Universität Konstanz

Introduction to PDE-Constrained Optimization

Part II

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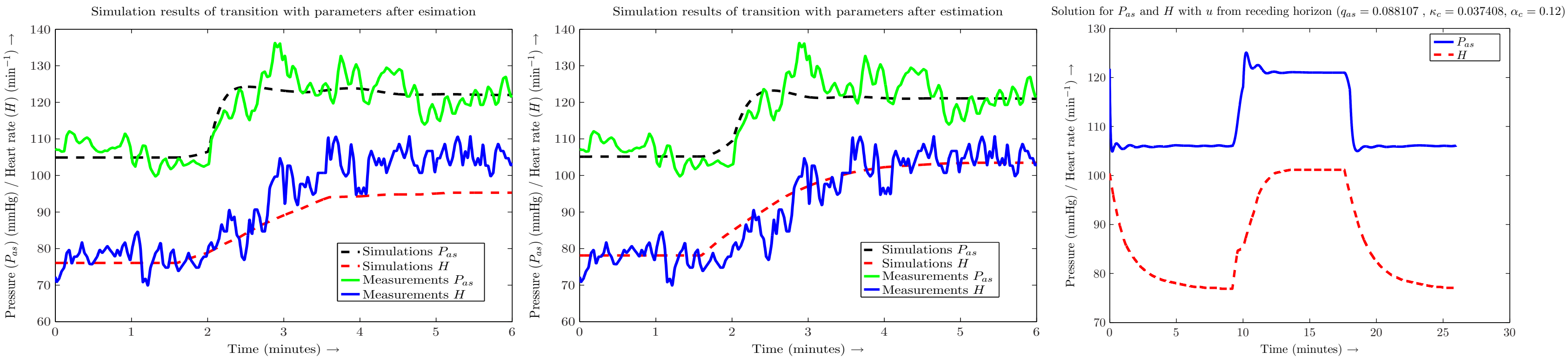
Outline of the Talk

1 Optimization of Linear Dynamical Systems – Continued



References:

[4] Gerdts: *Optimal Control of ODEs and DAEs*, De Gruyter, 2011

Linear Quadratic Problem

Consider:

$$\min_{(y,u)} J(y,u) = \frac{1}{2}\int_0^T y(t)^\top Q y(t) + u(t)^\top R u(t)\,\mathrm{d}t \quad \text{s.t.} \quad \begin{cases} \dot{y}(t) = Ay(t) + Bu(t) \text{ for } t \in (0,T] \\ y(0) = y_\circ \end{cases} \qquad (\mathbf{P}_2)$$

Equality Constraint: For $\mathcal{X} = \mathcal{Y} \times \mathcal{U}$ and $\mathcal{Z} = L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n$ let $e : \mathcal{X} \to \mathcal{Z}$ be given as

$$e(x) = \begin{pmatrix} e_1(x) \\ e_2(x) \end{pmatrix} = \begin{pmatrix} \dot{y} - Ay - Bu \\ y(0) - y_\circ \end{pmatrix} \in \mathcal{Z} \quad \text{for } x = (y,u) \in \mathcal{X}$$

Constrained Optimization Problem:

$$\min_x J(x) \quad \text{s.t.} \quad x \in \mathcal{X} \text{ satisfies } e(x) = 0 \text{ in } \mathcal{Z} = L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n \qquad (\mathbf{P}_2)$$

State Equation

Unique Optimal Solution: $\bar{x} = (\bar{y}, \bar{u}) \in \mathcal{X} = \mathcal{Y} \times \mathcal{U} = H^1(0,T;\mathbb{R}^n) \times L^2(0,T;\mathbb{R}^n)$

Lagrange Functional: For $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathcal{Z} = L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n$ we set

$$\mathscr{L}(\bar{x}, \bar{p}) = \frac{1}{2} \int_0^T \bar{y}(t)^\top Q \bar{y}(t) + \bar{u}(t)^\top R \bar{u}(t)\,\mathrm{d}t + \int_0^T \big(\dot{\bar{y}}(t) - A\bar{y}(t) - B\bar{u}(t)\big)^\top \bar{p}_1(t)\,\mathrm{d}t + \big(\bar{y}(0) - y_\circ\big)^\top \bar{p}_2$$

Optimality Conditions: There exists a unique $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathcal{Z}$ satisfying

$$\mathscr{L}_y(\bar{y}, \bar{u}, \bar{p}) \text{ in } \mathcal{Y}', \quad \mathscr{L}_u(\bar{y}, \bar{u}, \bar{p}) \text{ in } \mathcal{U}', \quad \mathscr{L}_p(\bar{y}, \bar{u}, \bar{p}) = 0 \text{ in } \mathcal{Z}'$$

State/Primal Equation: $\mathscr{L}_p(\bar{y}, \bar{u}, \bar{p}) = 0$ in $\mathcal{Z}' \Leftrightarrow \langle \mathscr{L}_p(\bar{y}, \bar{u}, \bar{p}), p \rangle_{\mathcal{Z}',\mathcal{Z}} = 0$ for all $p = (p_1, p_2) \in \mathcal{Z}$

$$0 = \langle \mathscr{L}_p(\bar{y}, \bar{u}, \bar{p}), p \rangle_{\mathcal{Z}',\mathcal{Z}} = \int_0^T \big(\dot{\bar{y}}(t) - A\bar{y}(t) - B\bar{u}(t)\big)^\top p_1(t)\,\mathrm{d}t + \big(\bar{y}(0) - y_\circ\big)^\top p_2$$

$$\Leftrightarrow \dot{\bar{y}} = A\bar{y} + B\bar{u} \text{ in } L^2(0,T;\mathbb{R}^n) \text{ and } \bar{y}(0) = y_\circ \text{ in } \mathbb{R}^n$$

Adjoint/Dual Equation

Lagrange Functional: For $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathcal{Z} = L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n$ we set

$$\mathscr{L}(\bar{x}, \bar{p}) = \frac{1}{2} \int_0^T \bar{y}(t)^\top Q \bar{y}(t) + \bar{u}(t)^\top R \bar{u}(t)\,\mathrm{d}t + \int_0^T \big(\dot{\bar{y}}(t) - A\bar{y}(t) - B\bar{u}(t)\big)^\top \bar{p}_1(t)\,\mathrm{d}t + \big(\bar{y}(0) - y_\circ\big)^\top \bar{p}_2$$

