_i(\partial_x y_i^{n+1}(1)) + \sigma y_i^{n+1}(1) - \mu p_i^{n+1}(1) = -d_{ij}\beta_j(\partial_x y_2^n(1)) + \sigma y_j^n(1) - \mu p_j^n(1) := \lambda_i^n$$

$$d_{ij}\beta_i'(\partial_x y_i^{n+1}(1))\partial_x p_i^{n+1} + \sigma p_i^{n+1}(1) + \mu y_i^{n+1}(1) = -d_{ij}\beta_j'(\partial_x y_j^n(1))\partial_x p_j^{n+1}(1) + \sigma p_j^n(1) + \mu y_j^n(1) =: \rho_i^n$$

Error evolution

We introduce the errors $\tilde{y}_i^n := y_i^n - y_i; \tilde{p}_i^n := p_i^n - p_i$ and subtract the equations:

$$\partial_t \tilde{y}_i - \partial_x(\beta_i(\partial_x \tilde{y}_i^{n+1} + \partial_x y_i)) - \partial_x(\beta_i(\partial_x y_i)) = \frac{1}{\nu}\tilde{p}_i^{n+1} \qquad i = 1,2,\ x \in I_i$$

$$\partial_t \tilde{p}_i^{n+1} + \partial_x(\beta_i'(\partial_x y_i^{n+1} + \partial_x y_i)\partial_x \tilde{p}_i^{n+1}) = (\beta_i'(\partial_x \tilde{y}_i^{n+1} + \partial_x y_i) - \beta_i'(\partial_x y_i))\partial_x p_i)$$
$$+ \kappa(y_i - y_i^d) =: g_i^n, \qquad i = 1,2,\ x \in I_i$$

$$\tilde{y}_1^{n+1}(0) = 0,\ \tilde{y}_2^{n+1}(2) = 0,$$
$$\tilde{p}_1^{n+1}(0) = 0,\ \tilde{p}_2^{n+1}(2) = 0,$$

Transmission conditions and fixed point map

$$
\begin{aligned}
&\beta_1(\partial_x y_1^{n+1}(1)) - \beta_1(\partial_x y_1(1)) + \sigma\tilde{y}_1^{n+1}(1) - \mu\tilde{p}_1^{n+1}(1)\\
&\qquad\qquad = \beta_2(\partial_x y_2^{n}(1)) - \beta_2(\partial_x y_2(1)) + \sigma\tilde{y}_2^{n}(1) - \mu\tilde{p}_2^{n}(1)\\
&-(\beta_2(\partial_x y_2^{n+1}(1)) - \beta_2(\partial_x y_2(1))) + \sigma\tilde{y}_2^{n+1}(1) - \mu\tilde{p}_2^{n+1}(1)\\
&\qquad\qquad = -(\beta_1(\partial_x y_1^{n}(1)) - \beta_1(\partial_x y_1(1))) + \sigma\tilde{y}_1^{n}(1) - \mu\tilde{p}_1^{n}(1)\\
&\beta_1'(\partial_x y_1^{n+1}(1))(\partial_x p_1^{n+1}(1)) - \beta_1'(\partial_x y_1)\partial_x p_1(1) + \sigma\tilde{p}_1^{n+1}(1) + \mu\tilde{y}_1^{n+1}(1)\\
&\qquad\quad = \beta_2'(\partial_x y_2^{n}(1))(\partial_x p_2^{n+1}(1)) - \beta_2'(\partial_x y_2(1))\partial_x p_2(1)) + \sigma\tilde{p}_2^{n}(1) + \mu\tilde{y}_2^{n}(1)\\
&-(\beta_2'(\partial_x y_2^{n+1}(1))(\partial_x p_2^{n+1}(1)) - \beta_2'(\partial_x y_2(1))\partial_x p_2(1))) + \sigma\tilde{p}_2^{n+1}(1) + \mu\tilde{y}_2^{n+1}(1)\\
&\qquad\quad = -(\beta_1'(\partial_x y_1^{n}(1))(\partial_x p_1^{n} + \partial_x p_1(1)) - \beta_1'(\partial_x y_1(1))\partial_x p_1(1))) + \sigma\tilde{p}_1^{n}(1) + \mu\tilde{y}_1^{n}(1).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{X}^n := &(\beta_1(\partial_x y_1^{n}(1)) - \beta_1(\partial_x y_1(1)) + \sigma\tilde{y}_1^{n}(1) - \mu\tilde{p}_1^{n}(1),\\
&-(\beta_2(\partial_x y_2^{n}(1)) - \beta_2(\partial_x y_2(1))) + \sigma\tilde{y}_2^{n+1}(1) - \mu\tilde{p}_2^{n+1}(1),\\
&\beta_1'(\partial_x y_1^{n}(1))\partial_x p_1^{n}(1) - \beta_1'(\partial_x y_1)\partial_x p_1(1) + \sigma\tilde{p}_1^{n}(1) + \mu\tilde{y}_1^{n}(1),\\
&-(\beta_2'(\partial_x y_2^{n}(1))\partial_x p_2^{n}(1) - \beta_2'(\partial_x y_2(1))\partial_x p_2(1))) + \sigma\tilde{p}_2^{n}(1) + \mu\tilde{y}_2^{n}(1)).
\end{aligned}
$$

Accordingly, the blue terms are collected in $\mathcal{T}X^n$.

