Convergence speed of Schwarz Waveform Relaxation Algorithms

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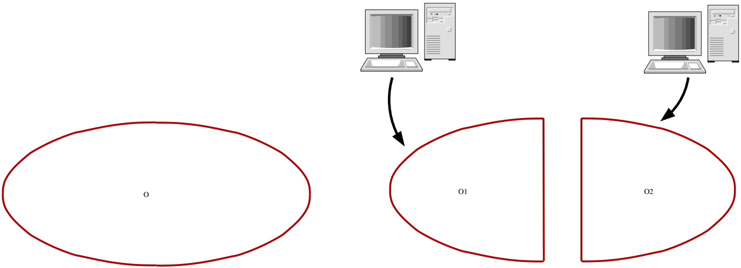
Outline

Outline

Motivations

DD for stationary problem



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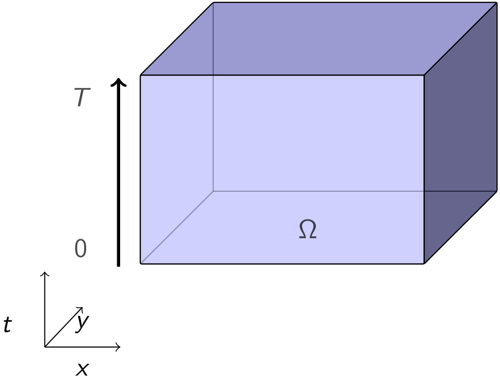
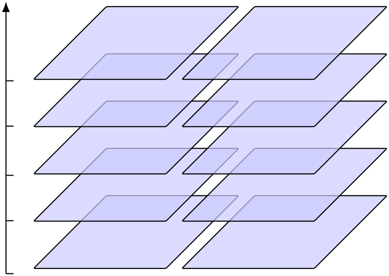


Schwarz Algo. versus Schwarz Waveform Relaxation Algo.



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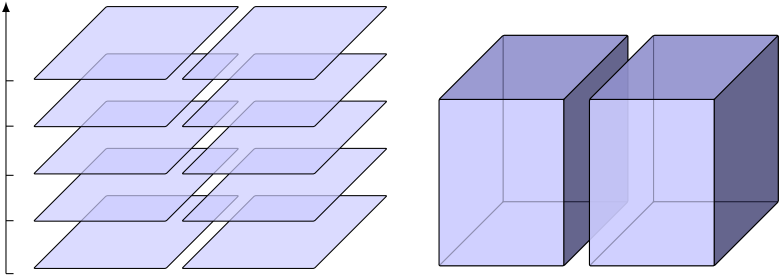
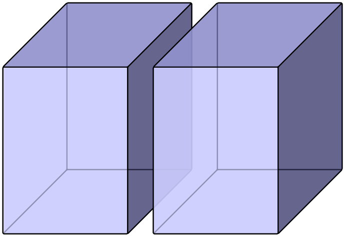


Motivations for Schwarz Waveform Relaxation Method



- Save time communication
- Adapt the time and space steps
- Adapt the model

About the name SWR

Picard's Fixed Point

$$\begin{cases} \dfrac{d}{dt}y(t) = f(t, y(t)) \text{ on } [0, T], \\ y(0) = y_0. \end{cases}$$

$$\frac{d}{dt}y^{n+1}(t) = f(t, y^n(t)) \Longrightarrow y^{n+1}(t) = y_0 + \int_0^t f(s, y^n(s))ds$$

Theorem (E. Lindelof (1894))

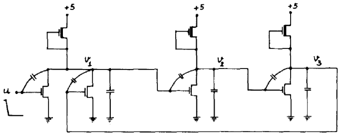
For f continuous and uniformly Lipschitz with respect to y (constant L) then the sequence $(y^n)_n$ is convergent and the error satisfies

$$\|y - y^n\|_\infty \leq \frac{(LT)^n}{n!}\|y - y^0\|_\infty.$$

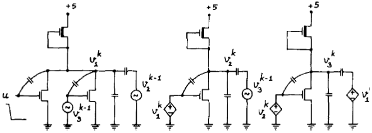
Systems of ODEs

The Waveform Relaxation Method for Time-Domain Analysis of Large Scale Integrated Circuits. E. Lelarasmee; A.E. Ruehli; A.L. Sangiovanni-Vincentelli. 1982.

$$\begin{cases} \dfrac{dv_1}{dt} = f_1(v_1, v_2, v_3) \\ \dfrac{dv_2}{dt} = f_2(v_1, v_2, v_3) \\ \dfrac{dv_3}{dt} = f_2(v_1, v_2, v_3) \end{cases}$$



$$\begin{cases} \dfrac{d{\color{red}v_1^{k+1}}}{dt} = f_1({\color{red}v_1^{k+1}}, v_2^k, v_3^k) \\ \dfrac{d{\color{blue}v_2^{k+1}}}{dt} = f_2(v_1^k, {\color{blue}v_2^{k+1}}, v_3^k) \\ \dfrac{d{\color{magenta}v_3^{k+1}}}{dt} = f_2(v_1^k, v_2^k, {\color{magenta}v_3^{k+1}}) \end{cases}$$



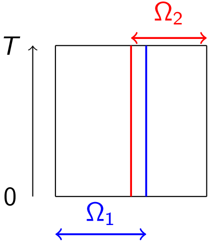
Outline

SWR for the heat equation

SWR Algorithm for $\mathcal{L}u = F$ on $[-L, L] \times [0, T]$

$$\begin{cases} \mathcal{L}u_1^{n+1} = F & \text{on } \Omega_1 \times (0, T), \\ u_1^{n+1}(\delta, t) = u_2^n(\delta, t), & \\ u_1^{n+1}(\cdot, 0) = u_0 & \end{cases} \tag{1}$$

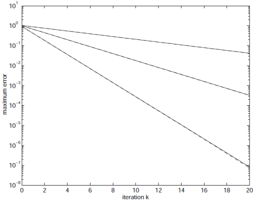
$$\begin{cases} \mathcal{L}u_2^{n+1} = F & \text{on } \Omega_2 \times (0, T), \\ u_2^{n+1}(0, t) = u_1^n(0, t), & \\ u_2^{n+1}(\cdot, 0) = u_0 & \end{cases} \tag{2}$$



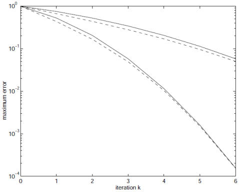
Convergence of the SWR Algorithm

- Gander M. J., Zhao H. 2002. Overlapping Schwarz Waveform Relaxation for the Heat Equation in n-Dimensions.
- E. Giladi, H. B. Keller Space-time 2002. Domain decomposition for parabolic problems.