Adjoint Equation: $\mathscr{L}_y(\bar{y}, \bar{u}, \bar{p}) = 0$ in $\mathcal{Y}' \Leftrightarrow \langle \mathscr{L}_y(\bar{y}, \bar{u}, \bar{p}), y \rangle_{\mathcal{Y}',\mathcal{Y}} = 0$ for all $y \in \mathcal{Y} = H^1(0,T;\mathbb{R}^n)$

$$0 = \langle \mathscr{L}_y(\bar{y}, \bar{u}, \bar{p}), y \rangle_{\mathcal{Y}',\mathcal{Y}} = \int_0^T \bar{y}(t)^\top Q y(t)\,\mathrm{d}t + \int_0^T \big(\dot{y}(t) - Ay(t)\big)^\top \bar{p}_1(t)\,\mathrm{d}t + y(0)^\top \bar{p}_2$$

Variational arguments:

$\int_0^T \dot{y}(t)^\top \bar{p}_1(t)\,\mathrm{d}t = -\int_0^T \dot{\bar{p}}_1(t)^\top y(t)\,\mathrm{d}t$ for all $y \in C_0^\infty(0,T;\mathbb{R}^n) \hookrightarrow \mathcal{Y} \hookrightarrow L^2(0,T;\mathbb{R}^n)$

$\Rightarrow \quad 0 = \int_0^T \big(-\dot{\bar{p}}_1(t) - A^\top \bar{p}_1(t) + Q\bar{y}(t)\big)^\top y(t)\,\mathrm{d}t$ for all $y \in C_0^\infty(0,T;\mathbb{R}^n) \hookrightarrow L^2(0,T;\mathbb{R}^n)$

$\Rightarrow \quad 0 = -\dot{\bar{p}}_1 - A^\top \bar{p}_1 + Q\bar{y}$ in $L^2(0,T;\mathbb{R}^n)$

$\Rightarrow \quad 0 = \langle \mathscr{L}_y(\bar{y}, \bar{u}, \bar{p}), y \rangle_{\mathcal{Y}',\mathcal{Y}} = \big(y(t)^\top \bar{p}_1(t)\big)\Big|_{t=0}^{t=T} + y(0)^\top \bar{p}_2 \quad \Rightarrow \quad \bar{p}_1(T) = 0$ and $\bar{p}_2 = \bar{p}_1(0)$

Optimality Condition

Lagrange Functional: For $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathcal{Z} = L^2(0,T;\mathbb{R}^n) \times \mathbb{R}^n$ we set

$$\mathscr{L}(\bar{x},\bar{p}) = \frac{1}{2}\int_0^T \bar{y}(t)^\top Q\bar{y}(t) + \bar{u}(t)^\top R\bar{u}(t)\,\mathrm{d}t + \int_0^T \big(\dot{\bar{y}}(t) - A\bar{y}(t) - B\bar{u}(t)\big)^\top \bar{p}_1(t)\,\mathrm{d}t + \big(\bar{y}(0) - y_\circ\big)^\top \bar{p}_2$$

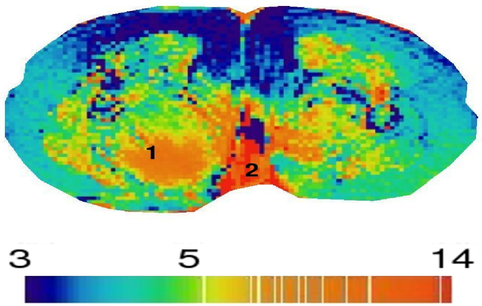
Optimality Condition: $\mathscr{L}_u(\bar{y},\bar{u},\bar{p}) = 0$ in $\mathcal{U}' \Leftrightarrow \langle \mathscr{L}_u(\bar{y},\bar{u},\bar{p}), u\rangle_{\mathcal{U}',\mathcal{U}} = 0$ for all $u \in \mathcal{U}$

$$\begin{aligned} 0 = \langle \mathscr{L}_u(\bar{y},\bar{u},\bar{p}), u\rangle_{\mathcal{U}',\mathcal{U}} &= \int_0^T \bar{u}(t)^\top Ru(t)\,\mathrm{d}t + \int_0^T \big(-Bu(t)\big)^\top \bar{p}_1(t)\,\mathrm{d}t \\ &= \int_0^T \big(R\bar{u}(t) - B^\top \bar{p}_1(t)\big)^\top u(t)\,\mathrm{d}t \quad \Rightarrow \quad \bar{u} = R^{-1}B^\top \bar{p}_1 \text{ in } \mathcal{U} = L^2(0,T;\mathbb{R}^m) \end{aligned}$$

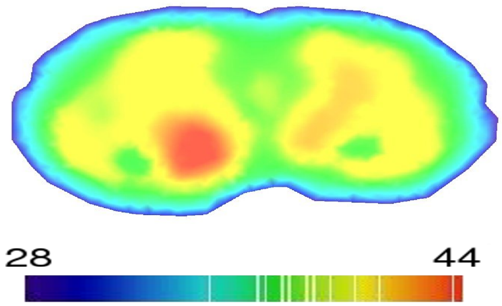
KKT System: coupled ODE system

$$\begin{aligned} \dot{\bar{y}}(t) &= A\bar{y}(t) + B\bar{u}(t) && \text{for } t \in (0,T], \quad \bar{y}(0) = y_\circ && \text{(state equation)} \\ \bar{u}(t) &= R^{-1}B^\top \bar{p}_1(t) && \text{for } t \in (0,T) && \text{(optimality condition)} \\ -\dot{\bar{p}}_1(t) &= A^\top \bar{p}_1(t) - Q\bar{y}(t) && \text{for } t \in [0,T), \quad \bar{p}_1(T) = 0, \quad \bar{p}_2 = \bar{p}_1(0) && \text{(adjoint equation)} \end{aligned}$$

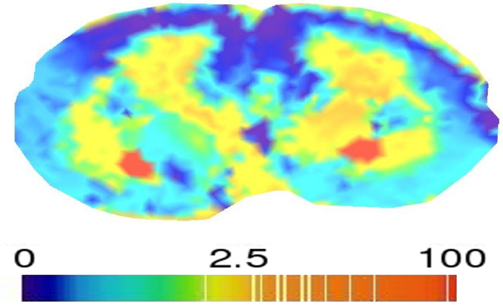
2 Optimization of Linear Elliptic PDEs



Measurements



identified temperature



optimal perfusion

References:

[5] Hinze/Pinnau/Ulbrich/Ulbrich: *Optimization with PDE Constraints*, Springer, 2009
[6] Lions: *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, 1971
[7] Tröltzsch: *Optimal Control pf Partial Differential Equations*, AMS, 2010

Infinite-Dimensional Optimization

Consider: $\Omega \subset \mathbb{R}^d$ bounded ($d \in \{1,2,3\}$), $y_\mathrm{d} \in L^2(\Omega)$, $\sigma > 0$, $V = H^1(\Omega)$, $\mathcal{U} = L^2(\Omega)$

$$\min_{(y,u)} J(y,u) = \frac{1}{2} \int_\Omega |y - y_\mathrm{d}|^2 + \sigma\, |u|^2 \,\mathrm{d}\mathbf{x} \quad \text{s.t.} \quad -\Delta y + y = u \text{ in } \Omega, \quad \frac{\partial y}{\partial \mathbf{n}} = 0 \text{ on } \Gamma = \partial\Omega \qquad (\mathbf{P}_3)$$

Setting: $\mathcal{X} = V \times \mathcal{U}$, cost functional $J : \mathcal{X} \to \mathbb{R}$, constraint operator $e : \mathcal{X} \to V'$ with

$$\langle e(x), \varphi \rangle_{V',V} = \int_\Omega \nabla y \cdot \nabla \varphi + y\varphi - u\varphi \,\mathrm{d}\mathbf{x} \quad \text{for } x = (y,u) \in \mathcal{X},\ \varphi \in V$$

i.e., $e(x) = 0$ in $V' \Leftrightarrow x = (y,u)$ satisfies $\int_\Omega \nabla y \cdot \nabla \varphi + y\varphi \,\mathrm{d}\mathbf{x} = \int_\Omega u\varphi \,\mathrm{d}\mathbf{x}$ for all $\varphi \in V$.

Optimization Problem with Equality Constraints:

$$\min J(x) \quad \text{s.t.} \quad x \in \mathcal{X} \text{ satisfies } e(x) = 0 \text{ in } V' \qquad (\mathbf{P}_3)$$

Unconstrained Reduced Problem

Neumann Problem: $V = H^1(\Omega)$, $\mathcal{U} = L^2(\Omega)$

$$\min_{(y,u)} J(y,u) = \frac{1}{2}\int_\Omega |y - y_\mathrm{d}|^2 + \sigma\,|u|^2\,\mathrm{d}\mathbf{x} \quad \text{s.t.} \quad \int_\Omega \nabla y \cdot \nabla \varphi + y\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega u\varphi\,\mathrm{d}\mathbf{x} \text{ for all } \varphi \in V \qquad (\mathbf{P}_3)$$

Control-to-State Mapping: For $u \in \mathcal{U}$ let $y(u) = \mathscr{S}u \in V$ be the unique solution to

$$\langle e(y(u),u),\varphi\rangle_{V',V} = \int_\Omega \nabla y(u)\cdot\nabla\varphi + y(u)\varphi - u\varphi\,\mathrm{d}\mathbf{x} \quad \text{for all } \varphi \in V$$

Then, $e(y(u),u) = 0$ in V' holds. Moreover, $\mathscr{S}$ is linear and bounded ($\|y(u)\|_V \le \|u\|_\mathcal{U}$)

Reduced Cost Functional: $\hat{J}(u) = J(y(u),u)$ for $u \in \mathcal{U}$

Reduced Problem:

$$\min_u \hat{J}(u) = \frac{1}{2}\int_\Omega |\mathscr{S}u - y_\mathrm{d}|^2 + \sigma\,|u|^2\,\mathrm{d}\mathbf{x} \qquad (\hat{\mathbf{P}}_3)$$

Existence of Optimal Controls

Constrained Optimization: $\min_{(y,u)} J(y,u) = \frac{1}{2}\int_\Omega |y - y_\mathrm{d}|^2 + \sigma\,|u|^2\,\mathrm{d}\mathbf{x}$ s.t. $e(y,u) = 0$ $\quad(\mathbf{P}_3)$

Unconstrained Reduced Problem: $\min_u \hat{J}(u) = \frac{1}{2}\int_\Omega |\mathscr{S}u - y_\mathrm{d}|^2 + \sigma\,|u|^2\,\mathrm{d}\mathbf{x}$ $\quad(\hat{\mathbf{P}}_3)$

Existence of a unique Optimal Control: $\mathcal{U} = L^2(\Omega)$

- $\{u^k\}_{k\in\mathbb{N}}$ minimizing sequence for $(\hat{\mathbf{P}}_3)$, i.e., $\lim_{k\to\infty}\hat{J}(u^k) = \inf\{\hat{J}(u)\,|\,u\in\mathcal{U}\} \geq 0$
- $\hat{J}$ radially unbounded, i.e., $\hat{J}(u)\to\infty$ for $\|u\|_\mathcal{U}\to\infty \Rightarrow \|u^k\|_\mathcal{U}$ bounded
- $\bar{u}$ solution to $(\hat{\mathbf{P}}_3)$, because $u^k \rightharpoonup \bar{u}$ in $\mathcal{U}$ and weakly lower semicontinuity of $\hat{J}$ imply

$$\infty > \inf\{\hat{J}(u)\,|\,u\in\mathcal{U}\} = \lim_{k\to\infty}\hat{J}(u^k) = \liminf_{k\to\infty}\hat{J}(u^k) \geq \hat{J}(\bar{u})$$

- $\hat{J}$ quadratic $\Rightarrow$ $(\hat{\mathbf{P}}_3)$ convex $\Rightarrow$ unique optimal solution $\bar{u}$
- $\bar{x} = (\bar{y},\bar{u})$ unique solution to $(\mathbf{P}_3)$ with $\bar{y} = \mathscr{S}\bar{u}$
- Utilized facts: linear, bounded control-to-state mapping $\mathscr{S}$, reflexivity of $\mathcal{U}$, $\sigma > 0$

