

Space-time method for controllability problems (toward a space-time DDM ???)

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We address the approximation of null controls for a **semi-linear wave equation**

$$y_{tt} - \Delta y + f(y) = v1_\omega, \quad \Omega \times (0, T) \quad (1)$$

Part 1: Find a sequence $(y_k, v_k)_{k \in \mathbb{N}}$ converging strongly to a control-state pair for (1)?

Typically, (y_k, v_k) solves a **linear controllability problem** for

$$z_{tt} - \Delta z + Az = u1_\omega + B, \quad \Omega \times (0, T) \quad (2)$$

Part 2: for each k , find a convergent numerical approximation $(y_{kh}, v_{kh})_{h>0}$ of (y_k, v_k) for (2)?

Rk. $\|v - v_{k,h}\| \leq \|v - v_k\| + \|v_{k,h} - v_k\|$

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Semilinear wave equation

- Let $\Omega \subset \mathbb{R}^d$, $\omega \subset \Omega$, $T > 0$. $Q_T := \Omega \times (0, T)$, $q_T := \omega \times (0, T)$.

$$\begin{cases} \partial_{tt}y - \Delta y + f(y) = v1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \quad (3)$$

- $(u_0, u_1) \in \mathbf{V} := H_0^1(\Omega) \times L^2(\Omega)$, $v \in L^2(q_T)$. $f \in C^1(\mathbb{R}; \mathbb{R})$.
- $|f(r)| \leq C(1 + |r|) \ln^2(2 + |r|) \forall r \in \mathbb{R}$
- $y \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ is unique.

Definition

(3) is **null controllable in time T** IFF for any $(u_0, u_1) \in \mathbf{V}$, $\exists$ a control function $v \in L^2(q_T)$ such that $(y(\cdot, T), \partial_t y(\cdot, T)) = (0, 0)$.

Theorem (Zuazua'93, Zhang '00, ...)

If T and ω are large enough and if f does not grow too fast at infinity

$$(H_1) \quad \limsup_{|r| \rightarrow \infty} \frac{|f(r)|}{|r| \ln^{1/2}|r|} < \beta$$

(for some $\beta > 0$ small enough) then (3) is exactly controllable in time T .

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(for some $\beta > 0$ small enough) then (3) is exactly controllable in time T .

Part 1: Construction of a sequence $(y_k, v_k)_{k \in \mathbb{N}}$ converging strongly to a solution of the semilinear pb.

- Bottois, Lemoine, M. *Constructive exact controls for semi-linear wave equations*, [arxiv](#).
- Trélat, M. *Constructive exact control of semilinear 1D wave equations by a least-squares approach*, [SICON 2022](#)
- Bhandari, Lemoine, M. *Exact boundary controllability of 1D semilinear wave equations through a constructive approach*, [AIMS EECT 2022](#)
- Lemoine, M, Sue. *Exact boundary controllability of semilinear wave equations through a constructive approach*, [arxiv](#).

Semilinear wave equation: Non constructive argument

The controllability proof given in Zuazua'93, Zhang'00 is based on a Leray Schauder **fixed point argument**.

Let $\Lambda : L^\infty(0, T; L^d(\Omega)) \rightarrow L^\infty(0, T; L^d(\Omega))$, where $y := \Lambda(\xi)$ is a controlled solution with the control v_ξ of the linear problem (assuming $f(0) = 0$)

$$\begin{cases} \partial_{tt}y - \Delta y + y \frac{f(\xi)}{\xi} = v_\xi 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \\ (y(\cdot, T), \partial_t y(\cdot, T)) = (0, 0) & \text{in } \Omega. \end{cases} \quad (4)$$

Then, Λ **has a fixed point**.

Useless in practice, since Λ is **not contracting**: the Picard sequence $y_{k+1} = \Lambda(y_k)$ is bounded but not convergent.

A first constructive method - Least-squares approach

We consider the Hilbert space

$$\mathcal{H} := \left\{ (y, v) \in L^2(Q_T) \times L^2(Q_T) \mid \partial_{tt}y - \Delta y \in L^2(Q_T), y = 0 \text{ on } \Sigma_T, \right. \\ \left. (y(\cdot, 0), \partial_t y(\cdot, 0)) \in \mathbf{V} \right\}$$

and the subspace

$$\mathcal{A} := \left\{ (y, v) \in \mathcal{H} \mid (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1), (y(\cdot, T), \partial_t y(\cdot, T)) = (0, 0) \text{ in } \Omega \right\}$$

We define the **least-squares functional** $E : \mathcal{A} \rightarrow \mathbb{R}$ by

$$E(y, v) := \frac{1}{2} \|\partial_{tt}y - \Delta y + f(y) - v1_\omega\|_{L^2(Q_T)}^2$$

and consider the **nonconvex minimization problem**

$$\inf_{(y, v) \in \mathcal{A}} E(y, v) \tag{5}$$

Proposition

$\forall (y, v) \in \mathcal{A}$,

$$\sqrt{E(y, v)} \leq C e^{C\sqrt{\|f'(y)\|_\infty}} \|E'(y, v)\|_{\mathcal{A}'_0}. \quad (6)$$

Consequence:

Any *critical point* $(y, v) \in \mathcal{A}$ of E (i.e., $E'(y, v) = 0$) is a zero of E , and thus is a pair solution of the controllability problem. Moreover:

given any sequence $(y_k, v_k)_{k \in \mathbb{N}}$ in $\mathcal{A}$ such that $\|E'(y_k, v_k)\|_{\mathcal{A}'_0} \xrightarrow{k \rightarrow +\infty} 0$ and such that $\|f'(y_k)\|_\infty$ is uniformly bounded, we have $E(y_k, v_k) \xrightarrow{k \rightarrow +\infty} 0$.