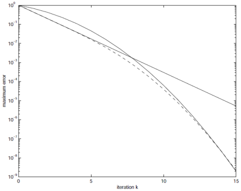
Two behaviors:



Large T



Small T



Intermediate T

Convergence of the SWR Algorithm

$$\begin{cases} \dfrac{\partial e_1^{n+1}}{\partial t} - \nu \dfrac{\partial^2 e_1^{n+1}}{\partial x^2} = 0 \text{ in } (-L, \delta) \times (0, T) \\ e_1^{n+1}(\delta, \cdot) = e_2^n(\delta, \cdot) \\ + B.C. \end{cases} \quad \begin{cases} \dfrac{\partial e_2^{n+1}}{\partial t} - \nu \dfrac{\partial^2 e_2^{n+1}}{\partial x^2} = 0 \text{ in } (0, L) \times (0, T) \\ e_2^{n+1}(0, \cdot) = e_1^n(0, \cdot) \\ + B.C. \end{cases}$$

- $T = \infty$: linear bound (use Maximum Principle):

$$\|e_1^n(0,\cdot)\|_{L^\infty} \leq \left(\frac{L-\delta}{L+\delta}\right)^n \|e_1^0(0,\cdot)\|_{L^\infty}$$

- $T < +\infty$: superlinear bound (use the heat kernel):

$$\|e_1^n(0,\cdot)\|_{L^\infty} \leq erfc\left(\frac{n\delta}{2\sqrt{\nu T}}\right)\|e_1^0(0,\cdot)\|_{L^\infty}$$

Linear convergence

The proof of the linear convergence relies on the maximum principle.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} = 0 \quad \text{on } (a,b) \times (0,+\infty), \\ u(\cdot,0) = 0, \\ u(a,\cdot) = g_1, \\ u(b,\cdot) = g_2. \end{cases}$$

Lemma

The solution of the heat equation on $(a,b) \times (0,+\infty)$ satisfies:

$$\|u(x,\cdot)\|_\infty \leq \frac{b-x}{b-a}\|g_1\|_\infty + \frac{x-a}{b-a}\|g_2\|_\infty, \; x \in [a,b].$$

Linear convergence

$$\begin{cases} \dfrac{\partial e_1^{n+1}}{\partial t} - \nu \dfrac{\partial^2 e_1^{n+1}}{\partial x^2} = 0 \text{ on } (-L,\delta)\times(0,T) \\ e_1^{n+1}(\delta,\cdot) = e_2^n(\delta,\cdot) \\ e_1^{n+1}(0,\cdot) = 0 \end{cases} \quad \begin{cases} \dfrac{\partial e_2^{n+1}}{\partial t} - \nu \dfrac{\partial^2 e_2^{n+1}}{\partial x^2} = 0 \text{ on } (0,L)\times(0,T) \\ e_2^{n+1}(0,\cdot) = e_1^n(0,\cdot) \\ e_2^{n+1}(L,\cdot) = 0 \end{cases}$$

We apply the maximum principle:

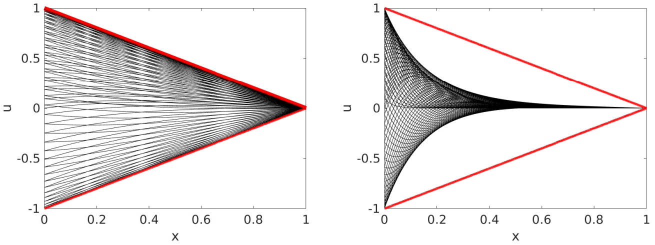
$$\|e_1^{n+1}(0,\cdot)\|_\infty \leq \frac{L}{L+\delta}\|e_2^n(\delta,\cdot)\|_\infty,$$
$$\|e_2^{n+1}(\delta,\cdot)\|_\infty \leq \frac{L-\delta}{L}\|e_1^n(0,\cdot)\|_\infty.$$

Combining gives the result:

$$\|e_1^{n+1}(0,\cdot)\|_\infty \leq \frac{L-\delta}{L+\delta}\|e_1^{n-1}(0,\cdot)\|_\infty.$$

Adimentionalized heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial u}{\partial x^2} = 0 \text{ on } [0, L] \times [0, T], \\ u(0, \cdot) = \sin(3\pi \dfrac{t}{T}), \\ u(L, \cdot) = 0, \\ u(\cdot, 0) = 0, \end{cases} \rightsquigarrow \begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\nu T}{L^2} \dfrac{\partial u}{\partial x^2} = 0 \text{ on } [0, 1] \times [0, 1], \\ u(0, \cdot) = \sin(3\pi t), \\ u(1, \cdot) = 0, \\ u(\cdot, 0) = 0, \end{cases}$$



$$\frac{\nu T}{L^2} = 10 \qquad\qquad \frac{\nu T}{L^2} = 0.1$$

Superlinear convergence

The solution of

$$\begin{cases} \dfrac{\partial w}{\partial t} - \nu \dfrac{\partial^2 w}{\partial x^2} = 0 & \text{on } (0, +\infty) \times (0, T), \\ w(\cdot, 0) = 0, \\ w(0, t) = \max\limits_{0 \le t \le \tau} |e_1^{k-1}(0, \tau)| := E_1^{k-1}(t), \end{cases}$$

is $w(x, t) = \displaystyle\int_0^t K(x, t-\tau) E_1^{k-1}(\tau)\, d\tau$ where

$$K(x, t) = \frac{x}{2\sqrt{\nu\pi}} \frac{e^{-\frac{x^2}{4\nu t}}}{t^{3/2}}.$$

We have

$$|e_2^k(\delta, t)| \le |w(\delta, t)| = \left| \int_0^t K(\delta, t-\tau) E_1^{k-1}(\tau)\, d\tau \right|.$$

Superlinear convergence

From $|e_2^k(\delta, t)| \leq \int_0^t K(\delta, t-\tau) E_1^{k-1}(\tau)\, d\tau$, we obtain

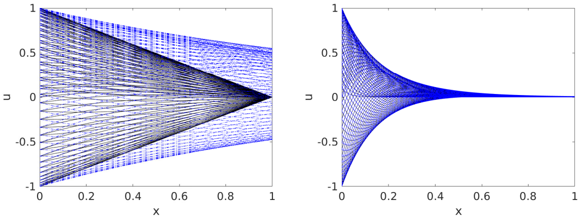
$$\begin{aligned}
\|e_2^k(\delta,\cdot)\|_\infty & \leq \|E_1^k\|_\infty \frac{1}{\sqrt{\pi}} \int_0^t \frac{\delta}{2\sqrt{\nu}} \frac{e^{-\frac{\delta^2}{4\nu(t-\tau)}}}{(t-\tau)^{3/2}}\, d\tau \\
& \leq \|E_1^k\|_\infty \underbrace{\frac{2}{\sqrt{\pi}} \int_{\frac{\delta}{2\sqrt{\nu t}}}^{+\infty} e^{-\tau^2}}_{=erfc(\frac{x}{2\sqrt{\nu t}})}\, d\tau \\
& \leq \|E_1^k\|_\infty \mathrm{erfc}\left(\frac{\delta}{2\sqrt{\nu T}}\right)
\end{aligned}$$

with the change of var $\tilde{\tau} = \dfrac{\delta}{\sqrt{\nu(t-\tau)}}$ ($d\tilde{\tau} = -\dfrac{\delta}{4\sqrt{\nu}(t-\tau)^{3/2}} d\tau$).