First-Order Sufficient Optimality Conditions

Neumann Problem: $V = H^1(\Omega)$, $\mathcal{U} = L^2(\Omega)$, $\mathcal{X} = V \times \mathcal{U}$

$$\min_{(y,u)} J(y,u) = \frac{1}{2}\int_\Omega |y - y_\mathrm{d}|^2 + \sigma\,|u|^2\,\mathrm{d}\mathbf{x} \quad \text{s.t.} \quad \int_\Omega \nabla y \cdot \nabla \varphi + y\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega u\varphi\,\mathrm{d}\mathbf{x} \text{ for all } \varphi \in V \quad (\mathbf{P}_3)$$

Lagrange Functional for $(\mathbf{P}_3)$:

$$\mathscr{L}(y,u,p) = J(y,u) + \langle e(y,u), p\rangle_{V',V} = \int_\Omega \Big(\frac{1}{2}|y - y_\mathrm{d}|^2 + \frac{\sigma}{2}|u|^2 + \nabla y \cdot \nabla p + yp - up\Big)\,\mathrm{d}\mathbf{x}$$

Optimality System: $\bar{x} = (\bar{y}, \bar{u}) \in \mathcal{X}$ optimal solution. Then there exists $\bar{p} \in V$ satisfying

$$\mathscr{L}_y(\bar{y},\bar{u},\bar{p}) = 0 \text{ in } V', \quad \mathscr{L}_u(\bar{y},\bar{u},p) = 0 \text{ in } \mathcal{U}', \quad \mathscr{L}_p(\bar{y},\bar{u},\bar{p}) = 0 \text{ in } V'$$

since a constrained qualification condition holds ($e'(\bar{x}) : \mathcal{X} \to V'$ is surjective)

Note: derivatives are functionals!

State Equation

Lagrange Functional for $(\mathbf{P}_3)$: $\mathscr{L}(\bar{y},\bar{u},\bar{p}) = \int_\Omega \Big(\frac{1}{2}|\bar{y}-y_\mathrm{d}|^2 + \frac{\sigma}{2}|\bar{u}|^2 + \nabla\bar{y}\cdot\nabla\bar{p} + \bar{y}\bar{p} - \bar{u}\bar{p}\Big)\,\mathrm{d}\mathbf{x}$

State/Primal Equation: $\mathscr{L}_p(\bar{y},\bar{u},\bar{p}) = 0$ in V' $\Leftrightarrow$ $\langle \mathscr{L}_p(\bar{y},\bar{u},\bar{p}), p\rangle_{V',V} = 0$ for all $p \in V = H^1(\Omega)$

$$0 = \langle \mathscr{L}_p(\bar{y},\bar{u},\bar{p}), p\rangle_{V',V} = \int_\Omega \big(\nabla\bar{y}\cdot\nabla p + \bar{y}p - \bar{u}p\big)\,\mathrm{d}\mathbf{x} \quad \text{for all } p\in V$$

i.e., $\bar{y} = y(\bar{u}) \in V$ is the weak solution to $-\Delta\bar{y} + \bar{y} = \bar{u}$ in Ω, $\frac{\partial\bar{y}}{\partial\mathbf{n}} = 0$ on Γ (state equation)

Stationarity: From

$$\int_\Omega \nabla y(u)\cdot\nabla\varphi + y(u)\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega u\varphi\,\mathrm{d}\mathbf{x} \quad \text{for all } \varphi\in V'$$

we infer $\mathscr{L}_p(y(u),u,p) = 0$ in V' for any $(u,p)\in\mathfrak{U}\times V$, because

$$\langle \mathscr{L}_p(y(u),u,p), \varphi\rangle_{V',V} = \int_\Omega \big(\nabla y(u)\cdot\nabla\varphi + y(u)\varphi - u\varphi\big)\,\mathrm{d}\mathbf{x} = 0 \quad \text{for all } \varphi\in V$$

Adjoint Equation

Lagrange Functional for $(\mathbf{P}_3)$: $\mathscr{L}(\bar{y},\bar{u},\bar{p}) = \int_\Omega \Big(\frac{1}{2}|\bar{y}-y_\mathsf{d}|^2 + \frac{\sigma}{2}|\bar{u}|^2 + \nabla\bar{y}\cdot\nabla\bar{p} + \bar{y}\bar{p} - \bar{u}\bar{p}\Big)\,\mathrm{d}\mathbf{x}$

Adjoint/Dual Equation: $\mathscr{L}_y(\bar{y},\bar{u},\bar{p}) = 0$ in V' $\Leftrightarrow$ $\langle \mathscr{L}_y(\bar{y},\bar{u},\bar{p}), y\rangle_{V',V} = 0$ for all $y \in V$

$$\begin{aligned} 0 = \langle \mathscr{L}_y(\bar{y},\bar{u},\bar{p}), y\rangle_{V',V} &= \int_\Omega \Big((\bar{y}-y_\mathsf{d})y + \nabla y\cdot\nabla\bar{p} + y\bar{p}\Big)\,\mathrm{d}\mathbf{x} \\ &= \int_\Omega \big(\nabla\bar{p}\cdot\nabla y + \bar{p}y - (y_\mathsf{d}-\bar{y})y\big)\,\mathrm{d}\mathbf{x} \quad \text{for all } y \in V \end{aligned}$$

i.e., $\bar{p} \in V$ is the weak solution to $-\Delta\bar{p} + \bar{p} = y_\mathsf{d} - \bar{y}$ in Ω, $\frac{\partial\bar{p}}{\partial\mathbf{n}} = 0$ on Γ (adjoint equation)

Stationarity: From

$$\int_\Omega \nabla p(u)\cdot\nabla\varphi + p(u)\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega (y_\mathsf{d} - y(u))\varphi\,\mathrm{d}\mathbf{x} \quad \text{for all } \varphi \in V'$$

we infer $\mathscr{L}_y(y(u),u,p(u)) = 0$ in V' for any $u \in \mathfrak{U}$

Optimality Condition

Lagrange Functional for $(\mathbf{P}_3)$: $\mathscr{L}(\bar{y},\bar{u},\bar{p}) = \int_\Omega \Big(\frac{1}{2}|\bar{y}-y_\mathrm{d}|^2 + \frac{\sigma}{2}|\bar{u}|^2 + \nabla\bar{y}\cdot\nabla\bar{p} + \bar{y}\bar{p} - \bar{u}\bar{p}\Big)\,\mathrm{d}\mathbf{x}$