Thanks to this instrumental property, a minimizing sequence for E **cannot be stuck** in a local minimum, even though E fails to be convex (it has multiple zeros).

Strong convergence of the LS method

$(y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k(Y_k, V_k)$ where (Y_k, V_k) solves

$$\begin{cases} \partial_{tt} Y_k - \Delta Y_k + f'(y_k) Y_k = V_k 1_\omega + (\partial_{tt} y_k - \Delta y_k + f(y_k) - v_k 1_\omega) & \text{in } Q_T, \\ Y_k = 0 & \text{on } \Sigma_T, \\ (Y_k(\cdot, 0), \partial_t Y_k(\cdot, 0)) = (0, 0), \quad (Y_k(\cdot, T), \partial_t Y_k(\cdot, T)) = (0, 0) & \text{in } \Omega, \end{cases} \quad (7)$$

and $\lambda_k \in (0, 1)$ the optimal (damped Newton) parameter.
Here, V_k is the control of minimal $L^2(q_T)$ -norm.

Théorème 1.1 (Münch, 2021, 2022) [1], [2]. *Edwards, Cambridge, MA, 2022 (to appear).*

Assume that $T, \omega \subset \Omega \subset \mathbb{R}^d$ large enough and f Loc Lip satisfies

$$|f'(r)| \leq \alpha + \beta \ln^2(1 + |r|) \quad \forall r \in \mathbb{R}. \quad (8)$$

For any $(y_0, v_0) \in \mathcal{A}$, the minimizing sequence $(y_k, v_k)_{k \in \mathbb{N}}$ for E converges strongly to a state-control (y, v) for the nonlinear wave eq. The convergence is sur-linear after a finite number of iterations.

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Theorem (M-Trélat 2021 ($d = 1$), Bottois-Lemoine-M 22 ($d > 1$))

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a second constructive method : simpler Linearization leading to contracting prop.

We introduce the operator $\Lambda_s : L^\infty(Q_T) \mapsto L^\infty(Q_T)$, $\Lambda_s(\hat{y}) = y$ where y solves

$$\begin{cases} y_{tt} - \Delta y = -f(\hat{y}) & \text{in } Q_T, \\ y = 0, \quad y_t = v \mathbf{1}_{\Gamma_0} & \text{in } \partial\Omega \times (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \\ (y(\cdot, T), y_t(\cdot, T)) = (0, 0) & \text{in } \Omega, \end{cases} \quad (9)$$

and (y, v) corresponds to the minimizer of a functional J_s

$$J_s(y, v) := s \int_{Q_T} \rho^2(s) y^2 + \int_{\Gamma_0} \int_0^T \eta^{-2} \rho_1^2(s) v^2 \quad (10)$$

$\rho(s), \rho_1(s) \approx e^{s\phi(x,t)}$ are Carleman weights; $s > 0$ is a Carleman parameter;

Theorem (Bhandari, Lempire, M. D., Lempire, Sue, M. D., 2019)

Let $\Omega \subset \mathbb{R}^d$; assume T, Γ_0 are large enough, and f Loc Lip satisfies

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If $s > 0$ is large enough, then Λ_s is contracting.

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Part 2: (Space-time Numerical) approximation of null controls for linear wave equation

$$\partial_{tt}y - \Delta y + Ay = v1_\omega + B \quad \text{in } Q_T, \quad (12)$$

- control of minimal $L^2(q_T)$ norm

$$\inf J(y, v) = \int_{q_T} v^2, \quad (y, v) \text{ solves } (12)$$

- state-control pair with weighted cost :

$$\inf J(y, v) = \int_{Q_T} \rho^2 y^2 + \int_{q_T} \rho_1^2 v^2, \quad (y, v) \text{ solves } (12)$$

Part 2: (space-time Numerical) approximation of null controls for linear wave equation

- **Burman, Feizmohammadi, M, Oksanen.** *Space-time stabilized finite element methods for a unique continuation problem subject to the wave equation*, **M2AN 2021**
- **Montaner, M.** *Approximation of controls for the linear wave equation: a first order mixed formulation*, **AIMS MCRF 2019**
- **Cindea, M.** *A mixed formulation for the direct approximation of the control of minimal L^2 -norm for linear type wave equations*, **Calcolo 2015**
- **Cindea, Fernandez-Cara, M, .** *Numerical controllability of the wave equation through primal methods and Carleman estimates*, **ESAIM COCV 2013**
- **M, .** *A uniformly controllable and implicit scheme for the 1-D wave equation*, **ESAIM M2AN 2005**

$$\begin{cases} \text{Minimize } J(y, v) = \frac{1}{2} \int_0^T \int_{\Gamma_0} |v|^2 dt \\ \text{Subject to } (y, v) \in \mathcal{C}(y_0, y_1; T) \end{cases} \quad (13)$$

where $\mathcal{C}(y_0, y_1; T)$ denotes the non-empty linear manifold

$$\mathcal{C}(y_0, y_1; T) = \{ (y, v) : v \in L^2(\Gamma_0 \times (0, T)), y \text{ solves (12)} \}.$$

Using the **Fenchel-Rockafellar theorem** [Ekeland-Temam 74], [Brezis 84] we get that

$$\inf_{(y, v) \in \mathcal{C}(y_0, y_1; T)} J(y, v) = - \min_{(\varphi_0, \varphi_1) \in \mathcal{V}} J^*(\varphi_0, \varphi_1)$$

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Negative Commutation diagram

The stable/constant centered finite difference scheme with $\Delta t < h$:

$$(\bar{S}_{h,\Delta t}) \begin{cases} \Delta_{\Delta t} y_{h,\Delta t} - \Delta_h y_{h,\Delta t} = 0, \\ + \text{Initial conditions and Boundary terms} \end{cases} \quad (15)$$

produces a non discrete uniformly bounded and converging control under the condition $\Delta t < h$.