Superlinear convergence

Adimentionalized heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial u}{\partial x^2} = 0 \text{ on } [0, L] \times [0, T], \\ u(0, \cdot) = \sin(3\pi \dfrac{t}{T}), \\ u(L, \cdot) = g_1, \\ u(\cdot, 0) = 0, \end{cases} \rightsquigarrow \begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\nu T}{L^2} \dfrac{\partial u}{\partial x^2} = 0 \text{ on } [0, 1] \times [0, 1], \\ u(0, \cdot) = \sin(3\pi t), \\ u(1, \cdot) = 0, \\ u(\cdot, 0) = 0, \end{cases}$$



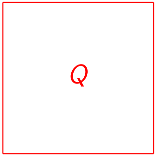
$$\frac{\nu T}{L^2} = 10 \qquad \frac{\nu T}{L^2} = 0.1$$

Outline

Schwarz Algorithm

We want to solve

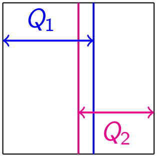
$$\mathcal{L}u = f \text{ in } Q$$
$$+ \text{ B.C.}$$



We use a Schwarz algorithm

$$\begin{cases} \mathcal{L}u_1^{n+1} = f & \text{in } Q_1 \\ u_1^{n+1} = u_2^n & \text{on } \Gamma_1 \\ +B.C. & \text{on } \partial Q_1 - \{\Gamma_1\} \end{cases}$$

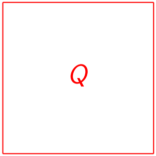
$$\begin{cases} \mathcal{L}u_2^{n+1} = f & \text{in } Q_2 \\ u_2^{n+1} = u_1^{n+1} & \text{on } \Gamma_2 \\ +B.C. & \text{on } \partial Q_2 - \{\Gamma_2\} \end{cases}$$



Schwarz Algorithm

We want to solve

$$\mathcal{L}u = f \text{ in } Q$$
$$+ \text{ B.C.}$$



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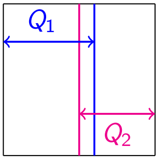
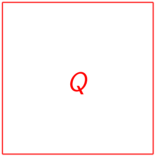
$$\begin{cases} \mathcal{L}e_2^{n+1} = 0 & \text{in } Q_2 \\ e_2^{n+1} = e_1^{n+1} & \text{on } \Gamma_2 \\ +B.C. & \text{on } \partial Q_2 - \{\Gamma_2\} \end{cases}$$



How fast is the convergence?

Using Fourier analysis for $\triangle u = f$

$$\begin{cases} \triangle u = f & \text{on } Q = [-L, L] \times [0, 1] \\ u = g & \text{on } \partial Q \end{cases}$$



Using Fourier analysis for $\triangle u = f$

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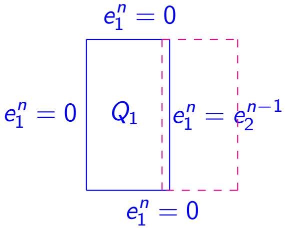


Solving at iteration n:

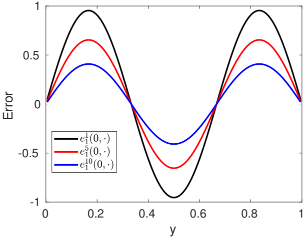
$$\begin{aligned} e_1^n(x,y) &= \sum_{k=0}^{\infty} \hat{e}_1^n(k) \frac{\sinh(k\pi(x+L))}{\sinh(k\pi(\delta+L))} \sin(k\pi y) \;\; \text{on } Q_1 = [-L, \delta] \times [0,1] \\ e_2^n(x,y) &= \sum_{k=0}^{\infty} \hat{e}_2^n(k) \frac{\sinh(k\pi(x-L))}{\sinh(k\pi L)} \sin(k\pi y) \;\; \text{on } Q_2 = [0, L] \times [0,1] \end{aligned}$$

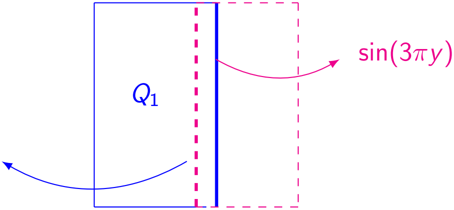
Using the transmission conditions $e_1^{n+1}(\delta,\cdot) = e_2^n(\delta,\cdot)$ and $e_2^n(0,\cdot) = e_1^n(0,\cdot)$

$$\hat{e}_1^{n+1}(k) = \underbrace{\frac{\sinh(k\pi(\delta-L))}{\sinh(k\pi(\delta+L))}}_{=\rho(k)} \hat{e}_1^n(k)$$

Using Fourier analysis for $\triangle u = f$

Introducing a pure sine frequency: $e_2^0(\delta, y) = \sin(3\pi y)$

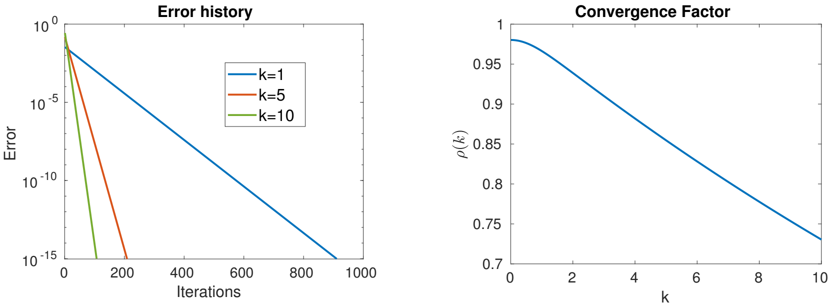




$\rightarrow$ **The sine is contracted as predicted by the Fourier convergence factor.**

Convergence factor

Domain: $Q = [-1/2, 1/2] \times [0, 1]$
Overlap: $\delta = \Delta x = 0.005$
First guess: $e_2^0(\delta, \cdot) = \sin(k\pi y),\ k = 1, 5$ or 10