Energy and non-expansiveness

$$\mathcal{E}^n := \int_0^T \sum_{i=1}^{2} \{(\beta_i(\partial_x y_i^n(1) - \beta_i(\partial_x y_i(1))^2 + (\beta_i'(\partial_x y_i^n(1))\partial_x p_i^n(1) - \beta_i'(\partial_x y_i(1))\partial_x p_i(1))^2$$
$$+(\sigma^2 + \mu^2)(\tilde{y}_i^n(1)^2 + \tilde{p}_i^n(1)^2)\}\, dt$$

$$\mathcal{F}^n := 2\sigma \int_0^T \{(\beta_1(\partial_x y_1^n(1)) - \beta_1(\partial_x y_1(1)))\tilde{y}_1^n(1) - (\beta_2(\partial_x y_2^n(1)) - \beta_2(\partial_x y_2(1)))\tilde{y}_2^n(1)$$
$$+(\beta_1'(\partial_x y_1^n(1))\partial_x p_1^n(1) - \beta_1'(\partial_x y_1(1))\partial_x p_1(1))\tilde{p}_1^n(1)$$
$$-(\beta_2'(\partial_x y_2^n(1))\partial_x p_2^n(1) - \beta_2'(\partial_x y_2(1))\partial_x p_2(1)))\tilde{p}_2^n(1)\}\, dt$$
$$+ 2\mu \int_0^T \{(-\beta_1(\partial_x y_1^n(1) - \beta_1(\partial_x y_1(1)))\tilde{p}_1^n(1) + (\beta_2(\partial_x y_2^n(1)) - \beta_2(\partial_x y_2(1)))\tilde{p}_2^n(1)$$
$$+(\beta_1'(\partial_x y_1^n(1))\partial_x p_1^n(1) - \beta_1'(\partial_x y_1(1))\partial_x p_1(1))\tilde{y}_1^n(1)$$
$$-(\beta_2'(\partial_x y_2^n(1))\partial_x p_2^n(1) - \beta_2'(\partial_x y_2(1))\partial_x p_2(1))\tilde{y}_2^n(1)\}\, dt$$

$$\|\mathcal{X}^n\|^2 = \mathcal{E}^n + \mathcal{F}^n, \ \|\mathcal{T}\mathcal{X}^n\|^2 = \mathcal{E}^n - \mathcal{F}^n, \ \|\mathcal{X}^{n+1}\|^2 = \|\mathcal{T}\mathcal{X}^n\|^2 = \|\mathcal{X}^n\|^2 - 2\mathcal{F}^n$$

Proof....

- We need to show that $\mathcal{F}$ is positive definite with respect to the norms of $\tilde{y}, \tilde{p}$.
- To do this, we multiply the state and the adjoint equation for the edge i individually by $\tilde{y}_i, \tilde{p}_i$ and perform integration by parts.
- This leads to 4 equations for the boundary values needed in $\mathcal{F}$
- For the estimates we strongly rely on the monotonicity of $\beta_p(\cdot)$.
- Moreover, in order to control mixed terms in $\tilde{y}_i, \tilde{p}_i$, we need to assume that the iteration starts in a possibly small neighborhood of the solution (locality as for Newton's method).
- Finally, we need a careful estimate of the parameters ν, κ (penalty parameters) and σ, μ (Robin-type parameters) in order to achieve the positiveness of $\mathcal{F}$.
- Then the crucial estimate $\|\mathcal{X}^{n+1}\|^2 = \|\mathcal{T}\mathcal{X}^{n+1}\|^2 = \|\mathcal{X}^n\|^2 - 2\mathcal{F}^n$ leads to the result.
- The estimations are quite technical!

Time-domain decomposition

- We introduce a coarse time discretization with

$$0 = T_0 < T_1 < \cdots < T_k < T_{k+1} < \cdots < T_K < T_{K+1} = T.$$

- We introduce the intervals $I_k := (T_k, T_{k+1})$.
- We take the optimality system and restrict the to time interval I_k.
- At the time-interfaces T_k, T_{k+1}, we employ continuity conditions $(y_k)(T_k) = (y_{k-1})(T_k)$ $k = 1, \ldots, K+1$, and similarly for the adjoint variables.