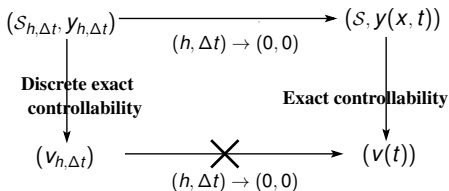


Figure: Non commuting diagram associated to the scheme $(\bar{S}_{h,\Delta t})$ for $\Delta t < h$.

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } Q_T, \\ y(0, t) = 0, y(1, t) = v(t) & \text{on } (0, T), \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (4x 1_{(0,1/2)}(x), 0) & \text{in } \Omega, \end{cases} \quad (16)$$

The control of minimal $L^2(0, T)$ norm is $v(t) = 2(1-t)1_{(1/2,3/2)}(t)$.

The corresponding controlled solution is

$$y(x, t) = \begin{cases} 4x & 0 \leq x+t < \frac{1}{2}, \\ 2(x-t) & -\frac{1}{2} < t-x < \frac{1}{2}, \quad x+t \geq \frac{1}{2}, \\ 0 & \text{elsewhere,} \end{cases} \quad (17)$$

The initial condition of the adjoint solution is

$(\phi_0, \phi_1) = (0, -2x 1_{(0,1/2)}(x)) \in H^1(\Omega) \times H^0(\Omega)$, which gives:

$$\phi(x, t) = \begin{cases} -2xt & 0 \leq x+t < \frac{1}{2}, \quad x \geq 0, t \geq 0, \\ \frac{(x-t)^2}{2} - \frac{1}{8} & \frac{1}{2} \leq x+t < \frac{3}{2}, \quad -\frac{1}{2} < x-t < \frac{1}{2}, \\ 2(x-1)(1-t) & \frac{3}{2} \leq x+t, \quad -\frac{1}{2} < x-t, \\ -\frac{(x+t-2)^2}{2} + \frac{1}{8} & \frac{3}{2} < x+t < \frac{5}{2}, \quad -\frac{3}{2} < x-t \leq -\frac{1}{2}, \\ 2x(2-t) & x-t \leq -\frac{3}{2}. \end{cases} \quad (18)$$

Usual scheme - control

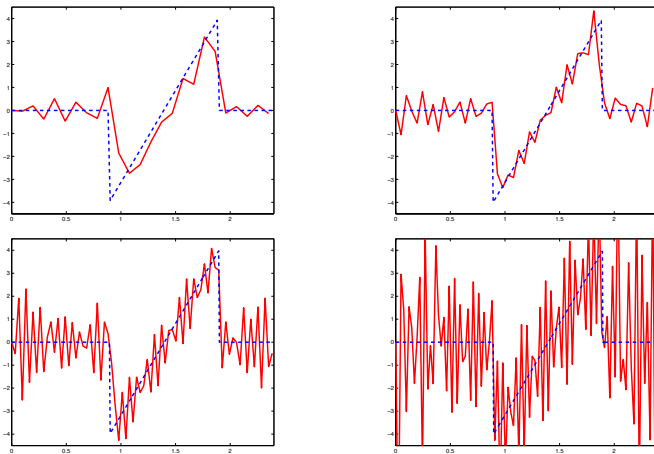


Figure: Control $P(\mathbf{v}_h)(t)$ vs. $t \in [0, T]$, $\Delta t = 0.98h$, $T = 2.4$ and $h = 1/10, h = 1/20, h = 1/30, h = 1/40$.

Minimization of J^* w.r.t. φ

We now replace the problem

$$\begin{cases} \text{Min } J^*(\varphi_0, \varphi_1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt + \langle y_0, \varphi_1 \rangle_{L^2} - \langle y_1, \varphi_0 \rangle_{H^{-1}, H_0^1} \\ \text{Subject to } (\varphi_0, \varphi_1) \in \mathbf{V} = H_0^1(\Omega) \times L^2(\Omega) \quad \text{where } L\varphi = 0 \end{cases} \quad (19)$$

by the equivalent problem

$$\begin{cases} \min J^*(\varphi) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt + \langle y_0, \varphi_t(\cdot, 0) \rangle_{L^2} - \langle y_1, \varphi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \\ \text{Subject to } \varphi \in \mathbf{W} := \left\{ \varphi : \varphi \in C^0(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), L\varphi = 0 \in L^2(Q_T) \right\} \end{cases} \quad (20)$$

Remark- If $\varphi \in \mathbf{W}$ then $\frac{\partial \varphi}{\partial \nu} \in L^2(\Gamma_T)$

Remark- $\mathbf{W}$ endowed with the norm $\|\varphi\|_{\mathbf{W}} := \left\| \frac{\partial \varphi}{\partial \nu} \right\|_{L^2(\Gamma_T)}$ is an Hilbert space.

