An introduction to domain decomposition methods

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Outline

Introduction

- Connection with the Block-Jacobi algorithm
- Discrete setting
- Convergence analysis
- Algebraic Schwarz Methods

Schwarz methods using Freefem++

- Schwarz algorithms as solvers
- Schwarz algorithms as preconditioners
- Schwarz preconditioners using FreeFEM++

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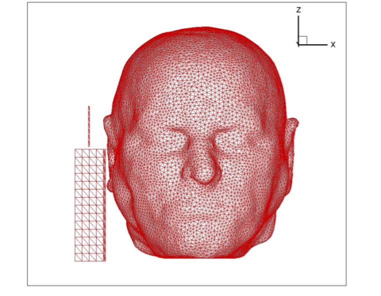
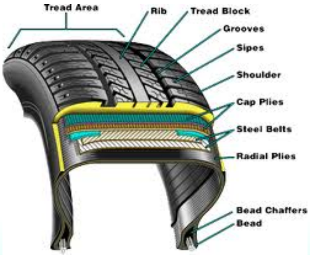


Time-harmonic Maxwell's equations

$$
\begin{aligned}
-i\omega\varepsilon \mathbf{E} + \nabla \times \mathbf{H} - \sigma \mathbf{E} &= \mathbf{J}, \\
i\omega\mu \mathbf{H} + \nabla \times \mathbf{E} &= \mathbf{0}.
\end{aligned}
$$



Linear elasticity

$$-\nabla \cdot (\sigma(\mathbf{u})) = \mathbf{f},$$

$$
\begin{aligned}
\sigma_{ij}(\mathbf{u}) &= 2\mu\varepsilon_{ij}(\mathbf{u}) + \lambda\delta_{ij}\nabla \cdot (\mathbf{u}), \\
\varepsilon_{ij}(\mathbf{u}) &= \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \\
\mu &= \tfrac{E}{2(1+\nu)}, \ \lambda = \tfrac{E\nu}{(1+\nu)(1-2\nu)}.
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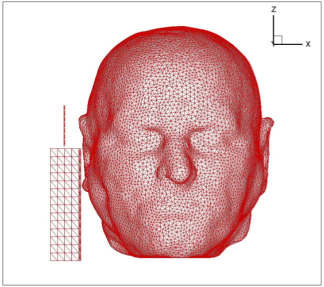

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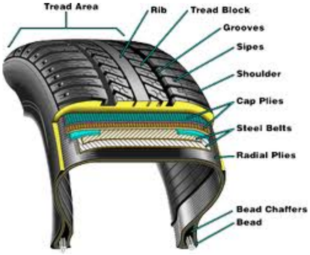

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$$\sigma_{ij}(\mathbf{u}) = 2\mu\varepsilon_{ij}(\mathbf{u}) + \lambda\delta_{ij}\nabla \cdot (\mathbf{u}),$$
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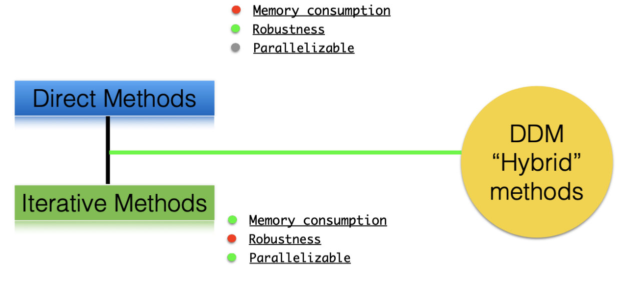
$$\mu = \frac{E}{2(1+\nu)}, \ \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

After discretisation $\Rightarrow$ large linear system $A\mathbf{u} = \mathbf{b}$. Matrix A inherits the properties of the underlying PDE (symmetric, positive definite, indefinite, etc...) A is in general sparse, large and ill conditioned.

Direct Solvers
MUMPS (J.Y. L'Excellent), SuperLU (Demmel, ...), PastiX, UMFPACK, PARDISO (O. Schenk):

Iterative Methods
- Fixed point iteration: Jacobi, Gauss-Seidel, SSOR
- Krylov type methods: Conjuguate Gradient (Stiefel-Hestenes), GMRES (Y. Saad), QMR (R. Freund), MinRes, BiCGSTAB (van der Vorst):



Motivation: pro and cons of direct solvers

Complexity of the Gauss factorisation (sparse linear algebra)

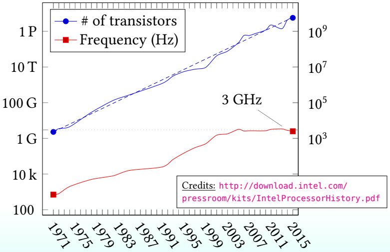
Gauss	$d=1$	$d=2$	$d=3$
dense matrix	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$
using band structure	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^{7/3})$
using sparsity	$\mathcal{O}(n)$	$\mathcal{O}(n^{3/2})$	$\mathcal{O}(n^2)$

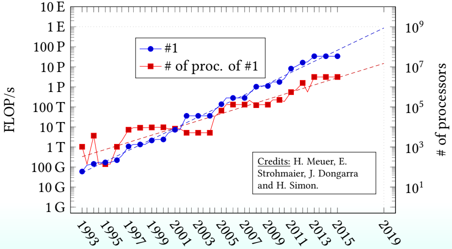
As of today: $n = 10^6$ (2d) and $n = 10^5$ (3d) is ok.

Different sparse direct solvers

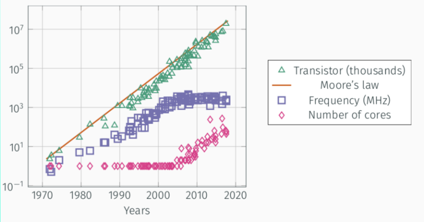
- PARDISO (`http://www.pardiso-project.org`)
- SUPERLU (`https://portal.nersc.gov/project/sparse/superlu/`)
- MUMPS (`http://graal.ens-lyon.fr/MUMPS/`)
- UMFPACK, in SuiteSparse (`https://people.engr.tamu.edu/davis/suitesparse.html`)

Moore's law: the number of transistors doubles every two years





Dennard scaling: the power density stays constant



Need & Opportunities for massively parallel computing

Parallel computers are more and more available to scientists and engineers

- Apple, Linux, Windows laptops, 4/8 cores
- Desktop Computers, 64/128 cores
- Laboratory cluster, 300 cores
- University cluster, $\sim$ 10 000 cores
- Cloud computing on Data Mining machines
- Supercomputers via academic or commercial providers $>$ 100k cores

All fields of computer science are impacted.