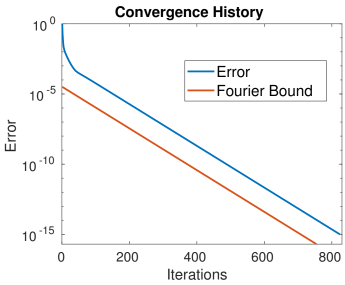


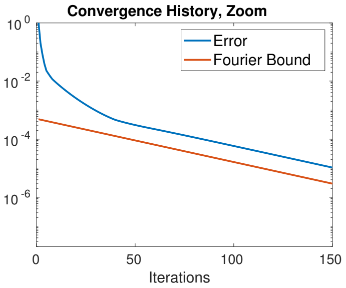
$\rightarrow$ **The convergence is linear**

$\rightarrow$ **High frequencies converge quickly**

Convergence factor

Domain: $Q = [-1/2, 1/2] \times [0, 1]$
Overlap: $\delta = \Delta x = 0.005$
First guess: $e_2^0(\delta, \cdot) = \text{rand}(-1, 1)$



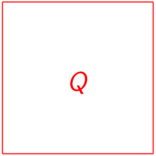


$$e_1^n(0, y) = \sum_{k=k_{min}}^{k_{max}} \lambda_k (\rho(k))^n \sin(k\pi y)$$

$\rightarrow$ **The convergence is guided by the lowest frequency** $\rho(k_{min})$

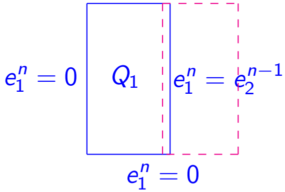
Using Fourier analysis for the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} = f \text{ on } Q = [-L, L] \times [0, T] \\ u = g \text{ at } x = -L, x = L \quad + I.C. \end{cases}$$



Using Fourier analysis for the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} = f \text{ on } Q = [-L, L] \times [0, T] \\ u = g \text{ at } x = -L, x = L \quad + I.C. \end{cases}$$



Laplace transform ($\hat{u}(x,s) = \int_0^{+\infty} u(x,t) e^{-st} dt$) of the equation:

$$\hat{e}_1^n(x,s) = A^n \frac{\sinh(\sqrt{s\nu}(x+L))}{\sinh(\sqrt{s\nu}(\delta+L))}$$

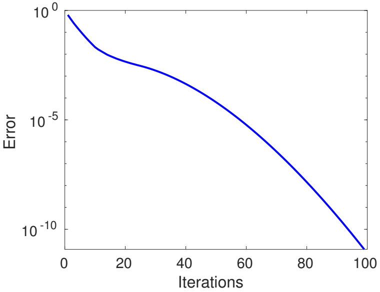
$$\hat{e}_2^n(x,s) = B^n \frac{\sinh(\sqrt{s}(x-L))}{\sinh(\sqrt{s}L)}$$

so that using the transmission conditions:

$$\hat{e}_1^n(x,s) = \underbrace{\frac{\sinh(\sqrt{s\nu}(\delta-L))}{\sinh(\sqrt{s\nu}(\delta+L))}}_{=\rho(s)} \hat{e}_1^{n-1}(x,s)$$

Using Fourier analysis for the heat equation

Domain: $Q = [-3.5, 3.5] \times [0, 1]$
Overlap: $\delta = 5\Delta x = 5 \times 0.05$
First guess: $e_2^0(\delta, t) = \sin(50t)$



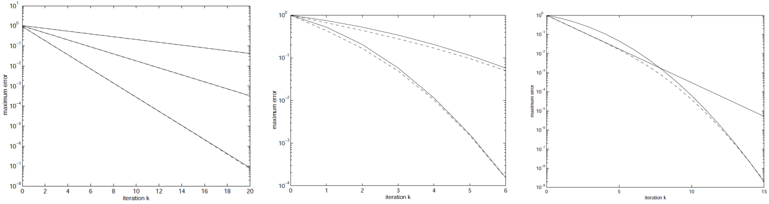
Convergence results for the heat equation

- $T = \infty$: linear bound (use Maximum Principle):

$$\|e_1^n(0,\cdot)\|_{L^\infty} \leq \left(\frac{L-\delta}{L+\delta}\right)^n \|e_1^0(0,\cdot)\|_{L^\infty}$$

- $T < +\infty$: superlinear bound (use the heat kernel):

$$\|e_1^n(0,\cdot)\|_{L^\infty} \leq erfc\left(\frac{n\delta}{2\sqrt{\nu T}}\right)\|e_1^0(0,\cdot)\|_{L^\infty}$$



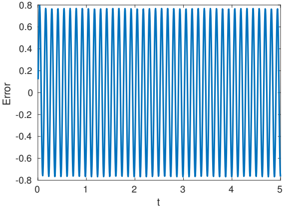
→ **What is the link with the Fourier analysis?**

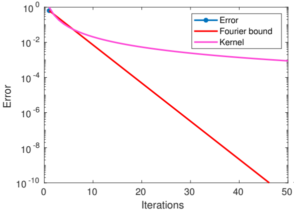
An additional phenomenon

Domain: $Q = [-10, 10] \times [0, 5]$

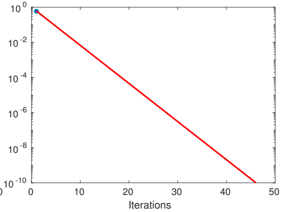
Overlap: $\delta = 10\Delta x = 10 \times 0.005$

First guess: $e_2^0(\delta, \cdot) = \sin(50t)$





$\|e_1^k(0,\cdot)\|_{L^\infty(0,T)}$



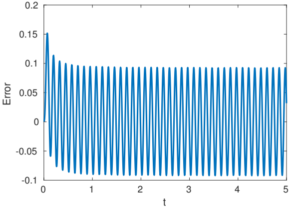
$\|e_1^k(0,\cdot)\|_{L^\infty(T/2,T)}$

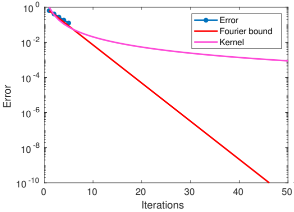
An additional phenomenon

Domain: $Q = [-10, 10] \times [0, 5]$

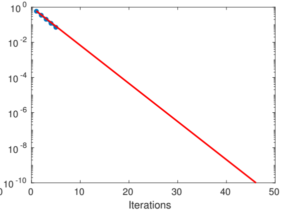
Overlap: $\delta = 10\Delta x = 10 \times 0.005$