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$$\begin{cases} \text{Min } J^*(\varphi_0, \varphi_1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt + \langle y_0, \varphi_1 \rangle_{L^2} - \langle y_1, \varphi_0 \rangle_{H^{-1}, H_0^1} \\ \text{Subject to } (\varphi_0, \varphi_1) \in \mathbf{V} = H_0^1(\Omega) \times L^2(\Omega) \quad \text{where } L\varphi = 0 \end{cases} \quad (19)$$

by the equivalent problem

$$\begin{cases} \min J^*(\varphi) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt + \langle y_0, \varphi_t(\cdot, 0) \rangle_{L^2} - \langle y_1, \varphi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \\ \text{Subject to } \varphi \in \mathbf{W} := \left\{ \varphi : \varphi \in C^0(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), L\varphi = 0 \in L^2(Q_T) \right\} \end{cases} \quad (20)$$

Remark- If $\varphi \in \mathbf{W}$ then $\frac{\partial \varphi}{\partial \nu} \in L^2(\Gamma_T)$

Remark- $\mathbf{W}$ endowed with the norm $\|\varphi\|_{\mathbf{W}} := \left\| \frac{\partial \varphi}{\partial \nu} \right\|_{L^2(\Gamma_T)}$ is an Hilbert space.

Minimization of J^* w.r.t. φ

We assume T and Γ_0 large enough. We now replace the problem

$$\begin{cases} \text{Min } J^*(\varphi_0, \varphi_1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt + \langle y_0, \varphi_1 \rangle_{L^2} - \langle y_1, \varphi_0 \rangle_{H^{-1}, H_0^1} \\ \text{Subject to } (\varphi_0, \varphi_1) \in \mathbf{V} = H_0^1(\Omega) \times L^2(\Omega) \text{ where } L\varphi = 0 \end{cases} \quad (21)$$

by the equivalent problem

$$\begin{cases} \min J_r^*(\varphi) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt + \frac{r}{2} \|L\varphi\|_{L^2(Q_T)}^2 + \langle y_0, \varphi_t(\cdot, 0) \rangle_{L^2} - \langle y_1, \varphi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \\ \text{Subject to } \varphi \in \mathbf{W} := \left\{ \varphi : \varphi \in C^0(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), L\varphi = 0 \in L^2(Q_T) \right\} \end{cases} \quad (22)$$

for all $r \geq 0$.

Remark- If $\varphi \in \mathbf{W}$ then $\frac{\partial \varphi}{\partial \nu} \in L^2(\Gamma_T)$

Remark- $\mathbf{W}$ endowed with the norm $\|\varphi\|_{\mathbf{W}} := \left\| \frac{\partial \varphi}{\partial \nu} \right\|_{L^2(\Gamma_T)}$ is an Hilbert space.

In order to address the $L^2(Q_T)$ constraint $L\varphi = 0$, we introduce a **Lagrange multiplier** $\lambda \in L^2(Q_T)$; we consider the **saddle point problem** :

$$\left\{ \begin{array}{l} \sup_{\lambda \in L^2(Q_T)} \inf_{\varphi \in \Phi} \mathcal{L}_r(\varphi, \lambda), \\ \mathcal{L}_r(\varphi, \lambda) := J_r(\varphi) + \langle L\varphi, \lambda \rangle_{L^2(Q_T)} \\ \Phi := \left\{ \varphi : \varphi \in C^0(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), L\varphi \in L^2(Q_T) \right\} \supset W \end{array} \right. \quad (23)$$

Remark- Φ is endowed with the inner product,

$$\langle \varphi, \bar{\varphi} \rangle_{\Phi} := \left\langle \frac{\partial \varphi}{\partial \nu}, \frac{\partial \bar{\varphi}}{\partial \nu} \right\rangle_{L^2(\Gamma_T)} + \langle L\varphi, L\bar{\varphi} \rangle_{L^2(Q_T)}, \quad \forall \varphi, \bar{\varphi} \in \Phi.$$

$\|\varphi\|_{\Phi} := \sqrt{\langle \varphi, \varphi \rangle_{\Phi}}$ is a norm and $(\Phi, \|\cdot\|_{\Phi})$ is an Hilbert space.

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$\|\varphi\|_{\Phi} := \sqrt{\langle \varphi, \varphi \rangle_{\Phi}}$ is a norm and $(\Phi, \|\cdot\|_{\Phi})$ is an Hilbert space.

Find $(\varphi, \lambda) \in \Phi \times L^2(Q_T)$ solution of

$$\begin{cases} a_r(\varphi, \bar{\varphi}) + b(\bar{\varphi}, \lambda) &= I(\bar{\varphi}), & \forall \bar{\varphi} \in \Phi \\ b(\varphi, \bar{\lambda}) &= 0, & \forall \bar{\lambda} \in L^2(Q_T), \end{cases} \quad (24)$$

where

$$a_r : \Phi \times \Phi \rightarrow \mathbb{R}, \quad a_r(\varphi, \bar{\varphi}) = \left\langle \frac{\partial \varphi}{\partial \nu}, \frac{\partial \bar{\varphi}}{\partial \nu} \right\rangle_{L^2(\Gamma_T)} + r \langle L\varphi, L\bar{\varphi} \rangle_{L^2(Q_T)} \quad (25)$$

$$b : \Phi \times L^2(Q_T) \rightarrow \mathbb{R}, \quad b(\varphi, \lambda) = \langle L\varphi, \lambda \rangle_{L^2(Q_T)} \quad (26)$$

$$I : \Phi \rightarrow \mathbb{R}, \quad I(\varphi) = - \langle y_0, \varphi_t(\cdot, 0) \rangle_{L^2} + \langle y_1, \varphi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \quad (27)$$

Rk. The continuity of the linear form I derives from generalized observability ineq.