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Hardware news

ARM (Advanced Risc Machines) and Soc

- A64FX (Fujitsu), M1 (Apple), Graviton (Amazon)
- Soc: System on a Chip, HMB: High Bandwidth Memory
- Fugaku HPC exclusively based on A64FX ranks on TOP500 1st for dense linear algebra, 1st for sparse linear algebra, 1st for AI and 1st on Graph500)
- A64FX $\gg$ GPU even for AI and ML thanks to its Scalable Vector Extension SVE
- Programmable with standard Fortran/C/C++, OpenMP and MPI

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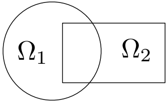
Software news

Software

- Some New High level languages support natively some degree of parallelism: Julia, Rust and wrap MPI library with a simpler interface
- Parallel linear algebra package: PETSc (USA), Scalapack (USA), HPDDM (France)
- Free Finite element/volume packages allow for parallel computing: OpenFoam (UK), FreeFem (France), Firedrake (UK), Dune (Germany),

The original Schwarz Method (H.A. Schwarz, 1870)



$$-\Delta(u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Schwarz Method : $(u_1^n, u_2^n) \to (u_1^{n+1}, u_2^{n+1})$ with

$$\begin{array}{ll} -\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1 & \qquad -\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2 \\ u_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega & \qquad u_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega \\ u_1^{n+1} = u_2^{n} \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}. & \qquad u_2^{n+1} = u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}. \end{array}$$

Parallel algorithm, converges but very slowly, overlapping subdomains only.

The parallel version is called **Jacobi Schwarz method (JSM)**.

The algorithm acts on the local functions $(u_i)_{i=1,2}$. To make things global, we need:

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• extension operators, E_i, s.t. for a function $w_i : \Omega_i \mapsto \mathbb{R}$, $E_i(w_i) : \Omega \mapsto \mathbb{R}$ is the extension of w_i by zero outside Ω_i.

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• partition of unity functions $\chi_i : \Omega_i \mapsto \mathbb{R}$, $\chi_i \geq 0$ and $\chi_i(x) = 0$ for $x \in \partial\Omega_i$ and s.t.

$$w = \sum_{i=1}^{2} E_i(\chi_i \, w_{|\Omega_i}) \, .$$

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Let u^n be an approximation to the solution to the global Poisson problem then u^{n+1} is computed by solving first local subproblems and then gluing them together.

Local problems to solve $(i = 1, 2)$

$$
\begin{array}{rcl}
-\Delta(u_i^{n+1}) = f & \text{in} & \Omega_i \\
u_i^{n+1} = 0 & \text{on} & \partial\Omega_i \cap \partial\Omega \\
u_i^{n+1} = u^n & \text{on} & \partial\Omega_i \cap \overline{\Omega}_{3-i}.
\end{array}
$$

Local problems to solve $(i = 1, 2)$

$$\begin{aligned} -\Delta(u_i^{n+1}) = f \quad & \text{in} \quad \Omega_i \\ u_i^{n+1} = 0 \quad & \text{on} \quad \partial\Omega_i \cap \partial\Omega \\ u_i^{n+1} = u^n \quad & \text{on} \quad \partial\Omega_i \cap \overline{\Omega}_{3-i}. \end{aligned}$$

Two ways to "glue" solutions

- Using the partition of unity functions

Restricted Additive Schwarz (RAS)

$$u^{n+1} := \sum_{i=1}^{2} E_i(\chi_i \, u_i^{n+1}) \, .$$

Local problems to solve $(i = 1, 2)$

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$$u^{n+1} := \sum_{i=1}^{2} E_i(\chi_i\, u_i^{n+1}) \,.$$

- Not based on the partition of unity

Additive Schwarz (ASM)

$$u^{n+1} := \sum_{i=1}^{2} E_i(u_i^{n+1}) \,.$$

The Jacobi algorithm

Let us consider a linear system:

$$A\mathbf{U} = \mathbf{F}$$

with a matrix A of size $m \times m$, a right handside $F \in \mathbb{R}^m$ and a solution $\mathbf{U} \in \mathbb{R}^m$ where m is an integer. Let D be the diagonal of A:

$$D\mathbf{U}^{n+1} = D\mathbf{U}^n + (b - A\mathbf{U}^n)\,,$$

or equivalently,

$$\mathbf{U}^{n+1} = \mathbf{U}^n + D^{-1}(b - A\mathbf{U}^n) = \mathbf{U}^n + D^{-1}\mathbf{r}^n\,,$$

where $\mathbf{r}^n$ is the residual of the equation.

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The block-Jacobi algorithm

The set of indices $\{1, \ldots, m\}$ is partitioned into two sets

$$\mathcal{N}_1 := \{1, \ldots, m_s\} \text{ and } \mathcal{N}_2 := \{m_s + 1, \ldots, m\}\,.$$

Let $\mathbf{U}_1 := \mathbf{U}_{|\mathcal{N}_1}$, $\mathbf{U}_2 := \mathbf{U}_{|\mathcal{N}_2}$ and similarly $\mathbf{F}_1 := \mathbf{F}_{|\mathcal{N}_1}$, $\mathbf{F}_2 := \mathbf{F}_{|\mathcal{N}_2}$.

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The linear system has the following block form:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{pmatrix}$$

where $A_{ij} := A_{|\mathcal{N}_i \times \mathcal{N}_j}$, $1 \le i,j \le 2$.

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The block-Jacobi algorithm reads:

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \mathbf{U}_1^{n+1} \\ \mathbf{U}_2^{n+1} \end{pmatrix} = \begin{pmatrix} F_1 - A_{12}\,\mathbf{U}_2^n \\ F_2 - A_{21}\,\mathbf{U}_1^n \end{pmatrix}.$$

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Let $\mathbf{U}^n = (\mathbf{U}_1^n, \mathbf{U}_2^n)^T$, algorithm becomes

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \mathbf{U}^{n+1} = F - \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \mathbf{U}^n\,.$$

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Or equivalently

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}^{-1} \mathbf{r}^n$$

where

$$\mathbf{r}^n := F - A\mathbf{U}^n, \mathbf{r}_i^n := \mathbf{r}^n_{|\mathcal{N}_i}, \; i = 1, 2.$$

Notations for a compact form

- R_1 the restriction operator from $\mathcal{N}$ into $\mathcal{N}_1$
- R_2 the restriction operator from $\mathcal{N}$ into $\mathcal{N}_2$.

The transpose operator R_1^T is an extension operator from $\mathcal{N}_1$ into $\mathcal{N}$ and the same holds for R_2^T.

Block-Jacobi in compact form

$$\mathbf{U}^{n+1} = \mathbf{U}^n + (R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)\mathbf{r}^n \,.$$

where

$$\mathbf{r}^n := \mathbf{F} - A\mathbf{U}^n \,, \mathbf{r}_i^n := \mathbf{r}^n_{|\mathcal{N}_i}, \; i = 1, 2 \,.$$

$$A_i = R_i A R_i^T, \; i = 1, 2 \,.$$

Let $\Omega := (0,1)$ and consider the following BVP

$$
\begin{aligned}
-\Delta u &= f \text{ in } \Omega\\
u(0) &= u(1) = 0\,.
\end{aligned}
$$

discretized by a three point FD on the grid $x_j := j\,h$, $1 \le j \le m$ where $h := 1/(m+1)$.
Let $u_j \simeq u(x_j)$, $f_j := f(x_j)$, $1 \le j \le m$ and the discrete problem

$$
A\mathbf{U} = F,\ \mathbf{U} = (u_j)_{1\le j\le m},\ F = (f_j)_{1\le j\le m}.
$$

where $A_{j\,j} := 2/h^2$ and
$A_{j\,j+1} = A_{j+1\,j} := -1/h^2$.
Let domains $\Omega_1 := (0, (m_s+1)\,h)$ and $\Omega_2 := (m_s\,h, 1)$ define an overlapping decomposition with a minimal overlap of width h.