The time-domain-decomposition algorithm

Algorithm

1. Given $\mu^n_{k,k-1}, \mu^n_{k,k+1}$,
2. solve the restricted $\mathrm{OS}|_{I_k}$ for y_k^{n+1}, p_k^{n+1}

$$(y_k^{n+1})(T_{k+1}) + \sigma\beta'(y_k^{n+1})(p_k^{n+1})(T_{k+1}) = \mu^n_{k,k+1}, \quad \beta(y_k^{n+1})(T_k) - \sigma p_k^{n+1}(T_k) = \mu^n_{k,k-1}, \tag{1}$$

   with

$$\begin{aligned} \mu^n_{k,k+1} &= (1-\varepsilon)\left(\beta(y^n_{k+1})(T_{k+1}) + \sigma p^n_{k+1}(T_{k+1})\right) + \varepsilon\left(y_k^{n+1}(T_{k+1}) + \sigma\beta'(y_k^{n+1})(p_k^{n+1})(T_{k+1})\right), \ k = 0, \dots, \\ \mu^n_{k,k-1} &= (1-\varepsilon)\left(y^n_{k-1}(T_k) - \sigma\beta'(y^n_{k-1})p^n_{k-1}(T_k)\right) + \varepsilon\left(\beta(y_k^{n+1})(T_k) - \sigma p_k^{n+1}(T_k)\right), \ k = 1, \dots, K. \end{aligned} \tag{2}$$

3. Update $\mu^{n+1}_{k,k-1}, \mu^{n+1}_{k,k+1}$ for $n \to n+1$.

Virtual control problem

The corresponding virtual optimal control problem for the generic interval I_k reads as follows. With

$$J_k^n(u_k, y_k, h_{k,k-1}) := \frac{\kappa}{2} \int_{T_k}^{T_{k+1}} \int_0^{\ell} (y_k - y_k^d)^2 dx dx t + \frac{\nu}{2} \int_{T_k}^{T_{k+1}} \int_0^{\ell} u_k^2 dx dt + \frac{1}{2\sigma} \int_0^{\ell} \left((y_k(T_{k+1}) - \mu_{k,k+1})^2 + (h_{k,k-1})^2 \right) dx,$$

we have

$$\begin{aligned} \min_{u_k, y_k, h_{k,k-1}} \; & J_k^n(u_k, y_k, h_{k,k-1}) \\ \text{s. t.} \; & \\ \partial_t \beta_k(y_k) - \partial_x(\beta_k(\partial_x(y_k))) &= u_k, \quad \text{in } (T_k, T_{k+1}) \times (0, \ell) \\ \beta_k(y_k)(T_k) &= h_{k,k-1} + \mu_{k,k-1}^n, \quad \text{in } (0, \ell), \end{aligned}$$

where $h_{k,k-1}$ serves as the *virtual control.*

Virtual control problem: first interval

This system has to be complemented by the problems on the first and the last interval.

$$\min_{u_0,y_0} J_0^n(u_0,y_0) := \frac{\kappa}{2}\int_{T_0}^{T_1}\int_0^{\ell}(y_0-y_0^d)^2dxdt + \frac{\nu}{2}\int_{T_0}^{T_1}\int_0^{\ell}u_0^2dxdt$$

$$+\frac{1}{2\sigma}\int_0^{\ell}(y_0(T_1)-\mu_{0,1})^2dx$$

s. t.

$$\partial_t\beta_0(y_0)-\partial_x(\beta_0(\partial_x(y_0))) = u_0, \quad \text{in } (T_0,T_1)\times(0,\ell)$$

$$\beta_k(y_0)(T_0) = y_0, \quad \text{in } (0,\ell),$$

Virtual control problem: last interval

$$\min_{u_K, y_K, h_{K,K-1}} J_K^n(u_K, y_K) := \frac{\kappa}{2} \int_{T_K}^{T_{K+1}} \int_0^\ell (y_K - y_K^d)^2 dx dt + \frac{\kappa}{2} \int_0^\ell (y_K(T_{K+1}) - y_T^d)^2 dx$$

$$\frac{\nu}{2} \int_{T_K}^{T_{K+1}} \int_0^\ell u_K^2 dx dt + \frac{1}{2\sigma} \int_0^\ell \left((y_K(T_K) - \mu_{K,K+1})^2 + h_{K,K-1}^2 \right) dx$$

s. t.

$$\partial_t \beta_K(y_K) - \partial_x(\beta_K(\partial_x(y_K))) = u_K, \quad \text{in } (T_K, T_{K+1}) \times (0, \ell)$$

$$\beta_k(y_K)(T_K) = \mu_{K,K-1}^n + h_{K,K-1}, \quad \text{in } (0, \ell),$$

Further results and outlook

1. We have a similar result for the time-domain-decomposition problem (again, the proof only for for $\alpha = 2$)

2. The simultaneous space-time-domain decomposition is open (fine for the p-elliptic case)

3. The (β_α, β_p)-problem is open (as far as the proof is concerned)

4. Constrained control can be included, however, this has not yet been proved (just a matter of writing it up)

5. State constraints are completely open.

6. One may use PINN (XPINN) on subnetworks as surrogate models and perform interface learning (in preparation)

7. Final goal: **Ne**twork **T**earing and **I**nterconnection, a formal analogue of FETI.

Thank you for your attention!