$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_{\mathbf{V}}^2 \leq C_{obs} \left(\left\| \frac{\partial \varphi}{\partial \nu} \right\|_{L^2(\Gamma_T)}^2 + \|L\varphi\|_{L^2(Q_T)}^2 \right), \quad \forall \varphi \in \Phi \quad (28)$$

Conformal Approximation

Let then Φ_h and Λ_h be two finite dimensional spaces parametrized by the variable h such that

$$\Phi_h \subset \Phi, \quad \Lambda_h \subset L^2(Q_T), \quad \forall h > 0.$$

Then, we can introduce the following approximated problems : find $(\varphi_h, \lambda_h) \in \Phi_h \times \Lambda_h$ solution of

$$\begin{cases} a_r(\varphi_h, \bar{\varphi}_h) + b(\bar{\varphi}_h, \lambda_h) &= I(\bar{\varphi}_h), & \forall \bar{\varphi}_h \in \Phi_h \\ b(\varphi_h, \bar{\lambda}_h) &= 0, & \forall \bar{\lambda}_h \in \Lambda_h. \end{cases} \quad (29)$$

For any $h > 0$, the well-posedness is again a consequence of two properties

the coercivity of the bilinear form a_r on the subset

$\mathcal{N}_h(b) = \{\varphi_h \in \Phi_h; b(\varphi_h, \lambda_h) = 0 \quad \forall \lambda_h \in \Lambda_h\}$. From the relation

$$a_r(\varphi, \varphi) \geq \frac{r}{\eta} \|\varphi\|_{\Phi}^2, \quad \forall \varphi \in \Phi$$

the form a_r is coercive on the full space Φ , and so *a fortiori* on $\mathcal{N}_h(b) \subset \Phi_h \subset \Phi$.

The second property is a discrete inf-sup condition : there exists $\delta > 0$ such that

$$\delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_h} \|\lambda_h\|_{\Lambda_h}} \geq \delta. \quad (30)$$

A necessary condition is: $\dim(\Phi_h) > \dim(\Lambda_h)$

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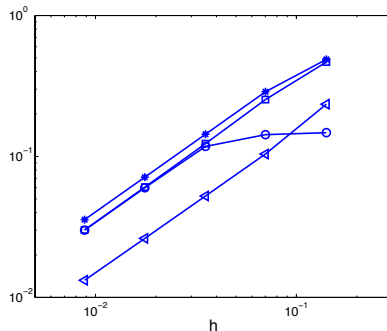


Figure: BFS finite element - Evolution of $\sqrt{r}\delta_{h,r}$ with respect to h for $r = 1$ ($\square$), $r = 10^{-2}$ ($\circ$), $r = h$ ($\star$) and $r = h^2$ ($\triangleleft$).

$$\delta_h \approx C_r \frac{h}{\sqrt{r}} \quad \text{as } h \rightarrow 0^+ \quad (31)$$

Stabilized mixed formulation "à la Barbosa-Hughes"

1

$\alpha > 0$

$$\begin{cases} \sup_{\lambda \in \Lambda} \inf_{\varphi \in \Phi} \mathcal{L}_{r,\alpha}(\varphi, \lambda), \\ \mathcal{L}_{r,\alpha}(\varphi, \lambda) := \mathcal{L}_r(\varphi, \lambda) - \frac{\alpha}{2} \|L\lambda\|_{L^2(H^{-1}(\Omega))}^2 - \frac{\alpha}{2} \|\lambda - \partial_\nu \varphi\|_{L^2(\Gamma_T)}^2. \end{cases} \quad (32)$$

$$\Lambda := \left\{ \lambda : \lambda \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega)), \right. \\ \left. L\lambda \in L^2([0, T]; H^{-1}(\Omega)), \lambda(\cdot, 0) = \lambda_t(\cdot, 0) = 0, \lambda|_{\Gamma_T} \in L^2(\Gamma_T) \right\}.$$

Λ is a Hilbert space endowed with the following inner product

$$\langle \lambda, \bar{\lambda} \rangle_\Lambda := \int_0^T \langle L\lambda(t), L\bar{\lambda}(t) \rangle_{H^{-1}(\Omega)} dt + \iint_{\Gamma_T} \lambda \bar{\lambda} d\sigma dt, \quad \forall \lambda, \bar{\lambda} \in \Lambda$$

using notably that

$$\|\lambda\|_{L^2(Q_T)} \leq C_{\Omega, T} \sqrt{\langle \lambda, \lambda \rangle_\Lambda}, \quad \forall \lambda \in \Lambda \quad (33)$$

for some positive constant $C_{\Omega, T}$. We denote $\|\lambda\|_\Lambda := \sqrt{\langle \lambda, \lambda \rangle_\Lambda}$.

¹H. Barbosa, T. Hugues : **The finite element method with Lagrange multipliers on the boundary: circumventing the Babuska-Brezzi condition**, 1991

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¹H. Barbosa, T. Hugues : [The finite element method with Lagrange multipliers on the boundary: circumventing the Babuska-Brezzi condition](#), 1991

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } Q_T, \\ y(0, t) = 0, y(1, t) = v(t) & \text{on } (0, T), \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (4x1_{(0,1/2)}(x), 0) & \text{in } \Omega, \end{cases} \quad (34)$$

The control of minimal $L^2(0, T)$ norm is $v(t) = 2(1 - t)1_{(1/2, 3/2)}(t)$.