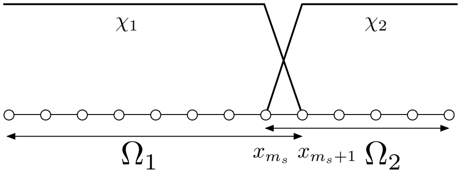
The discretization of the Jacobi-Schwarz for domain Ω_1 reads

$$
\left\{
\begin{array}{l}
-\dfrac{u_{1,j-1}^{n+1} - 2u_{1,j}^{n+1} + u_{1,j+1}^{n+1}}{h^2} = f_j,\ 1 \le j \le m_s\\
u_{1,0}^{n+1} = 0\\
u_{1,m_s+1}^{n+1} = u_{2,m_s+1}^{n}
\end{array}
\right. .
$$

Solving for $\mathbf{U}_1^{n+1} = (u_{1,j}^{n+1})_{1\le j\le m_s}$ corresponds to solving a Dirichlet BVP in subdomain Ω_1 with Dirichlet data taken from the other subdomain at the previous step.

$$
\begin{aligned}
A_{11}\mathbf{U}_1^{n+1} + A_{12}\mathbf{U}_2^{n} &= F_1,\\
A_{22}\mathbf{U}_2^{n+1} + A_{21}\mathbf{U}_1^{n} &= F_2\,.
\end{aligned}
$$

The discrete counterpart of the extension operator E_1 (resp. E_2) is defined by $E_1(U_1) = (U_1, 0)^T$ (resp. $E_2(U_2) = (0, U_2)^T$).



then $E_1(U_1) + E_2(U_2) = E_1(\chi_1 U_1) + E_2(\chi_2 U_2) = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$.

When the overlap is minimal, **the discrete counterparts of the three Schwarz methods (AS, RAS, JS) are equivalent to the same block Jacobi algorithm.**

Continuous level

- Domain Ω and an overlapping decomposition

$$\Omega = \cup_{i=1}^{N} \Omega_i.$$

- A function $u : \Omega \to \mathbb{R}$.
- Restriction of $u : \Omega \to \mathbb{R}$ to Ω_i, $1 \leq i \leq N$.
- The extension operator E_i of a function

$$u_i : \Omega_i \to \mathbb{R}$$

to a function $\Omega \to \mathbb{R}$.

- Partition of unity functions χ_i, $1 \leq i \leq N$.

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Discrete level

- A set of d.o.f. $\mathcal{N}$ and a decomposition

$$\mathcal{N} = \cup_{i=1}^{N} \mathcal{N}_i.$$

- A vector $U \in \mathbb{R}^{\#\mathcal{N}}$.
- Restriction R_i - $\#\mathcal{N}_i \times \#\mathcal{N}$ boolean matrix
- The extension matrix R_i^T.
- Partition of unity diagonal matrices with positive entries, of size $\#\mathcal{N}_i \times \#\mathcal{N}_i$ s. t.

$$Id = \sum_{i=1}^{N} R_i^T D_i R_i.$$

Let $(\mathcal{T}_{h,i})_{1\leq i\leq N}$ define an overlapping mesh decomposition of $\mathcal{T}_h$ and $\Omega_i := \cup_{K\in\mathcal{T}_{h,i}} K$.
Define $V_{h,i}$ the finite element space on $\mathcal{T}_{h,i}$.
Let $\mathcal{T}_h$ be a mesh of a domain Ω and u_h some discretization of a function u which is the solution of an elliptic Dirichlet BVP.

$$\text{Find } \mathbf{U} \in \mathbb{R}^{\#\mathcal{N}} \text{s.t. } A\,\mathbf{U} = F\,.$$

Define the restriction operator $r_i = E_i^T$:

$$r_i : u_h \mapsto u_{h|\Omega_i}$$

R_i boolean matrix corresponding to r_i:

$$R_i := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \end{bmatrix}$$

$$R_i : \mathbb{R}^{\#\mathcal{N}} \longmapsto \mathbb{R}^{\#\mathcal{N}_i}$$

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Partition of unity defined by matrices D_i

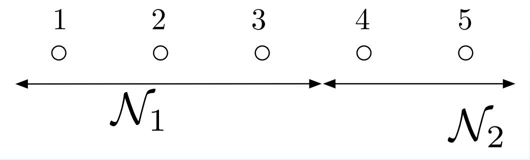
$$D_i : \mathbb{R}^{\#\mathcal{N}_i} \longmapsto \mathbb{R}^{\#\mathcal{N}_i}$$

$$\sum_{i=1}^{N} R_i^T D_i R_i = Id$$

Examples. Two subdomain case: 1d algebraic setting

Let $\mathcal{N} := \{1, \ldots, 5\}$ be partitioned into

$\mathcal{N}_1 := \{1, 2, 3\}$ and $\mathcal{N}_2 := \{4, 5\}$.



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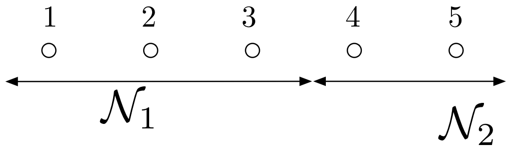

Restriction/partition of unity matrices R_1, R_2, D_1 and D_2:

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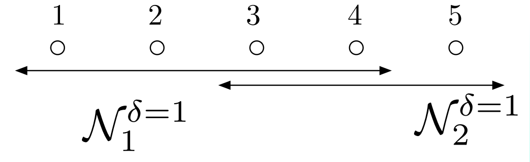
$$\mathcal{N}_1^{\delta=1} := \{1,2,3,4\} \text{ and } \mathcal{N}_2^{\delta=1} := \{3,4,5\}\,.$$



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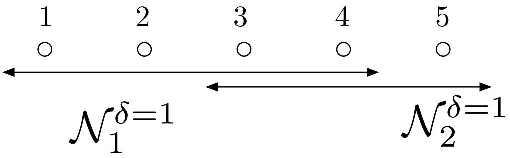

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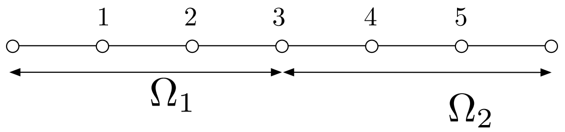

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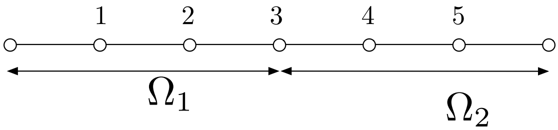