First guess: $e_2^0(\delta, \cdot) = \sin(50t)$





$\|e_1^k(0,\cdot)\|_{L^\infty(0,T)}$



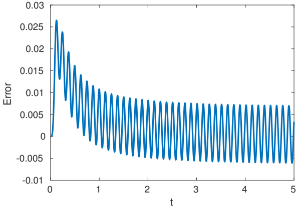
$\|e_1^k(0,\cdot)\|_{L^\infty(T/2,T)}$

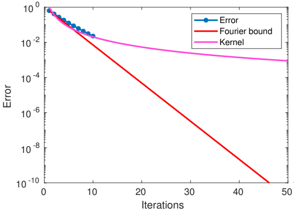
An additional phenomenon

Domain: $Q = [-10, 10] \times [0, 5]$

Overlap: $\delta = 10\Delta x = 10 \times 0.005$

First guess: $e_2^0(\delta, \cdot) = \sin(50t)$





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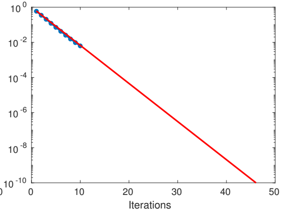

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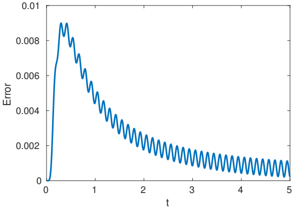

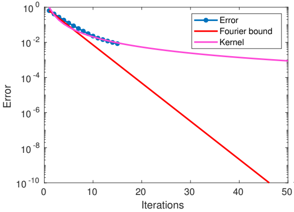

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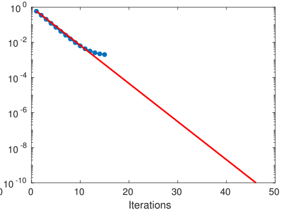

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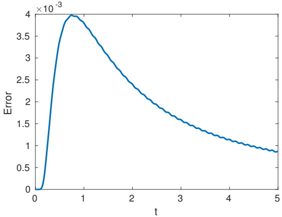

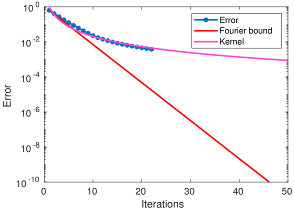

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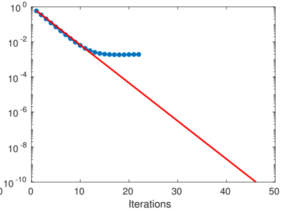

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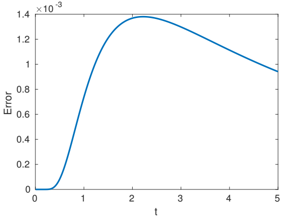

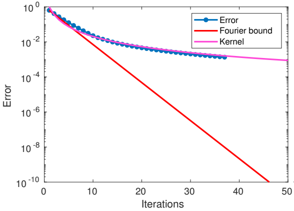

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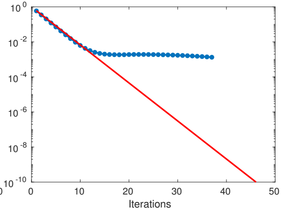

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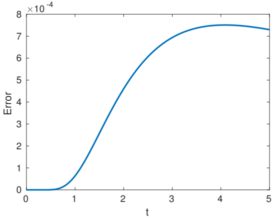
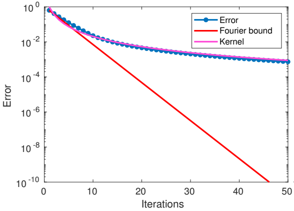
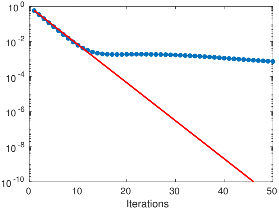




$\|e_1^k(0,\cdot)\|_{L^\infty(0,T)}$

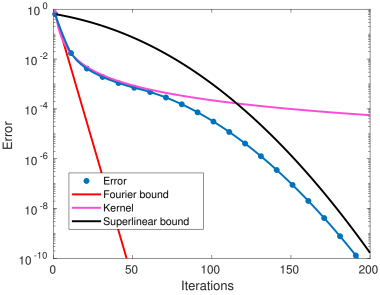


$\|e_1^k(0,\cdot)\|_{L^\infty(T/2,T)}$

An additional phenomenon

We can see:

- The Fourier convergence factor
- The behavior of the bump
- The superlinear convergence



An additional phenomenon

◊ When $\mathcal{L} = \triangle$, if the algorithm is initalized with the pure sine frequency $e_2^0(\delta, y) = \sin(\lambda y)$ we have:

$$e_1^n(0, y) = \rho(\lambda)^{2n-1} \sin(\lambda y)$$

◊ When $\mathcal{L} = \partial_t - \nu\partial_{xx}$ and $L = \infty$, if the algorithm is initalized with the pure sine frequency $e_2^0(\delta, t) = \sin(\lambda t)$, then for large λ, we have

$$\begin{aligned} e_1^n(0, t) \quad &= \rho(\lambda)^{2n-1} \sin(\lambda t - (2n-1)L\sqrt{\frac{\lambda}{2\nu}}) \\ &+ \frac{1}{\lambda} K((2n-1)\frac{L}{\sqrt{\nu}}, t) + \mathcal{O}(\frac{1}{\lambda^3}). \end{aligned}$$

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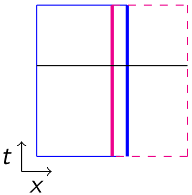
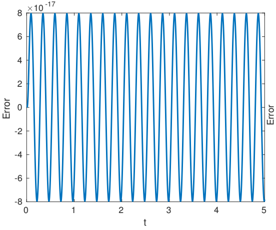
Proof (M.J. Gander, V.M. 2022):

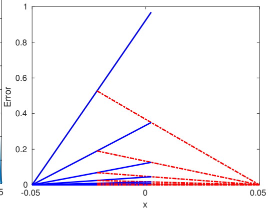
$$e_1^n(0, t) = \frac{1}{\sqrt{\pi}} \int_0^t \sin(\lambda(t - \tau)) K(\frac{(2n-1)L}{2}), \tau) d\tau$$

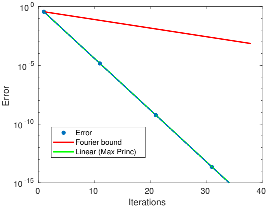
To sum up

Domain: $Q = [-0.05, 0.05] \times [0, 5]$
Overlap: $\delta = 5\Delta x = 5 \times 0.005$
First guess: $e_2^0(\delta, \cdot) = \sin(25t)$