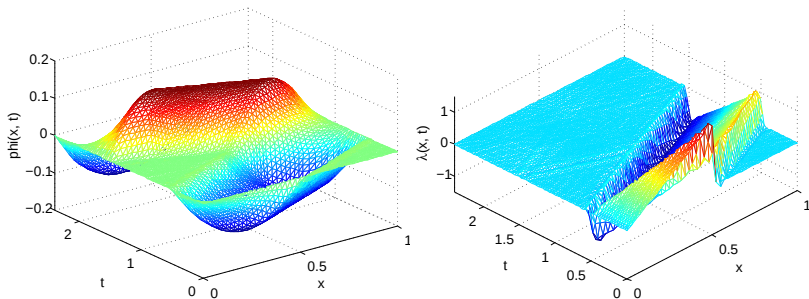


Figure: The dual variable φ_h (Left) and primal variable λ_h (Right) in Q_T ; $h = 2.46 \times 10^{-2}$; $r = 10^{-2}$.

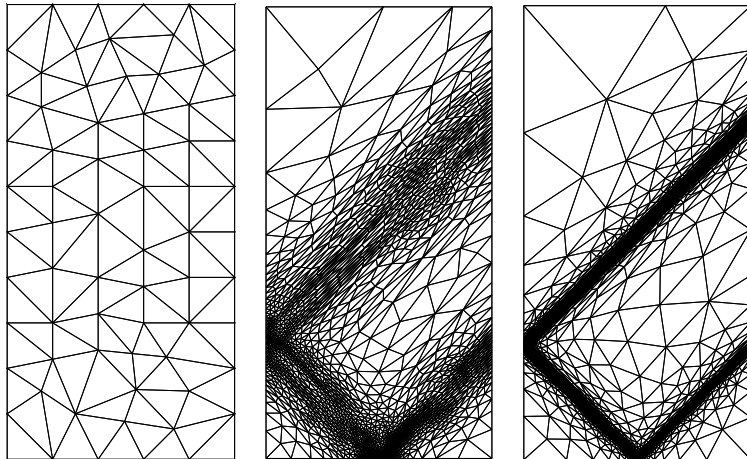


Figure: Iterative refinement of the triangular mesh over Q_T with respect to the variable λ_h : 110, 2 880 and 8 636 triangles.

Example 1 - $N = 1$ - Numerical experiments

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } Q_T, \\ y(0, t) = 0, y(1, t) = v(t) & \text{on } (0, T), \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (4x1_{(0,1/2)}(x), 0) & \text{in } \Omega, \end{cases} \quad (35)$$

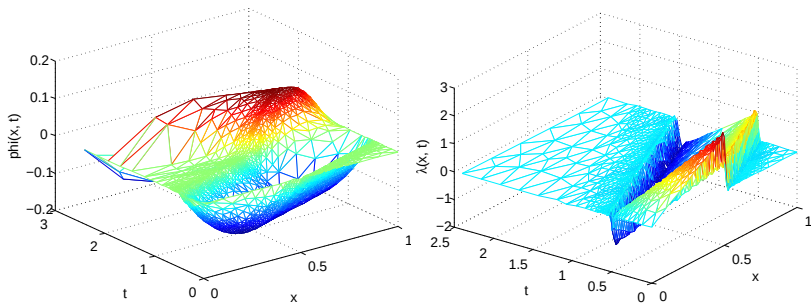


Figure: The variable φ_h and λ_h in Q_T corresponding to the finer mesh; $r = 2 \times 10^{-3}$.

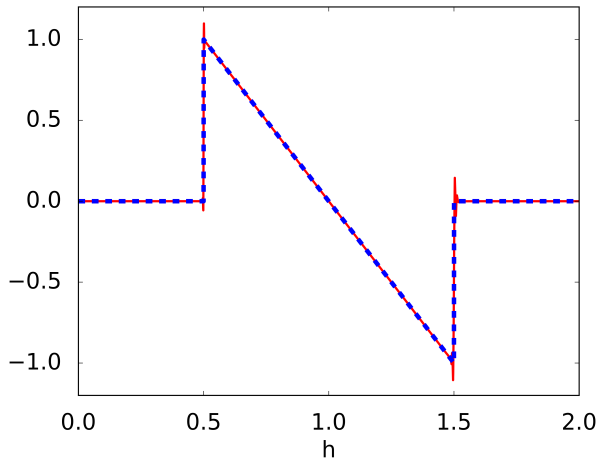


Figure: Control of minimal L^2 -norm v (dashed blue line) and its approximation $\lambda_h(1, \cdot)$ (red line) on $(0, T)$. Third adapted mesh, $r = 10^{-6}$.