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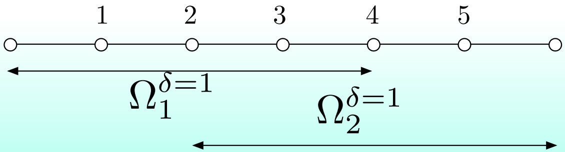

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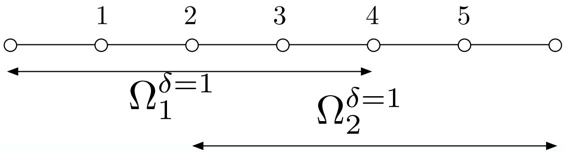

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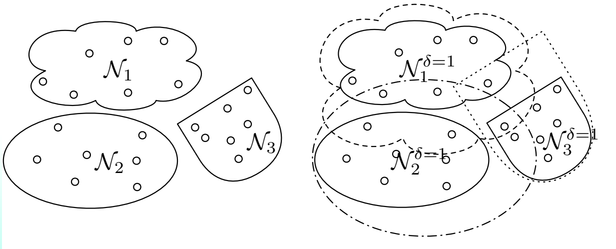
- The number of adjacent elements assigned to different processors is minimized (minimize the communication between different processors).

Extend each disjoint subset $\mathcal{N}_i$ with its direct neighbors to form $\mathcal{N}_i^{\delta=1}$.

$$\mathcal{N} := \cup_{i=1}^{N} \mathcal{N}_i, \quad \mathcal{N}_i \cap \mathcal{N}_j = \emptyset \text{ for } i \neq j\,.$$

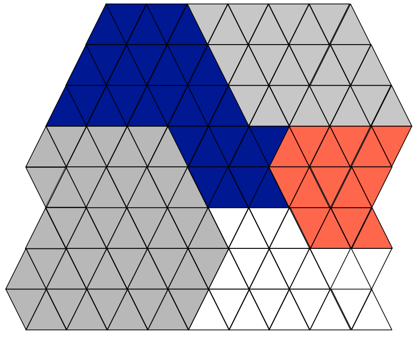


R_i be the restriction matrix from set $\mathcal{N}$ to the subset $\mathcal{N}_i^{\delta=1}$.
Partition of unity: D_i a diagonal matrix of size $\#\mathcal{N}_i^{\delta=1} \times \#\mathcal{N}_i^{\delta=1}$, $1 \leq i \leq N$ such that

$$(D_i)_{jj} := 1/\#\mathcal{M}_j,\ \mathcal{M}_j := \{1 \leq i \leq N \mid j \in \mathcal{N}_i^{\delta=1}\}\,.$$

Multi-D algebraic finite element (FE) decomposition

Computational domain



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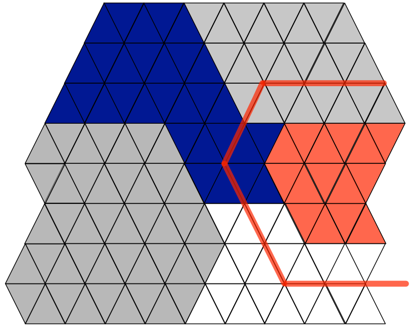

Create overlapping subdomains resolved by the finite element meshes:

$$\Omega_i = \bigcup_{\tau \in \mathcal{T}_{i,h}} \tau \quad \text{for } 1 \leq i \leq N\,. \tag{1}$$

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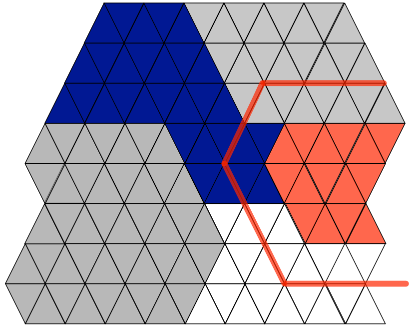

Let $\{\phi_k\}_{k\in\mathcal{N}}$ be a basis of the finite element space.

$$\mathcal{N}_i := \{k \in \mathcal{N} : \mathrm{supp}(\phi_k) \cap \Omega_i \neq \emptyset\}\,.$$

For all degree of freedom $k \in \mathcal{N}$, define

$$\mu_k := \#\,\{j : 1 \leq j \leq N \text{ and } \mathrm{supp}(\phi_k) \cap \Omega_j \neq \emptyset\}\,.$$

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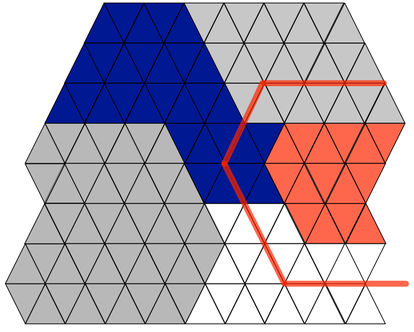

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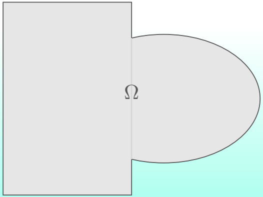
Let the discretised Poisson problem: $A\mathbf{U} = \mathbf{F} \in \mathbb{R}^n$.



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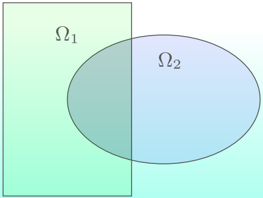

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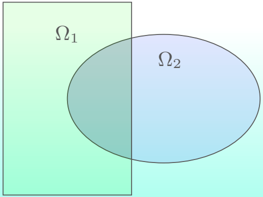
Find $\mathbf{U}^m \longrightarrow \mathbf{U}^{m+1}$ by solving concurrently:

$$\mathbf{U}_j^{m+1} = \mathbf{U}_j^m + A_j^{-1} R_j(\mathbf{F} - A\mathbf{U}^m),\ j = 1, 2$$

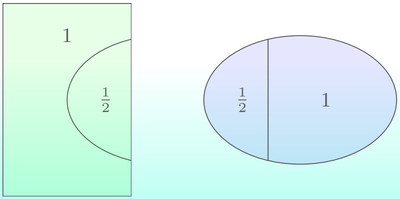
where $\mathbf{U}_i^m = R_i \mathbf{U}^m$ and $A_i := R_i A R_i^T$.



We have effectively divided, but we have yet to conquer.

Duplicated unknowns coupled via a partition of unity:

$$I = \sum_{i=1}^{N} R_i^T D_i R_i.$$



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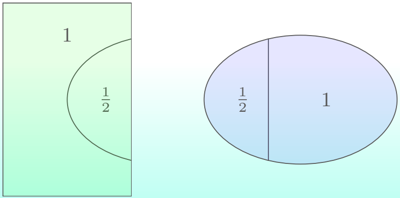

Then, $\mathbf{U}^{m+1} = \sum_{i=1}^{N} R_i^T D_i \mathbf{U}_i^{m+1}$.