Optimality: $\mathscr{L}_u(\bar{y},\bar{u},\bar{p}) = 0$ in $\mathfrak{U}'$ $\Leftrightarrow$ $\langle \mathscr{L}_u(\bar{y},\bar{u},\bar{p}),u\rangle_{\mathfrak{U}',\mathfrak{U}} = 0$ for all $u \in \mathfrak{U}$

$$0 = \langle \mathscr{L}_u(\bar{y},\bar{u},\bar{p}),u\rangle_{\mathfrak{U}',\mathfrak{U}} = \int_\Omega \big(\sigma\bar{u}u - u\bar{p}\big)\,\mathrm{d}\mathbf{x} = \int_\Omega \big(\sigma\bar{u} - \bar{p}\big)u\,\mathrm{d}\mathbf{x} = \langle \sigma\bar{u}-\bar{p},v\rangle_{\mathfrak{U}} \quad \text{for all } v \in \mathfrak{U}$$

$\Rightarrow$ $\bar{u} = \bar{p}/\sigma$ in $\mathfrak{U}$; moreover, $\bar{u} \in V$ follows from $\bar{p} \in V$

Gradient: Riesz representant $\nabla_u\mathscr{L}(\bar{y},\bar{u},\bar{p}) \in \mathfrak{U}$ of $\mathscr{L}_u(\bar{y},\bar{u},\bar{p}) \in \mathfrak{U}'$ is given as

$$\nabla_u\mathscr{L}(\bar{y},\bar{u},\bar{p}) = \sigma\bar{u} - \bar{p} \in \mathfrak{U}$$

Relation to the Reduced Cost Functional

Reduced Cost: For a chosen $u \in \mathcal{U}$ we infer from $\hat{J}(u) = J(y(u),u)$ and $e(y(u),u) = 0$ in V'

$$\hat{J}(u) = J(y(u),u) = J(y(u),u) + \underbrace{\langle e(y(u),u),p\rangle_{V',V}}_{=0} = \mathscr{L}(y(u),u,p) \quad \text{for any } p \in V$$

Chain Rule: For every direction $v \in \mathcal{U}$ we have

$$\langle \hat{J}'(u),v\rangle_{\mathcal{U}',\mathcal{U}} = \langle \mathscr{L}_y(y(u),u,p),y'(u)v\rangle_{V',V} + \langle \mathscr{L}_u(y(u),u,p),v\rangle_{\mathcal{U}',\mathcal{U}} \quad \text{for any } p \in V \tag{1}$$

Recall: $\mathscr{L}_y(y(u),u,p(u)) = 0$ in V'

Gradient of the Reduced Cost: Choosing $p = p(u) \in V$ in (1) imply for all $v \in \mathcal{U}$

$$\langle \hat{J}'(u),v\rangle_{\mathcal{U}',\mathcal{U}} = \langle \mathscr{L}_u(y(u),u,p(u)),v\rangle_{\mathcal{U}',\mathcal{U}} = \int_\Omega \big(\sigma u - p(u)\big) v \,\mathrm{d}\mathbf{x} = \langle \sigma u - p(u),v\rangle_{\mathcal{U}} = \langle \nabla \hat{J}(u),v\rangle_{\mathcal{U}}$$

First-Order Optimality System / KKT-System

Optimality system:

$$-\Delta\bar{y}+\bar{y}=\bar{u} \quad \text{in } \Omega, \qquad \frac{\partial\bar{y}}{\partial\mathbf{n}}=0 \quad \text{on } \Gamma \qquad \text{(state equation)}$$

$$\sigma\bar{u}-\bar{p}=0 \quad \text{in } \Omega, \qquad \text{(optimality conditon)}$$

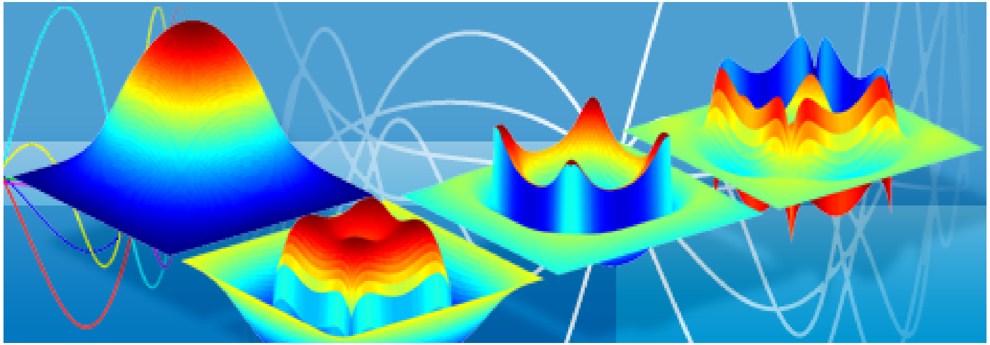
$$-\Delta\bar{p}+\bar{p}=y_{\mathrm{d}}-\bar{y} \quad \text{in } \Omega, \qquad \frac{\partial\bar{p}}{\partial\mathbf{n}}=0 \quad \text{on } \Gamma \qquad \text{(adjoint equation)}$$

Reduction:

$$-\Delta\bar{y}+\bar{y}+\frac{\bar{p}}{\sigma}=0 \quad \text{in } \Omega, \qquad \frac{\partial\bar{y}}{\partial\mathbf{n}}=0 \quad \text{on } \Gamma \qquad \text{(state equation)}$$

$$-\Delta\bar{p}+\bar{p}+\bar{y}=y_{\mathrm{d}} \quad \text{in } \Omega, \qquad \frac{\partial\bar{p}}{\partial\mathbf{n}}=0 \quad \text{on } \Gamma \qquad \text{(adjoint equation)}$$

3 Optimization of the Heat equation



References:

[5] Hinze/Pinnau/Ulbrich/Ulbrich: *Optimization with PDE Constraints*, Springer, 2009
[6] Lions: *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, 1971
[7] Tröltzsch: *Optimal Control pf Partial Differential Equations*, AMS, 2010

Infinite-Dimensional Optimization

Consider:

$$\min_{(y,u)} J(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y-y_\mathsf{d}|^2 + \sigma\,|u|^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \quad \text{s.t.} \quad \begin{cases} y_t - \Delta y = u & \text{in } Q \\ \frac{\partial y}{\partial \mathbf{n}} = 0 & \text{on } \Sigma \\ y(0) = y_\circ & \text{in } \Omega \end{cases} \qquad (\mathbf{P}_4)$$