Error vs time at iter. 38

Error vs x
First iter. at $t = 3.7037$

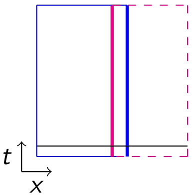
Error vs iterations

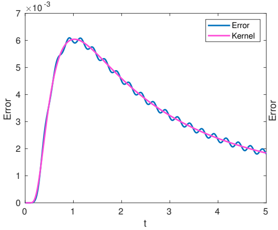
To sum up

Domain: $Q = [-10, 10] \times [0, 5]$

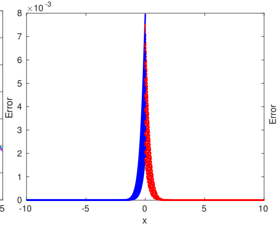
Overlap: $\delta = 5\Delta x = 5 \times 0.005$

First guess: $e_2^0(\delta, \cdot) = \sin(25t)$



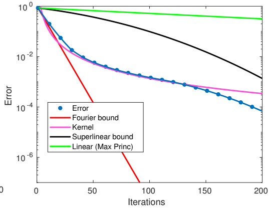


Error vs time at iter. 50



Error vs x

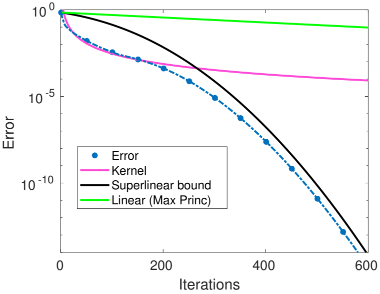
First iter. at $t = 0.3499$



Error vs iterations

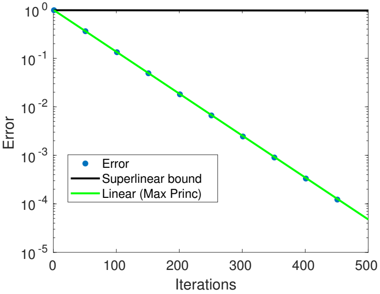
When all the frequencies are present

Domain: $Q = [-30, 30] \times [0, 30]$
Overlap: $\delta = 5\Delta x = 5 \times 0.05$
First guess: $e_2^0(\delta, \cdot) = \text{rand}(-1, 1)$



When all the frequencies are present

Domain: $Q = [-10^{-3}, 10^{-3}] \times [0, 0.1]$
Overlap: $\delta = \Delta x = 10^{-5}$
First guess: $e_2^0(\delta, \cdot) = \text{rand}(-1, 1)$



Conclusions and future works

- Fourier analysis is not sufficient to explain the convergence.
- What about Robin boundary conditions?

$$\begin{cases} \mathcal{L}u_1^{n+1} = f & \text{in } Q_1 \\ \dfrac{\partial u_1^{n+1}}{\partial x} - pu_1^{n+1} = \dfrac{\partial u_2^n}{\partial x} - pu_2^n & \text{on } \Gamma_1 \\ +B.C. & \text{on } \partial Q_1 - \{\Gamma_1\} \end{cases}$$

$$\begin{cases} \mathcal{L}u_2^{n+1} = f & \text{in } Q_2 \\ \dfrac{\partial u_2^{n+1}}{\partial x} + pu_2^{n+1} = \dfrac{\partial u_1^n}{\partial x} + pu_1^n & \text{on } \Gamma_1 \\ +B.C. & \text{on } \partial Q_2 - \{\Gamma_2\} \end{cases}$$

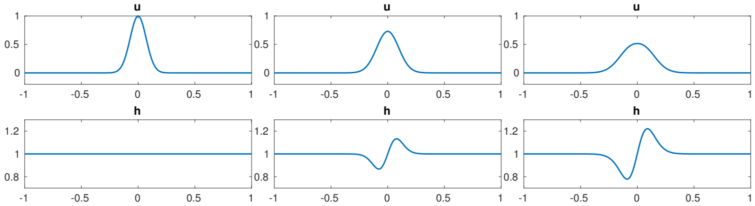
Outline

Linearized Viscous Shallow Water equations

Linearized adimensionalized equations $\mathcal{L}W = F$ are

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{Fr^2}\dfrac{\partial h}{\partial x} - \dfrac{1}{Re}\dfrac{\partial^2 u}{\partial x^2} = f & \text{on } \mathbb{R} \times (0, +\infty) \\ \dfrac{\partial h}{\partial t} + \dfrac{\partial u}{\partial x} = 0 & \text{on } \mathbb{R} \times (0, +\infty) \\ u(\cdot, 0) = u_0, h(\cdot, 0) = h_0 & \text{on } \mathbb{R} \end{cases}$$

where $W = (u, h)$, $Fr = U/c$, $Re = UL/\nu$.



SWR Algorithm with Robin boundary conditions

If $\mathcal{B}(u,h) = -\dfrac{1}{Re}\dfrac{\partial u}{\partial x} + \dfrac{1}{Fr^2}h$ the DD algorithm reads

$$\begin{cases} \mathcal{L}W_1^{n+1} = 0 & \text{on } \Omega_1 \times (0,+\infty), \\ (\mathcal{B}(u_1^{n+1}, h_1^{n+1}) - pu_1^{n+1})(0,t) & = (\mathcal{B}(u_2^n, h_2^n) - pu_2^n)(0,t), \\ u_1^{n+1}(\cdot,0) = u_0,\ h_1^{n+1}(\cdot,0) = h_0 & \end{cases} \tag{3}$$

$$\begin{cases} \mathcal{L}W_2^{n+1} = 0 & \text{on } \Omega_2 \times (0,+\infty), \\ (\mathcal{B}(u_2^{n+1}, h_2^{n+1}) + pu_2^{n+1})(0,t) & = (\mathcal{B}(u_1^n, h_1^n) + pu_1^n)(0,t), \\ u_2^{n+1}(\cdot,0) = u_0,\ h_2^{n+1}(\cdot,0) = h_0 & \end{cases} \tag{4}$$

How to choose p such that the convergence is fast?