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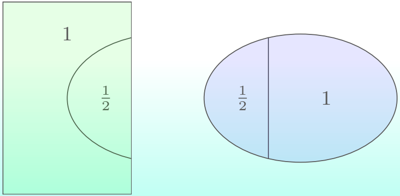

Then, $\mathbf{U}^{m+1} = \sum_{i=1}^{N} R_i^T D_i \mathbf{U}_i^{m+1}$. $\quad M_{RAS}^{-1} = \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i.$

RAS algorithm (Cai & Sarkis, 1999)

Algebraic formulation - RAS and ASM

Schwarz algorithm iterates on a pair of local functions (u_m^1, u_m^2)
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Schwarz and RAS (Efstathiou and Gander, 2002)
Discretization of the classical Schwarz algorithm and the iterative RAS algorithm:

$$\mathbf{U}^{n+1} = \mathbf{U}^n + M_{RAS}^{-1}\mathbf{r}^n\,, \mathbf{r}^n := \mathbf{F} - A\,\mathbf{U}^n.$$

are equivalent

$$\mathbf{U}^n = R_1^T D_1 \mathbf{U}_1^n + R_2^T D_2 \mathbf{U}_2^n\,.$$

Operator M_{RAS}^{-1} is used as a preconditioner in Krylov methods for non symmetric problems.

$$\boxed{M_{RAS}^{-1} := \sum_{i=1}^{N} R_i^T\, D_i\, A_i^{-1}\, R_i\,.} \tag{1}$$

A symmetrized version: Additive Schwarz Method (ASM),

$$\boxed{M_{ASM}^{-1} := \sum_{i=1}^{N} R_i^T\, A_i^{-1}\, R_i} \tag{2}$$

are used as a preconditioner for the conjugate gradient (CG) method. Later on, we introduce

$$\boxed{M_{SORAS}^{-1} := \sum_{i=1}^{N} R_i^T\, D_i\, B_i^{-1}\, D_i\, R_i} \tag{3}$$

where $(B_i)_{1\le i\le N}$ are some local invertible matrices.
Although RAS is more efficient, ASM is amenable to theory.

Let $L > 0$, $\Omega = (0, L)$ be decomposed into two subodmains $\Omega_1 := (0, L_1)$ and $\Omega_2 := (l_2, L)$ with $l_2 \leq L_1$.

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Errors are affine functions in each subdomain:

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Thus, we have

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Let $\delta := L_1 - l_2$ (overlap). Interface iteration

$$\begin{aligned} e_2^{n+1}(L_1) &= \tfrac{l_2}{l_2+\delta}\,\tfrac{L-l_2-\delta}{L-l_2}\,e_2^n(L_1) \\ &= \tfrac{1-\delta/(L-l_2)}{1+\delta/l_2}\,e_2^n(L_1)\,. \end{aligned}$$

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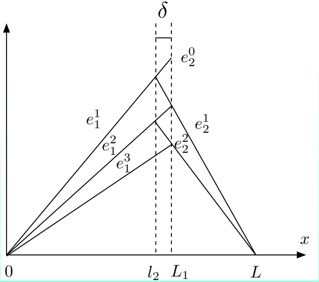
It is clear that $\delta > 0$ is sufficient and necessary to have convergence.



Let $\mathbb{R}^2$ decomposed into two half-planes $\Omega_1 = (-\infty, \delta) \times \mathbb{R}$ and $\Omega_2 = (0, \infty) \times \mathbb{R}$ with an overlap of size $\delta > 0$ and the problem

$$\begin{aligned} (\eta - \Delta)(u) \quad &= f \quad \text{in} \quad \mathbb{R}^2, \\ u \quad & \text{is bounded at infinity}\,, \end{aligned}$$

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$$\begin{aligned} (\eta - \Delta)(u) &= f \quad \text{in} \quad \mathbb{R}^2, \\ u & \quad \text{is bounded at infinity}, \end{aligned}$$

By linearity, the errors $e_i^n := u_i^n - u|_{\Omega_i}$ satisfy

$$\begin{aligned} (\eta - \Delta)(e_1^{n+1}) &= 0 \quad \text{in} \quad \Omega_1 \\ e_1^{n+1} & \quad \text{is bounded at infinity} \\ e_1^{n+1}(\delta, y) &= e_2^n(\delta, y), \end{aligned}$$

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By linearity, the errors $e_i^n := u_i^n - u|_{\Omega_i}$ satisfy

$$\begin{aligned} (\eta - \Delta)(e_2^{n+1}) &= 0 \quad \text{in} \quad \Omega_2 \\ e_2^{n+1} & \quad \text{is bounded at infinity} \\ e_2^{n+1}(0, y) &= e_1^n(0, y). \end{aligned}$$

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Partial Fourier transform of the equation in the y direction:

$$\left(\eta - \frac{\partial^2}{\partial x^2} + k^2\right)(\hat{e}_j^{n+1}(x,k)) = 0 \quad \text{in} \quad \Omega_j.$$

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$$\left(\eta - \frac{\partial^2}{\partial x^2} + k^2\right)(\hat{e}_j^{n+1}(x,k)) = 0 \quad \text{in} \quad \Omega_j.$$

For a given k, solutions

$$\hat{e}_j^{n+1}(x,k) = \gamma_+^{n+1}(k)^{\lambda^+(k)x} + \gamma_-^{n+1}(k)e^{\lambda^-(k)x)},$$

must be bounded at $x = \mp\infty$.

Fourier analysis in 2d

Let $\mathbb{R}^2$ decomposed into two half-planes $\Omega_1 = (-\infty, \delta) \times \mathbb{R}$ and $\Omega_2 = (0, \infty) \times \mathbb{R}$ with an overlap of size $\delta > 0$ and the problem

$$\begin{aligned} (\eta - \Delta)(u) &= f \quad \text{in} \quad \mathbb{R}^2, \\ u & \quad \text{is bounded at infinity}, \end{aligned}$$

Partial Fourier transform of the equation in the y direction:

$$\left(\eta - \frac{\partial^2}{\partial x^2} + k^2\right)(\hat{e}_j^{n+1}(x,k)) = 0 \quad \text{in} \quad \Omega_j.$$

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$$\hat{e}_j^{n+1}(x,k) = \gamma_+^{n+1}(k)^{\lambda^+(k)x} + \gamma_-^{n+1}(k)e^{\lambda^-(k)x)},$$

must be bounded at $x = \mp\infty$.

This implies

$$\hat{e}_1^{n+1}(x,k) = \gamma_+^{n+1}(k)e^{\lambda^+(k)x}$$

$$\hat{e}_2^{n+1}(x,k) = \gamma_-^{n+1}(k)e^{\lambda^-(k)x}$$

Let $\mathbb{R}^2$ decomposed into two half-planes $\Omega_1 = (-\infty, \delta) \times \mathbb{R}$ and $\Omega_2 = (0, \infty) \times \mathbb{R}$ with an overlap of size $\delta > 0$ and the problem

$$\begin{aligned} (\eta - \Delta)(u) &= f \quad \text{in} \quad \mathbb{R}^2, \\ u & \quad \text{is bounded at infinity}, \end{aligned}$$

Partial Fourier transform of the equation in the y direction:

$$\left(\eta - \frac{\partial^2}{\partial x^2} + k^2\right)(\hat{e}_j^{n+1}(x,k)) = 0 \quad \text{in} \quad \Omega_j.$$

For a given k, solutions

$$\hat{e}_j^{n+1}(x,k) = \gamma_+^{n+1}(k)^{\lambda^+(k)x} + \gamma_-^{n+1}(k)e^{\lambda^-(k)x)},$$

must be bounded at $x = \mp\infty$.