The approach leads to a simple and short Freefem++ code !!

```
1 real L = 1; // Size of the spatial domain
2 int N = 10; // Fineness of the mesh
3 real T = 2; // Final time
4 mesh Th = square(N,2*N,[L*x,T*y]); //Uniform space-time mesh
5 fespace Wh(Th,P2); //P2 Finite Element Space for the Dual variables
6 fespace Mh(Th,P1); //P1 Finite Element Space for the Primal variables
7 Wh w1, q1; //Declaration of the Dual variables (solution functions)
8 Wh w2, q2; //Declaration of the Dual variables (test functions)
9 Mh lambda1,mu1; //Declaration of the Primal variables (solution functions)
10 Mh lambda2,mu2; //Declaration of the Primal variables (test functions)
11 func u0 = 4*x*(x>=0 && x<0.5); //Initial data to be controlled
12 real r = 10e-2; // Augmentation parameter 'r'
13 real alpha = 10e-2; // Stabilization parameter 'alpha'
14
15 //Definition of the stabilized mixed variational formulation
16 problem ControlWave([w1,q1,lambda1,mu1],[w2,q2,lambda2,mu2]) =
17     // Initial conjugate functional terms
18     int1d(Th,2) (q1*q2) - int1d(Th,1) (u0*w2)
19
20     // Primal-dual bilinear terms
21     + int2d(Th) ((dy(w2)-dx(q2))*lambda1 + (dy(q2)-dx(w2))*mu1
22                 + (dy(w1)-dx(q1))*lambda2 + (dy(q1)-dx(w1))*mu2)
23
24     // Augmentation terms
25     + int2d(Th) (r*((dy(w1)-dx(q1))*(dy(w2)-dx(q2)) + (dy(q1)-dx(w1))*(dy(q2)-dx(w2))))
26
27     // Stabilization terms
28     - int1d(Th,2) (alpha*q1*q2 + alpha*(q2*lambda1 + q1*lambda2))
29     - int1d(Th,2,4) (alpha*lambda1*lambda2)
30     - int2d(Th) (alpha*((dy(lambda1)-dx(mu1))*(dy(lambda2)-dx(mu2))
31                 + (dy(mu1)-dx(lambda1))*(dy(mu2)-dx(lambda2))))
32
```

```
33 // Boundary conditions
34 +on(2,w1=0)+on(4,w1=0) + on(4,lambd1=0);
35
36 //The following instruction solves the mixed formulation
37 ControlWave;
38 //The solution for the dual variables are stored in (w1,q1)
39 //The solution for the primal variables are stored in (lambd1,mu1)
```


Non conformal approximation

Stabilization technics may also be employed in the context of non-conformal approximations. Let

$$V_h^q = \{p_h \in C(\overline{Q_T}); (p_h)|_K \in \mathbb{P}_q(K), \forall K \in \mathcal{T}_h\}$$

and consider the discrete Lagrangian $\mathcal{L}_h : V_h^p \times V_h^q \rightarrow \mathbb{R}$, given by

$$\begin{aligned} \mathcal{L}_h(\phi_h, \lambda_h) := & J^*(\phi_h) + \frac{h^2}{2} \|L\phi_h\|_{L^2(Q_T)}^2 + \frac{h}{2} \sum_{K \in \mathcal{T}_h} \int_{\partial K} [[\partial_\nu \phi_h]]^2 + h^{-1} \int_{\Sigma_T} \phi_h^2 \\ & + \int_{Q_T} (-\partial_t \phi_h \partial_t \lambda_h + \nabla \phi_h \nabla \lambda_h) - \frac{h}{2} \sum_{K \in \mathcal{T}_h} \int_{\partial K} [[\partial_\nu \lambda_h]]^2 - \frac{h^2}{2} \|L\lambda_h\|_{L^2(Q_T)}^2 \\ & - h^{-1} \int_{\Sigma_T} \lambda_h^2 - \frac{h^2}{2} \|\lambda_h - \chi \partial_\nu \phi_h\|_{L^2(Q_T)}^2 \end{aligned}$$

$[[\partial_\nu \phi_h]]$ denotes the jump of the normal derivative of ϕ_h across the internal edges of the triangulation.

The terms $h^2 \|L\phi_h\|_{L^2(Q_T)}^2$ and $-h^2 \|L\lambda_h\|_{L^2(Q_T)}^2$ play a symmetric role. Both vanish at the continuous level. The jump terms somehow aim to control the regularity of the approximation.

2

Moreover, if the saddle-point (λ, ϕ) of $\mathcal{L}_r$ is smooth enough, then the following approximation result holds true

Theorem (Burman, Feizmohammadi, M, Oksanen 2022)

Assume the geometric control condition. Let $p, q \geq 1$ and $h > 0$. Let $(\lambda_h, \phi_h) \in V_h^p \times V_h^q$ be the saddle point of $\mathcal{L}_h$ and assume that the saddle point (λ, ϕ) of $\mathcal{L}_r$ belongs to $H^{p+1}(Q_T) \times H^{q+1}(Q_T)$. Then, there exists a positive constant C independent of h such that

$$\|\chi(\partial_\nu \phi - \partial_\nu \phi_h)\|_{L^2(\Sigma_T)} \leq C(h^{p+\frac{1}{2}} \|u\|_{H^{p+1}(Q_T)} + h^{q-\frac{1}{2}} \|\phi\|_{H^{q+1}(Q_T)}), \quad (36)$$

where χ is a cut-off function $\chi(x, t) = \chi_0(x)\chi_1(t)$, with $\chi_0 \in C_0^\infty(\omega)$, $\chi_1 \in C_0^\infty(0, T)$.