Equality constraints: $e = (e_1, e_2) : \mathcal{X} \to \mathcal{Z}' \simeq L^2(0,T;V') \times H$, where for $x = (y,u) \in \mathcal{Y}$

$$\langle e_1(x), \varphi \rangle = \int_0^T \langle y_t, \varphi \rangle_{V',V} + \int_\Omega \nabla y \cdot \nabla \varphi - u\varphi \,\mathrm{d}\mathbf{x} \text{ for } \varphi \in L^2(0,T;V), \quad e_2(y,u) = y(0) - y_\circ \text{ in } H$$

Optimization Problem with Equality Constraints:

$$\min J(x) \quad \text{s.t.} \quad x \in \mathcal{X} \text{ satisfies } e(x) = 0 \text{ in } \mathcal{Z}' \qquad (\mathbf{P}_4)$$

Problem Setting: $T > 0$, $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$, $H = L^2(\Omega)$, $V = H^1(\Omega)$, $W(0,T) = L^2(0,T;V) \cap H^1(0,T;V')$, $y_\circ \in H$, $y_\mathsf{d} \in \mathcal{H}$

Optimization Setting: $\mathcal{Y} = W(0,T)$, $\mathcal{U} = L^2(0,T;H)$, $\mathcal{X} = \mathcal{Y} \times \mathcal{U}$, $\mathcal{Z} = L^2(0,T;V) \times H$, $\mathcal{Z}' \simeq L^2(0,T;V') \times H$, $J : \mathcal{X} \to \mathbb{R}$

Unconstrained Reduced Problem

Optimization Problem: $\mathcal{Z}' \simeq L^2(0,T;V') \times H$

$$\min_{(y,u)} J(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y - y_\mathrm{d}|^2 + \sigma\,|u|^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \quad \text{s.t.} \quad e(y(u),u) = 0 \text{ in } \mathcal{Z}' \qquad (\mathbf{P}_4)$$

Control-to-State Mapping: For $u \in \mathcal{U} = L^2(0,T;H)$ let $y(u) = \mathcal{S}u \in \mathcal{Y}$ uniquely solve

$$\langle e(y(u),u),\varphi\rangle = 0 \text{ for all } \varphi \in L^2(0,T;V) \quad \Leftrightarrow \quad e(y(u),u) = \begin{pmatrix} e_1(y(u),u) \\ e_2(y(u),u) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \mathcal{Z}'$$

Moreover, $\mathcal{S}$ is affin-linear and bounded ($\|y(u)\|_\mathcal{Y} \le C(\|y_\circ\|_H + \|u\|_\mathcal{U})$)

Reduced Cost Functional: $\hat{J}(u) = J(y(u),u)$ for $u \in \mathcal{U}$

Reduced Problem:

$$\min_u \hat{J}(u) = \frac{1}{2}\int_0^T \int_\Omega |\mathcal{S}u - y_\mathrm{d}|^2 + \sigma\,|u|^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \qquad (\hat{\mathbf{P}}_4)$$

Existence of Optimal Controls

Constrained Optimization: $\min_{(y,u)} J(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y-y_\mathrm{d}|^2 + \sigma\,|u|^2\,\mathrm{d}\mathbf{x}\mathrm{d}t$ s.t. $e(y,u)=0$ $\quad(\mathbf{P}_4)$

Equality constraints: $e=(e_1,e_2):\mathcal{X}\to\mathcal{Z}'\simeq L^2(0,T;V')\times H$, where for $x=(y,u)\in\mathcal{X}$

$$\langle e_1(x),\varphi\rangle = \int_0^T \langle y_t,\varphi\rangle_{V',V} + \int_\Omega \nabla y\cdot\nabla\varphi - u\varphi\,\mathrm{d}\mathbf{x} \text{ for } \varphi\in L^2(0,T;V), \quad e_2(y,u)=y(0)-y_\circ \text{ in } H$$

Unconstrained Reduced Problem: $\min_u \hat{J}(u) = \frac{1}{2}\int_0^T \int_\Omega |\mathcal{S}u-y_\mathrm{d}|^2 + \sigma\,|u|^2\,\mathrm{d}\mathbf{x}\mathrm{d}t$ $\quad(\hat{\mathbf{P}}_4)$

Existence of a unique Optimal Control:

- $\mathcal{U}=L^2(0,T;H)$ reflexive, $(\hat{\mathbf{P}}_4)$ strictly convex, $\mathcal{S}$ affin-linear and bounded
- same arguments as in the elliptic case leads the existence of a unique solution $\bar{u}$
- $\bar{x}=(\bar{y},\bar{u})$ unique solution to $(\mathbf{P}_1)$ with $\bar{y}=\mathcal{S}\bar{u}$

First-Order Sufficient Optimality Conditions

Constrained Optimization: $\min_{(y,u)} J(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y-y_\mathrm{d}|^2 + \sigma\,|u|^2\,\mathrm{d}\mathbf{x}\mathrm{d}t$ s.t. $e(y,u)=0$ $\quad(\mathbf{P}_4)$

Lagrange Functional for $(\mathbf{P}_4)$:

$$
\begin{aligned}
\mathscr{L}(\bar y,\bar u,\bar p) &= J(\bar y,\bar u) + \langle e(\bar y,\bar u),\bar p\rangle = J(\bar y,\bar u) + \langle e_1(\bar y,\bar u),\bar p_1\rangle + \langle e_2(\bar y,\bar u),\bar p_2\rangle \\
&= \int_0^T \left(\langle \bar y_t,\bar p_1\rangle_{V',V} + \int_\Omega \left(\frac{1}{2}\,|\bar y - y_\mathrm{d}|^2 + \frac{\sigma}{2}\,|\bar u|^2 + \nabla \bar y\cdot\nabla\bar p_1 - \bar u\bar p_1\right)\mathrm{d}\mathbf{x}\right)\mathrm{d}t + \int_\Omega (\bar y(0)-y_\circ)\bar p_2\,\mathrm{d}\mathbf{x}
\end{aligned}
$$

for $\bar x = (\bar y,\bar u)\in\mathcal{X} = W(0,T)\times L^2(0,T;H)$, $\bar p = (\bar p_1,\bar p_2)\in\mathcal{Z} = L^2(0,T;V)\times H$ with $H = L^2(\Omega)$