From the interface conditions we get

$$\gamma_+^{n+1}(k) = \gamma_-^n(k)\, e^{\lambda^-(k)x},\ \gamma_-^{n+1}(k) = \gamma_+^n(k)e^{-\lambda^+(k)x}.$$

Combining these two and denoting $\lambda(k) = \lambda^+(k) = -\lambda^-(k)$, we get for $i = 1, 2$,

$$\gamma_\pm^{n+1}(k) = \rho(k; \eta, \delta)^2\, \gamma_\pm^{n-1}(k)$$

The convergence factor ρ is given by:

$$\rho(k; \eta, \delta) = e^{-\lambda(k)\delta},\ \lambda(k) = \sqrt{\eta + k^2}$$

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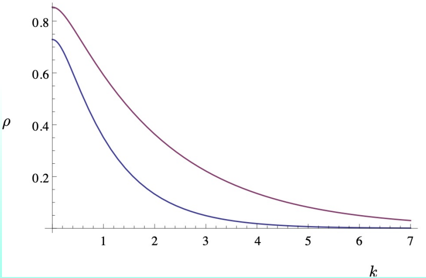
$$\rho(k;\eta,\delta) = e^{-\lambda(k)\delta},\ \lambda(k) = \sqrt{\eta + k^2}$$



Properties

- For all $k \in \mathbb{R}$, $\rho(k) < e^{-\sqrt{\eta}\,\delta} < 1$ so that $\gamma_j^n(k) \to 0$ uniformly as n goes to infinity.
- $\rho \to 0$ as k tends to infinity, high frequency modes of the error converge very fast.
- When there is no overlap ($\delta = 0$), $\rho = 1$ and there is stagnation of the method.

Outline

Introduction

Schwarz methods using Freefem++

- Schwarz algorithms as solvers
- Schwarz algorithms as preconditioners
- Schwarz preconditioners using FreeFEM++

About FreeFem++ (survival kit)

`FreeFem++` allows a very simple and natural way to solve a great variety of variational problems.

It is possible to have access to the underlying linear algebra such as the stiffness or mass matrices.

A very detailed documentation of `FreeFem++` is available on the official website
`http://www.freefem.org/`
Codes and lecture notes for the domain decomposition part available here:
`http://www.victoritadolean.com/p/book.html`

Let a homogeneous Dirichlet boundary value problem for a Laplacian defined on a unit square $\Omega =]0,1[^2$:

$$\begin{cases} -\Delta u = f & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \end{cases} \tag{4}$$

The variational formulation of the problem

$$\text{Find } u \in H_0^1(\Omega) := \{w \in H^1(\Omega) \ : \ w = 0, \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4\}$$

such that

$$\int_\Omega \nabla u.\nabla v dx - \int_\Omega f\, v\, dx = 0, \forall v \in H_0^1(\Omega)\,.$$

Feature of `Freefem++`: penalization of Dirichlet BC. Let TGV (*Très Grande Valeur* in French) be a very large value, the above variational formulation is approximated by
Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u.\nabla v dx + TGV \int_{\cup_{i=1,\dots,4}\Gamma_i} u\, v - \int_\Omega f v\, dx = 0, \forall v \in H^1(\Omega)\,.$$

The following FreeFem++ script is solving this problem

```
// Number of mesh points in x and y directions
int Nbnoeuds=10;
```

The text after // symbols are comments ignored by the FreeFem++ language.

Each new variable must be declared with its type (here int designs integers).

```
//Mesh definition
mesh Th=square(Nbnoeuds,Nbnoeuds,[x,y]);
```

The function square returns a structured mesh of the square, the sides of the square are labelled from 1 to 4 in trigonometrical sense.

Define the function representing the right hand side

```
// Function of x and y
func f=x*y;
```

and the P_1 finite element space `Vh` over the mesh `Th`.

```
// Finite element space on the mesh Th
fespace Vh(Th,P1);
//uh and vh are of type Vh
Vh uh,vh;
```

The functions u_h and v_h belong to the P_1 finite element space V_h which is an approximation to $H^1(\Omega)$.

```
// variational problem definition
problem heat(uh,vh,solver=LU)=
        int2d(Th)(dx(uh)*dx(vh)+dy(uh)*dy(vh))
       -int2d(Th)(f*vh)
       +on(1,2,3,4,uh=0);
```

The keyword `problem` allows the definition of a variational problem (without solving it)

$$\int_{\Omega} \nabla u_h . \nabla v_h dx + TGV \int_{\cup_{i=1,\dots,4}\Gamma_i} u_h\, v_h - \int_{\Omega} f v_h dx = 0, \forall v_h \in V_h\,.$$

where TGV is equal to 10^{30}.

The parameter `solver` sets the method that will be used to solve the resulting linear system. To solve the problem we need

```
//Solving the problem
heat;
// Plotting the result
plot(uh,wait=1);
```

The `Freefem++` script can be saved with your favourite text editor (e.g. under the name `heat.edp`). In order to execute the script write the shell command

```
FreeFem++ heat.edp
```

The result will be displayed in a graphic window.

Solve Neumann or Fourier boundary conditions such as

$$\begin{cases} -\Delta u + u = f & \text{dans } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{sur } \Gamma_1 \\ u = 0 & \text{sur } \Gamma_2 \\ \frac{\partial u}{\partial n} + \alpha u = g & \text{sur } \Gamma_3 \cup \Gamma_4 \end{cases} \tag{5}$$

The new variational formulation consists in determining $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h.\nabla v_h dx + \int_{\Gamma_3 \cup \Gamma_4} \alpha u_h v_h + TGV \int_{\Gamma_2} u_h.v_h$$
$$- \int_{\Gamma_3 \cup \Gamma_4} g v_h - \int_\Omega f v_h dx = 0, \forall v_h \in V_h.$$

The `Freefem++` definition of the problem

```
problem heat(uh,vh)=
        int2d(Th)(dx(uh)*dx(vh)+dy(uh)*dy(vh))
       +int1d(Th,3,4)(alpha*uh*vh)
       -int1d(Th,3,4)(g*vh)
       -int2d(Th)(f*vh)
       +on(2,uh=0);
```

In order to use some **linear algebra** package, we need the matrices. The keyword `varf` allows the definition of a variational formulation

```
varf heat(uh,vh)=
        int2d(Th)(dx(uh)*dx(vh)+dy(uh)*dy(vh))
       +int1d(Th,3,4)(alpha*uh*vh)
       -int1d(Th,3,4)(g*vh)
       -int2d(Th)(f*vh)
       +on(2,uh=0);
matrix Aglobal; // stiffness sparse matrix
Aglobal = heat(Vh,Vh,solver=UMFPACK);// UMFPACK solver
Vh rhsglobal;  //right hand side vector
rhsglobal[] = heat(0,Vh);
```

Here `rhsglobal` is a FE function and the associated vector of d.o.f. is `rhsglobal[]`.
The linear system is solved by using UMFPACK

```
// Solving the problem by a sparse LU sover
uh[] = Aglobal^-1*rhsglobal[];
```

Decomposition into overlapping domains

To build the overlapping decomposition and the associated algebraic call the routine `SubdomainsPartitionUnity`.