Solving in Laplace variables

We use the Laplace transform

$$\hat{u}(s) = \int_0^{+\infty} u(t) e^{-st} dt.$$

The SW equations $\mathcal{L}W = 0$ in Laplace variables are

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{Fr^2}\dfrac{\partial h}{\partial x} - \dfrac{1}{Re}\dfrac{\partial^2 u}{\partial x^2} = 0 \\ \dfrac{\partial h}{\partial t} + \dfrac{\partial u}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} s\hat{u} - (\dfrac{1}{Re} + \dfrac{1}{sFr^2})\dfrac{\partial^2 \hat{u}}{\partial x^2} = 0 \\ \hat{h} = -\dfrac{1}{s}\dfrac{\partial \hat{u}}{\partial x} \end{cases}$$

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The solution of the algorithm is

$$\begin{cases} \mathcal{L}(u_1^{n+1}, h_1^{n+1}) = 0 \quad \text{in } \mathbb{R}^- \times \mathbb{R}^+ \\ \mathcal{L}(u_2^{n+1}, h_2^{n+1}) = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \end{cases} \qquad \begin{cases} \hat{u}_1^{n+1}(x,s) = \alpha^{n+1}(s) e^{\mu(s)x} \\ \hat{u}_2^{n+1}(x,s) = \beta^{n+1}(s) e^{-\mu(s)x} \end{cases}$$

where $\mu(s) = \dfrac{sFr\sqrt{Re}}{\sqrt{sFr^2 + Re}}$.

Solving in Laplace variables

We have obtained

$$\hat{u}_1^n(x,s) = \alpha^n(s)e^{\mu(s)x} \text{ and } \hat{u}_2^n(x,s) = \beta^n(s)e^{-\mu(s)x}.$$

The transmission conditions at $\{0\} \times (0, +\infty)$ are

$$\left\{ \begin{array}{lcl} -\frac{1}{Re}\partial_x u_1^{n+1} + \frac{1}{Fr^2}h_1^{n+1} - pu_1^{n+1} & = & -\frac{1}{Re}\partial_x u_2^n + \frac{1}{Fr^2}h_2^n - pu_2^n \\ -\frac{1}{Re}\partial_x u_2^n + \frac{1}{Fr^2}h_2^n + pu_2^n & = & -\frac{1}{Re}\partial_x u_1^{n-1} + \frac{1}{Fr^2}h_1^{n-1} + pu_1^{n-1} \end{array} \right.$$

In Laplace variables they become

$$\begin{array}{lcl} (\frac{1}{Re} + \frac{1}{Fr^2 s})\partial_x \hat{u}_1^{n+1} + p\hat{u}_1^{n+1} & = & (\frac{1}{Re} + \frac{1}{Fr^2 s})\partial_x \hat{u}_2^n + p\hat{u}_2^n \\ -(\frac{1}{Re} + \frac{1}{Fr^2 s})\partial_x \hat{u}_2^n + p\hat{u}_2^n & = & -(\frac{1}{Re} + \frac{1}{Fr^2 s})\partial_x \hat{u}_1^{n-1} + p\hat{u}_1^{n-1} \end{array}$$

Solving in Laplace variables

Obtaining the convergence factor

We have the relation

$$\hat{u}_1^{n+1}(0,s) := \rho(p,s)\hat{u}_1^{n-1}(0,s)$$

with

$$\rho(\mathrm{p},\omega) = \left|\frac{\sqrt{i\omega + A} - \mathrm{p}}{\sqrt{i\omega + A} + \mathrm{p}}\right|^2 = \frac{\mathrm{p}^2 - \sqrt{2}\mathrm{p}\sqrt{A + \sqrt{A^2+\omega^2}} + \sqrt{A^2+\omega^2}}{\mathrm{p}^2 + \sqrt{2}\mathrm{p}\sqrt{A + \sqrt{A^2+\omega^2}} + \sqrt{A^2+\omega^2}}$$

where $A = Re/Fr^2$ et $\mathrm{p} = \sqrt{Re}p$.

Optimizing the convergence factor

$$\min_{p\in\mathbb{R}} \max_{\omega\in[\omega_{min},\omega_{max}]} \rho(p,\omega).$$

Optimization of the convergence factor for AD equation

The convergence factor of the OSWR algorithm applied to the advection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} + cu = 0.$$

is

$$\rho_{ad}(p,\omega) = \left|\frac{\sqrt{i\omega + A} - p}{\sqrt{i\omega + A} + p}\right|^2 \text{ with } A = \frac{a^2 + 4\nu c}{4\nu}.$$

Theorem (M.J. Gander, L. Halpern, 2007)

The solution of the min-max computed on $[0, \omega_{max}]$ problem is

$$q_1 = (A^2 + \omega_{max}^2)^{1/4} \qquad \text{if } \omega_{max}^2 > 4A^2(2+\sqrt{5})$$

$$q_2 = \left(A + \sqrt{2A}\sqrt{A + \sqrt{A^2 + \omega_{max}^2}}\right)^{1/2} \qquad \text{if } \omega_{max}^2 \leq 4A^2(2+\sqrt{5})$$

Optimization of the convergence factor

Let q_1 et q_2 defined by

$$q_1 = (A^2 + \omega_{max}^2)^{1/4}$$

$$q_2 = \left(A + \sqrt{A + \sqrt{A^2 + \omega_{min}^2}} \sqrt{A + \sqrt{A^2 + \omega_{max}^2}} \right)^{1/2}$$

Theorem

Optimizing the convergence factor gives a unique solution:
$p^\star = \min(q_1, q_2)$,
If $q_1 < q_2$ then the optimized solution satisfies

$$\min_{p_0 \in \mathbb{R}} \max_{\omega \in [\omega_{min}, \omega_{max}]} \rho(p_0, \omega) = \rho(q_1, \omega_{max}).$$

If $q_1 > q_2$ then the optimized solution satisfies

$$\min_{p \in \mathbb{R}} \max_{\omega \in [\omega_{min}, \omega_{max}]} \rho(p, \omega) = \rho(q_2, \omega_{min}) = \rho(q_2, \omega_{max}).$$

Optimizing the convergence factor

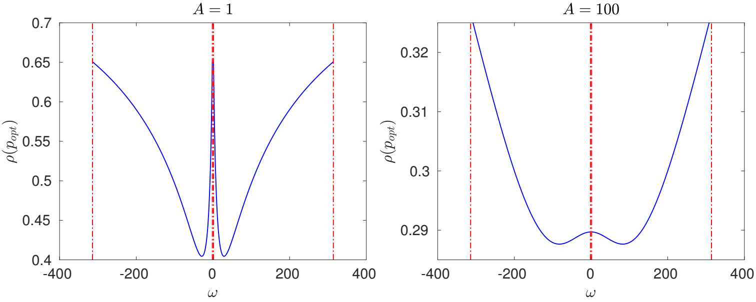
Theorem (D. Bennequin, M.J. Gander, L. Halpern 08)

The solution of the min-max problem

$$\min_{p\in\mathbb{C}} \max_{\omega\in[-\omega_{max},-\omega_{min}]\cup[\omega_{min},\omega_{max}]} \rho(p,\omega) \to p^\star$$

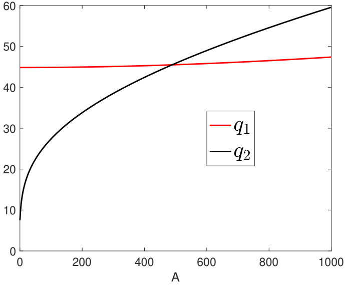
is unique, real and $\omega \to \rho(p^\star,\omega)$ equioscillates at, at least, 2 points.