Optimality System: $\bar x = (\bar y,\bar u)\in\mathcal{X}$ optimal solution. Then there exists $\bar p = (\bar p_1,\bar p_2)$ satisfying

$$
\mathscr{L}_y(\bar y,\bar u,\bar p) = 0 \text{ in } \mathcal{Y}', \quad \mathscr{L}_u(\bar y,\bar u,\bar p) = 0 \text{ in } \mathcal{U}', \quad \mathscr{L}_p(\bar y,\bar u,\bar p) = 0 \text{ in } \mathcal{Z}'
$$

State Equation

Lagrange Functional for $(\mathbf{P}_4)$:

$$
\begin{aligned}
\mathscr{L}(\bar y,\bar u,\bar p) &= J(\bar y,\bar u) + \langle e(\bar y,\bar u),\bar p\rangle \\
&= \int_0^T \left(\langle \bar y_t,\bar p_1\rangle_{V',V} + \int_\Omega \left(\frac{1}{2}|\bar y - y_\mathrm{d}|^2 + \frac{\sigma}{2}|\bar u|^2 + \nabla\bar y\cdot\nabla\bar p_1 - \bar u\bar p_1\right)\mathrm{d}\mathbf{x}\right)\mathrm{d}t + \int_\Omega (\bar y(0) - y_\circ)\bar p_2\,\mathrm{d}\mathbf{x}
\end{aligned}
$$

for $\bar x = (\bar y,\bar u) \in \mathcal{X} = \mathcal{Y}\times\mathcal{U}$, $\bar p = (\bar p_1,\bar p_2) \in \mathcal{Z} = L^2(0,T;V)\times H$

State: $\mathscr{L}_p(\bar y,\bar u,\bar p) = 0$ in $\mathcal{Z}'$ $\Leftrightarrow$ $\langle \mathscr{L}_p(\bar y,\bar u,\bar p),p\rangle = 0$ for all $p = (p_1,p_2)\in\mathcal{Z}$

$$
0 = \langle \mathscr{L}_p(\bar y,\bar u,\bar p),p\rangle = \int_0^T \left(\langle \bar y_t,p_1\rangle_{V',V} + \int_\Omega \left(\nabla\bar y\cdot\nabla p_1 - \bar u p_1\right)\mathrm{d}\mathbf{x}\right)\mathrm{d}t + \int_\Omega (\bar y(0) - y_\circ)p_2\,\mathrm{d}\mathbf{x}
$$

$\Rightarrow$ $\bar y = y(\bar u) \in V$ is the weak solution to $\bar y_t - \Delta\bar y = \bar u$ in Q, $\frac{\partial\bar y}{\partial\mathbf{n}} = 0$ on Σ, $\bar y(0) = y_\circ$ in Ω

Adjoint Equation

Lagrange Functional for $(\mathbf{P}_4)$:

$$\begin{aligned}\mathscr{L}(\bar{y},\bar{u},\bar{p}) &= J(\bar{y},\bar{u}) + \langle e(\bar{y},\bar{u}),\bar{p}\rangle \\ &= \int_0^T \left(\langle \bar{y}_t,\bar{p}_1\rangle_{V',V} + \int_\Omega \left(\frac{1}{2}\,|\bar{y}-y_\mathrm{d}|^2 + \frac{\sigma}{2}\,|\bar{u}|^2 + \nabla\bar{y}\cdot\nabla\bar{p}_1 - \bar{u}\bar{p}_1\right)\mathrm{d}\mathbf{x}\right)\mathrm{d}t + \int_\Omega (\bar{y}(0)-y_\circ)\bar{p}_2\,\mathrm{d}\mathbf{x}\end{aligned}$$

for $\bar{x} = (\bar{y},\bar{u}) \in \mathcal{X} = \mathcal{Y}\times\mathcal{U}$, $\bar{p} = (\bar{p}_1,\bar{p}_2) \in \mathcal{Z} = L^2(0,T;V)\times H$

Adjoint/Dual Equation: $\mathscr{L}_y(\bar{y},\bar{u},\bar{p}) = 0$ in $\mathcal{Y}' \Leftrightarrow \langle \mathscr{L}_y(\bar{y},\bar{u},\bar{p}),y\rangle = 0$ for all $y \in \mathcal{Y} = W(0,T)$

$$0 = \langle \mathscr{L}_y(\bar{y},\bar{u},\bar{p}),y\rangle = \int_0^T \left(\langle y_t,\bar{p}_1\rangle_{V',V} + \int_\Omega \left((\bar{y}-y_\mathrm{d})y + \nabla y\cdot\nabla\bar{p}_1\right)\mathrm{d}\mathbf{x}\right)\mathrm{d}t + \int_\Omega y(0)\bar{p}_2\,\mathrm{d}\mathbf{x}$$

$$0 = \langle \mathscr{L}_y(\bar{y}, \bar{u}, \bar{p}), y \rangle = \int_0^T \Big(\langle y_t, \bar{p}_1 \rangle_{V',V} + \int_\Omega \big((\bar{y} - y_\mathrm{d}) y + \nabla y \cdot \nabla \bar{p}_1 \big) \,\mathrm{d}\mathbf{x} \Big) \mathrm{d}t + \int_\Omega y(0) \bar{p}_2 \,\mathrm{d}\mathbf{x} \tag{2a}$$

Variational techniques:

- Choose $y = \chi(t)\varphi \in \mathscr{Y}$ ($\chi \in C_0^\infty(0,T)$, $\varphi \in H_0^1(\Omega)$), use integration by parts and $H \hookrightarrow V'$

$$\int_0^T \int_\Omega \big((\bar{y} - y_\mathrm{d}) y + \nabla y \cdot \nabla \bar{p}_1 \big) \,\mathrm{d}\mathbf{x} \Big) \mathrm{d}t + \int_\Omega y(0) \bar{p}_2 \,\mathrm{d}\mathbf{x} = \int_0^T \langle -\Delta \bar{p}_1 - (y_\mathrm{d} - \bar{y}), y \rangle_{V',V} \,\mathrm{d}t \tag{2b}$$