Output:

- overlapping meshes `aTh[i]`
- the restriction/interpolation operators `Rih[i]` from the local finite element space `Vh[i]` to the global one `Vh`
- the diagonal local matrices `Dih[i]` from the partition of unity.

```
include "./data.edp"
include "./decomp.idp"
include "./createPartition.idp"
SubdomainsPartitionUnity(Th,part[],sizeovr,aTh,Rih,Dih,Ndeg,AreaThi);
```

We first need to define the global data.

```
 Aglobal = vaglobal(Vh,Vh,solver = UMFPACK);  // global matrix
rhsglobal[] = vaglobal(0,Vh);                   // global rhs
uglob[] = Aglobal^-1*rhsglobal[];
plot(uglob,value=1,fill=1,wait=1,cmm="Solution by a direct method",dim=3);
```

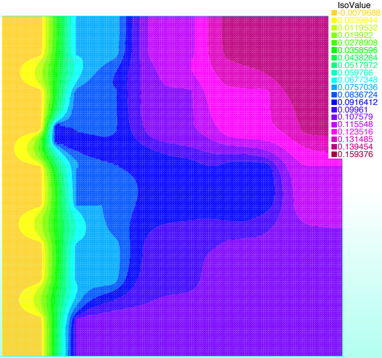
And then the local problems

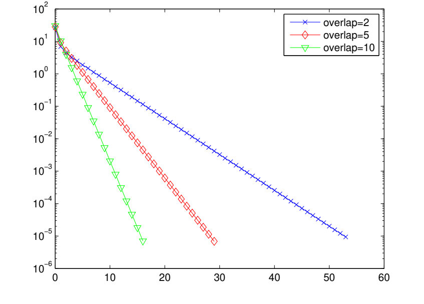
```
 for(int i = 0;i<npart;++i)
  {
    cout << " Domain :" << i << "/" << npart << endl;
    matrix aT = Aglobal*Rih[i]';
    aA[i] = Rih[i]*aT;
    set(aA[i],solver = UMFPACK);// direct solvers
  }
```

RAS and ASM : Schwarz iteration

```
ofstream filei("Conv.m");
Vh un = 0;                              // initial guess
Vh rn = rhsglobal;
for(int iter = 0;iter<maxit;++iter)
  { real err = 0, res;
    Vh er = 0;
    for(int i = 0;i<npart;++i)
      { real[int] bi = Rih[i]*rn[];     // restriction to the local domain
        real[int] ui = aA[i] ^-1 * bi; // local solve
        bi = Dih[i]*ui;
        // bi = ui;  // uncomment this line to test the ASM method as a solver
        er[] += Rih[i]'*bi;}
    un[] += er[];                  // build new iterate
    rn[] = Aglobal*un[];           // computes global residual
    rn[] = rn[] - rhsglobal[];
    rn[] *= -1;
    err = sqrt(er[]'*er[]);
    res = sqrt(rn[]'*rn[]);
    cout << "Iteration: " << iter << " Correction = " << err << " Residual = " << res << endl;
    plot(un,wait=1,value=1,fill=1,dim=3,cmm="Approximate solution at step " + iter);
    int j = iter+1;
    // Store the error and the residual in Matlab/Scilab/Octave form
    filei << "Convhist("+j+",:)=[" << err << " " << res <<"];" << endl;
    if(err < tol) break;}
plot(un,wait=1,value=1,fill=1,dim=3,cmm="Final solution");
```

Convergence history of the RAS solver for different values of the overlapping parameter.





Note that this convergence, not very fast even in a simple configuration of 4 subdomains.

The iterative version of ASM does not converge. For this reason, the ASM method is always used a preconditioner for a Krylov method such as CG, GMRES or BiCGSTAB.

Fixed point method

Consider the linear system

$$A\,x = b$$

A possible iterative method is a fixed point algorithm

$$x^{n+1} = x^n + B^{-1}(b - A\,x^n)$$

and x is a fixed point of the operator:

$$x \longmapsto x + B^{-1}(b - A\,x)\,.$$

Let $r_0 := b - Ax^0$ and $C := B^{-1}\,A$, then

$$x^n = \sum_{i=0}^{n}(I_d - C)^i\,B^{-1}r_0 + x^0\,.$$

We have convergence iff the spectral radius of the matrix $I_d - C$ is smaller than one.

Krylov method

Consider a preconditioned linear system:

$$B^{-1}\,A\,x = B^{-1}\,b$$

Let x^0 an initial guess and $r^0 := B^{-1}\,b - C\,x^0$ the initial residual. Then $y := x - x^0$ solves

$$C\,y = r^0\,.$$

The basis for Krylov methods is the following

Lemma
Let C be an invertible matrix of size $N \times N$. Then, there exists a polynomial $\mathcal{P}$ of degree $p < N$ such that

$$C^{-1} = \mathcal{P}(C)\,.$$

By a constructive proof

$$x = x^0 + \sum_{i=1}^{d} \left(-\frac{a_i}{a_0} \right) C^{i-1}\, r^0 \, .$$

Thus, it makes sense to introduce Krylov spaces, $\mathcal{K}^n(C, r^0)$

$$\mathcal{K}^n(C, r^0) := Span\{r^0, \, Cr^0, \ldots, C^{n-1} r^0\}, \; n \geq 1.$$

to seek y^n an approximation to y.

Why Krylov methods

By a constructive proof

$$x = x^0 + \sum_{i=1}^{d} \left(-\frac{a_i}{a_0}\right) C^{i-1}\, r^0\,.$$

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to seek y^n an approximation to y.

Example: The CG methods applies to symmetric positive definite (SPD) matrices and minimizes the A^{-1}-norm of the residual when solving $Ax = b$:

$$\text{CG} \begin{cases} \text{Find } y^n \in \mathcal{K}^n(A, r^0) \text{ such that} \\ \|A\, y^n - r^0\|_{A^{-1}} = \min\limits_{w \in \mathcal{K}^n(A, r^0)} \|A\, w - r^0\|_{A^{-1}}\,. \end{cases}$$

By a constructive proof

$$x = x^0 + \sum_{i=1}^{d} \left(-\frac{a_i}{a_0} \right) C^{i-1}\, r^0\,.$$

Thus, it makes sense to introduce Krylov spaces, $\mathcal{K}^n(C, r^0)$

$$\mathcal{K}^n(C, r^0) := Span\{r^0,\, Cr^0, \ldots, C^{n-1}r^0\},\ n \geq 1.$$

to seek y^n an approximation to y.