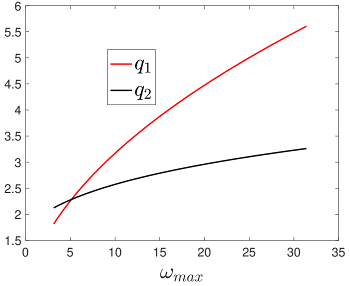
Physical and numerical data: $\omega_{min} = \frac{\pi}{2}$, $\omega_{max} = \frac{\pi}{dt} \simeq 2011.$



Dependance of q_1 and q_2 w.r. to the parameters

If $q_1 > q_2$ $p^\star = q_2$ (equi-oscillation)
If $q_1 < q_2$ $p^\star = q_1$ (no equi-oscillation)





q_1, q_2 as functions of A

q_1, q_2 as functions of $\omega_{max} = \dfrac{\pi}{\Delta t}$

Case $\Delta t \to 0$

Corollary 1

If $\Delta t \to 0$ then the optimized parameter behaves like

$$p^\star \simeq C_p \Delta t^{-1/4}, C_p = \frac{\pi^{1/4}}{\sqrt{ReFr}}(Re + \sqrt{Re^2 + \omega_{min}^2 Fr^4})^{1/4}$$

and the convergence factor like

$$\min_{p \in [0,+\infty[} \max_{\omega \in [\omega_{min}, \omega_{max}]} \rho(p,\omega) = 1 - 2\sqrt{2Re}\frac{C_p}{\sqrt{\pi}}\Delta t^{1/4} + o(\Delta t^{1/4})$$

Proof:

$q_1 \simeq \omega_{max}^{1/2}$

$q_2 \simeq C\omega_{max}^{1/4}$

Case $q_1 > q_2$.

Case $Re \to +\infty$

Corollary 2

If $Re \to +\infty$ then the optimized parameter behaves like

$$p^\star \simeq \frac{1}{Fr}$$

and the convergence factor behaves like

$$\min_{p \in [0,+\infty[} \max_{\omega \in [\omega_{min},\omega_{max}]} \rho(p,\omega) \simeq \frac{\omega_{max}^2 Fr^4}{16 Re^2}.$$

Proof:

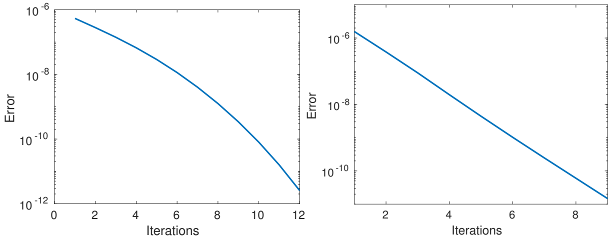
$$q_1 \simeq \sqrt{A} \qquad q_2 \simeq \sqrt{3A}$$

Case $q_2 > q_1$.

Influence of the time interval

Physical and numerical data

$\Delta x = 1.56 \cdot 10^{-4}, \Delta t = 5 \cdot 10^{-3}$
$u(x,t) = 0,\ h(x,t) = 0$
$u_2^0(0,\cdot) = \text{rand}(-1,1)$



$T = 0.2$ $\qquad$ $T = 2$

Error versus iterations

How sharp is the optimized parameter?

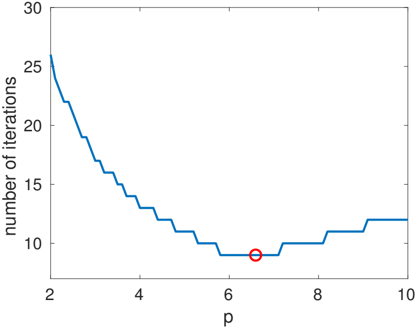
Physical and numerical data

$T = 2$

$\Delta x = 1.56 \cdot 10^{-4}, \Delta t = 5 \cdot 10^{-3}$

$u(x,t) = 0,\ h(x,t) = 0$

$u_2^0(0,\cdot) = \text{rand}(-1,1)$

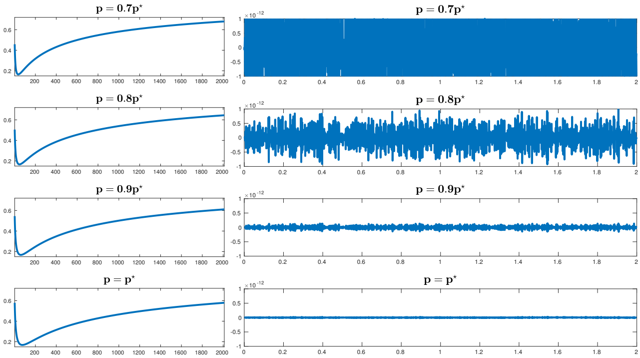


Number of iterations needed to reach an error of 10^{-10}.

Effect of p for a sum of frequencies

Physical and numerical data

$$T = 2$$
$$\Delta x = 1.56 \cdot 10^{-4}, \Delta t = 1.6 \cdot 10^{-3}$$
$$u_2^0(0, \cdot) = \text{rand}(-1, 1)$$



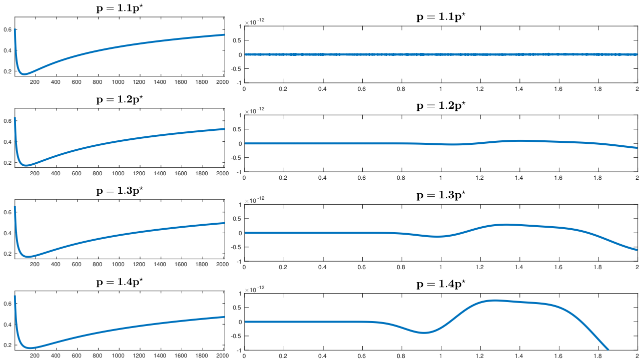
Convergence Factor

Error $u^{20}(0, t)$ w.r.t. t

Effect of p for a sum of frequencies

Physical and numerical data:

$$T = 2$$
$$\Delta x = 1.56 \cdot 10^{-4}, \Delta t = 1.6 \cdot 10^{-3}$$
$$u_2^0(0, \cdot) = \text{rand}(-1, 1)$$



Convergence Factor

Error $u^{20}(0, t)$ w.r.t. t

Conclusion and future works

- For large time interval the optimized parameter is sharp
- What about small time interval?
- What about 2D problem?