- Let $\bar{p}_{1,t}$ denote the distributional derivative of $\bar{p}_1$ satisfying

$$\int_0^T \langle \Delta \bar{p}_1 + (y_\mathrm{d} - \bar{y}), \varphi \rangle_{V',V} \,\mathrm{d}t \overset{(2)}{=} \int_0^T \langle y_t, \bar{p}_1 \rangle_{V',V} \,\mathrm{d}t = -\int_0^T \langle \bar{p}_{1,t}, y \rangle_{V',V} \,\mathrm{d}t$$

$\Rightarrow -\bar{p}_{1,t} = \Delta \bar{p}_1 + (y_\mathrm{d} - \bar{y}) \in L^2(0,T;V')$ which implies $\bar{p}_1 \in W(0,T) \hookrightarrow \mathscr{Y}$

Boundary terms: $-\bar{p}_{1,t} = \Delta\bar{p}_1 + (y_\mathrm{d} - \bar{y}) \in L^2(0,T;V')$

$$
\begin{aligned}
0 = \langle \mathscr{L}_y(\bar{y},\bar{u},\bar{p}), y\rangle &= \int_0^T \Big(\langle y_t, \bar{p}_1\rangle_{V',V} + \int_\Omega \big((\bar{y}-y_\mathrm{d})y + \nabla y\cdot\nabla\bar{p}_1\big)\,\mathrm{d}\mathbf{x}\Big)\mathrm{d}t + \int_\Omega y(0)\bar{p}_2\,\mathrm{d}\mathbf{x} \\
&= \langle y(T),\bar{p}_1(T)\rangle_H - \langle y(0),\bar{p}_1(0)\rangle_H + \int_0^T \langle -\bar{p}_{1,t}, y\rangle_{V',V}\,\mathrm{d}t \\
&\quad + \int_0^T \Big(\langle \Delta\bar{p}_1 - (y_\mathrm{d}-\bar{y})\rangle_{V',V} + \int_\Gamma \frac{\partial\bar{p}_1}{\partial\mathbf{n}} y\,\mathrm{d}\mathbf{s}\Big)\mathrm{d}t + \int_\Omega y(0)\bar{p}_2\,\mathrm{d}\mathbf{x} \\
&= \langle y(T),\bar{p}_1(T)\rangle_H - \langle y(0),\bar{p}_1(0)\rangle_H + \int_0^T\int_\Gamma \frac{\partial\bar{p}_1}{\partial\mathbf{n}} y\,\mathrm{d}\mathbf{s}\mathrm{d}t + \int_\Omega y(0)\bar{p}_2\,\mathrm{d}\mathbf{x} \quad \text{for all } y\in\mathscr{Y}
\end{aligned}
$$

Variational techniques:

- Choose $y\in\mathscr{Y}$ with $y(0,\cdot) = y(T,\cdot) = 0 \Rightarrow \frac{\partial\bar{p}_1}{\partial\mathbf{n}} = 0$ (boundary condition for $\bar{p}_1$
- Choose $y\in\mathscr{Y}$ with $y(0,\cdot) = 0 \Rightarrow \bar{p}_1(T) = 0$ (terminal condition for $\bar{p}_1$
- Choose $y\in\mathscr{Y}$ with $y(T,\cdot) = 0 \Rightarrow \bar{p}_2 = \bar{p}_1(0)$

Recall: $-\bar{p}_{1,t} = \Delta\bar{p}_1 + (y_\mathrm{d}-\bar{y})$ and $\langle y(T),\bar{p}_1(T)\rangle_H - \langle y(0),\bar{p}_1(0)\rangle_H = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle y,\bar{p}_1\rangle_H\,\mathrm{d}t = \int_0^T \big(\langle y_t,\bar{p}_1\rangle_{V',V} + \langle\bar{p}_{1,t},y\rangle_{V',V}\big)\,\mathrm{d}t$

Optimality Condition

Lagrange Functional for $(\mathbf{P}_4)$: $\bar{x} = (\bar{y}, \bar{u}) \in \mathcal{X} = \mathcal{Y} \times \mathcal{U}$, $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathcal{Z} = L^2(0,T;V) \times H$

$$\begin{aligned}\mathscr{L}(\bar{y},\bar{u},\bar{p}) &= J(\bar{y},\bar{u}) + \langle e(\bar{y},\bar{u}),\bar{p}\rangle \\ &= \int_0^T \left(\langle \bar{y}_t, \bar{p}_1 \rangle_{V',V} + \int_\Omega \left(\frac{1}{2}|\bar{y}-y_\mathrm{d}|^2 + \frac{\sigma}{2}|\bar{u}|^2 + \nabla\bar{y}\cdot\nabla\bar{p}_1 - \bar{u}\bar{p}_1 \right)\mathrm{d}\mathbf{x} \right)\mathrm{d}t + \int_\Omega (\bar{y}(0)-y_\circ)\bar{p}_2\,\mathrm{d}\mathbf{x}\end{aligned}$$

Optimality Condition: $\mathscr{L}_u(\bar{y},\bar{u},\bar{p}) = 0$ in $\mathcal{U}' \Leftrightarrow \langle \mathscr{L}_u(\bar{y},\bar{u},\bar{p}),u\rangle = 0$ for all $u \in \mathcal{U} = L^2(0,T;H)$

$$0 = \langle \mathscr{L}_u(\bar{y},\bar{u},\bar{p}),u\rangle = \int_0^T \int_\Omega (\sigma\bar{u}-\bar{p}_1)\,u\,\mathrm{d}\mathbf{x}\mathrm{d}t \quad \Leftrightarrow \quad \bar{u} = \frac{\bar{p}_1}{\sigma} \text{ in } \mathcal{U}$$

Optimality System:

$$\begin{aligned} \bar{y}_t - \Delta\bar{y} &= \bar{p}_1/\sigma \quad \text{in } Q, & \frac{\partial\bar{y}}{\partial\mathbf{n}} &= 0 \quad \text{on } \Sigma, & \bar{y}(0) &= y_\circ \quad \text{in } \Omega \quad \text{(state equation)} \\ -\bar{p}_{1,t} - \Delta\bar{p}_1 &= y_\mathrm{d} - \bar{y} \quad \text{in } Q, & \frac{\partial\bar{p}_1}{\partial\mathbf{n}} &= 0 \quad \text{on } \Sigma, & \bar{p}_1(T) &= 0 \quad \text{in } \Omega \quad \text{(adjoint equation)} \end{aligned}$$