A detailed analysis reveals that $x^n = y^n + x_0$ can be obtained by the quite cheap recursion formula:

for $i = 1, 2, \ldots$ **do**
  $\rho_{i-1} = (r_{i-1}, r_{i-1})_2$
  if $i = 1$ **then**
    $p_1 = r_0$
  else
    $\beta_{i-1} = \rho_{i-1}/\rho_{i-2}$
    $p_i = r_{i-1} + \beta_{i-1} p_{i-1}$
  end if
  $q_i = A p_{i-1}$
  $\alpha_i = \dfrac{\rho_{i-1}}{(p_i, q_i)_2}$
  $x_i = x_{i-1} + \alpha_i p_i$
  $r_i = r_{i-1} - \alpha_i q_i$
  check convergence; continue if necessary
end for

By solving an optimization problem:

$$
\mathrm{G}MRES \left\{ \begin{array}{l} \text{Find } y^n \in \mathcal{K}^n(C, r^0) \text{ such that} \\ \|C\, y^n - r^0\|_2 = \min\limits_{w \in \mathcal{K}^n(C, r^0)} \|C\, w - r^0\|_2 \end{array} \right.
$$

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a preconditioned Krylov solve will generate an optimal x_K^n in

$$\mathcal{K}^n(C, B^{-1} r_0) := x_0 + Span\{B^{-1} r_0,\, C\, B^{-1} r_0, \ldots, C^{n-1}\, B^{-1} r_0\}\,.$$

By solving an optimization problem:

$$\text{GMRES} \left\{ \begin{array}{l} \text{Find } y^n \in \mathcal{K}^n(C, r^0) \text{ such that} \\ \|C\, y^n - r^0\|_2 = \displaystyle\min_{w \in \mathcal{K}^n(C, r^0)} \|C\, w - r^0\|_2 \end{array} \right.$$

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$$\mathcal{K}^n(C, B^{-1} r_0) := x_0 + Span\{B^{-1} r_0,\, C\, B^{-1} r_0, \ldots, C^{n-1}\, B^{-1} r_0\}\,.$$

Remark. This minimization problem is of size n. When n is small w.r.t. N, its solving has a marginal cost. Thus, x_K^n has a computing cost similar to that of x^n. But, since $x^n \in \mathcal{K}^n(B^{-1}A, B^{-1} r_0)$ as well but with "frozen" coefficients, we have that x_n is less optimal (actually much much less) than x_K^n.

In the previous Krylov methods we can use as preconditioner

- RAS (in conjunction with BiCGStab or GMRES)

$$B^{-1} := M_{RAS}^{-1} = \sum_{i=1}^{N} R_i^T \, D_i \, (R_i \, A \, R_i^T)^{-1} \, R_i$$

- ASM (in a CG methods)

$$B^{-1} := M_{ASM}^{-1} = \sum_{i=1}^{N} R_i^T \, (R_i \, A \, R_i^T)^{-1} \, R_i$$

- M_{ASM}^{-1} as a preconditioner
- a Krylov method: conjugate gradient since M_{ASM}^{-1} and A are symmetric.

At iteration m the error for the PCG method is bounded by:

$$||\bar{x} - x_m||_{M_{ASM}^{-\frac{1}{2}} A M_{ASM}^{-\frac{1}{2}}} \leq 2\left[\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right]^m ||\bar{x} - x_0||_{M_{ASM}^{-\frac{1}{2}} A M_{ASM}^{-\frac{1}{2}}}.$$

where κ is the condition number of $M_{ASM}^{-1}A$ and $\bar{x}$ is the exact solution.

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where κ is the condition number of $M_{ASM}^{-1}A$ and $\bar{x}$ is the exact solution.

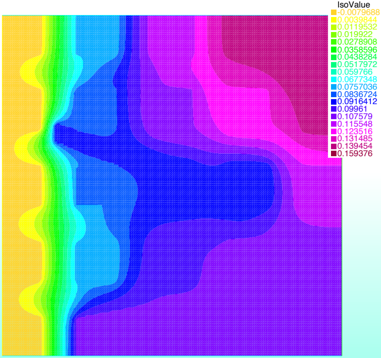
The CG with the ASM preconditioner becomes:

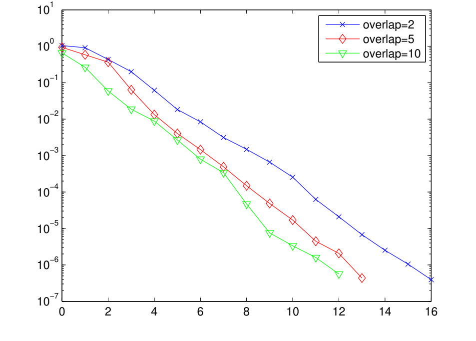
for $i = 1, 2, \ldots$ **do**
  $\rho_{i-1} = (r_{i-1}, M_{ASM}^{-1} r_{i-1})_2$
  if $i = 1$ **then**
    $p_1 = M_{ASM}^{-1} r_0$
  else
    $\beta_{i-1} = \rho_{i-1}/\rho_{i-2}$
    $p_i = M_{ASM}^{-1} r_{i-1} + \beta_{i-1} p_{i-1}$
  end if
  $q_i = A p_{i-1}$
  $\alpha_i = \dfrac{\rho_{i-1}}{(p_i, q_i)_2}$
  $x_i = x_{i-1} + \alpha_i p_i$
  $r_i = r_{i-1} - \alpha_i q_i$
  check convergence; continue if necessary
end for

The action of the global matrix and preconditioner

```
func real[int] A(real[int] &x)
{
        // Matrix vector product with the global matrix
        Vh Ax;
        Ax[]= Aglobal*x;
        return Ax[];
}
// and the application of the preconditioner
func real[int] AS(real[int] &l)
{
   // Application of the ASM preconditioner
   // M^{-1}*y = sum Ri^T*Ai^{-1}*Ri*y
   // Ri restriction operators, Ai  =Ri*A*Ri^T local matrices
   Vh s = 0;
   for(int i=0;i<npart;++i)
     {
       real[int] bi = Rih[i]*l;          // restricts rhs
       real[int] ui = aA[i] ^-1 * bi;    // local solves
       s[] += Rih[i]'*ui;                // prolongation
      }
   return s[];
}
```

The Krylov method applied in this case is the CG. The performance is now less sensitive to the overlap size.





We can also use RAS as a preconditioner, by taking into account the partition of unity

```
func real[int] RAS(real[int] &l)
{
   // Application of the RAS preconditioner
   // M^{-1}*y = \sum Ri^T*Di*Ai^{-1}*Ri*y
   // Ri restriction operators, Ai =Ri*A*Ri^T local matrices
   Vh s = 0;
   for(int i=0;i<npart;++i) {
       real[int] bi = Rih[i]*l;          // restricts rhs
       real[int] ui = aA[i] ^-1 * bi;  // local solves
       bi = Dih[i]*ui;                        // partition of unity
       s[] += Rih[i]'*bi;                     // prolongation
     }
   return s[];
}
```

this time in conjuction with BiCGStab since we deal with non-symmetric problems.