If $(u_0, u_1) \in H^{k+1}(\Omega) \times H^k(\Omega)$ satisfies the compatibility conditions of order k at $\partial\Omega \times \{0\}$. then the solution (u, ϕ) satisfies

$$(u, \phi) \in H^{k+1}(Q_T) \times H^{k+2}(Q_T).$$

²Burman, Feizmohammadi, Munch, Oksanen, Spacetime finite element methods for control problems subject to the wave equation, Arxiv2021

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } Q_T, \\ y(0, t) = 0, y(1, t) = v(t) & \text{on } (0, T), \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (4x1_{(0,1/2)}(x), 0) & \text{in } \Omega, \end{cases} \quad (37)$$

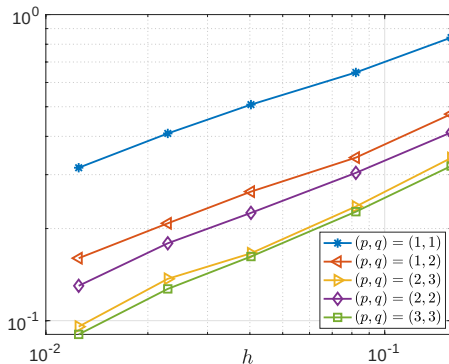


Figure: Relative error on the approximation of the boundary control $\frac{\|\partial_\nu \phi_h(1, \cdot) - v\|_{L^2(0, T)}}{\|v\|_{L^2(0, T)}}$ with respect to h for different approximations.

Non conformal approximation

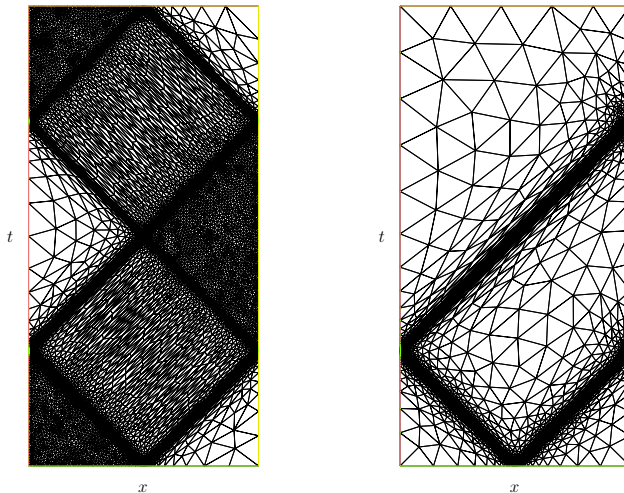


Figure: Locally refine spacetime meshes with respect to ϕ_h (Left) and λ_h (Right).

Theorem

$$\sup_{\lambda \in L^2} \inf_{\varphi \in \Phi} \mathcal{L}_r(\varphi, \lambda) = - \inf_{\lambda \in L^2} J_r^{**}(\lambda) + \mathcal{L}_r(\varphi_0, 0)$$

where $\varphi_0 \in \Phi$ solves $a_r(\varphi_0, \bar{\varphi}) = l(\bar{\varphi}), \forall \bar{\varphi} \in \Phi$ and $J_r^{**} : L^2 \rightarrow \mathbb{R}$ defined by

$$J_r^{**}(\lambda) := \frac{1}{2} \langle \mathcal{P}_r \lambda, \lambda \rangle_{L^2(Q_T)} - b(\varphi_0, \lambda)$$

Lemma

Let $\mathcal{P}_r$ be the linear operator from L^2 into L^2 defined by

$$\mathcal{P}_r \lambda := L\varphi, \quad \forall \lambda \in L^2 \quad \text{where} \quad \varphi \in \Phi \quad \text{solves} \quad a_r(\varphi, \bar{\varphi}) = b(\bar{\varphi}, \lambda), \quad \forall \bar{\varphi} \in \Phi.$$

For any $r > 0$, the operator $\mathcal{P}_r$ is a **strongly elliptic, symmetric** isomorphism from L^2 into L^2 .

Rk. The control problem is reduced to the minimization of an **unconstrained** functional with respect to the control state within a space-time framework!!!

Theorem

$$\sup_{\lambda \in L^2} \inf_{\varphi \in \Phi} \mathcal{L}_r(\varphi, \lambda) = - \inf_{\lambda \in L^2} J_r^{**}(\lambda) + \mathcal{L}_r(\varphi_0, 0)$$

where $\varphi_0 \in \Phi$ solves $a_r(\varphi_0, \bar{\varphi}) = l(\bar{\varphi}), \forall \bar{\varphi} \in \Phi$ and $J_r^{**} : L^2 \rightarrow \mathbb{R}$ defined by

$$J_r^{**}(\lambda) := \frac{1}{2} \langle \mathcal{P}_r \lambda, \lambda \rangle_{L^2(Q_T)} - b(\varphi_0, \lambda)$$

Lemma

Let $\mathcal{P}_r$ be the linear operator from L^2 into L^2 defined by

$$\mathcal{P}_r \lambda := L\varphi, \quad \forall \lambda \in L^2 \quad \text{where} \quad \varphi \in \Phi \quad \text{solves} \quad a_r(\varphi, \bar{\varphi}) = b(\bar{\varphi}, \lambda), \quad \forall \bar{\varphi} \in \Phi.$$

For any $r > 0$, the operator $\mathcal{P}_r$ is a **strongly elliptic, symmetric** isomorphism from L^2 into L^2 .

Rk. The control problem is reduced to the minimization of an **unconstrained** functional with respect to the control state within a space-time framework!!!

The situation is much simpler with cost involving both y and v

$$\left\{ \begin{array}{l} \text{Minimize } J(y, v) = \frac{1}{2} \iint_{Q_T} |y|^2 dx dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} |v|^2 d\sigma dt \\ \text{Subject to } (y, v) \in \mathcal{C}(y_0, y_1; T) \end{array} \right. \quad (38)$$

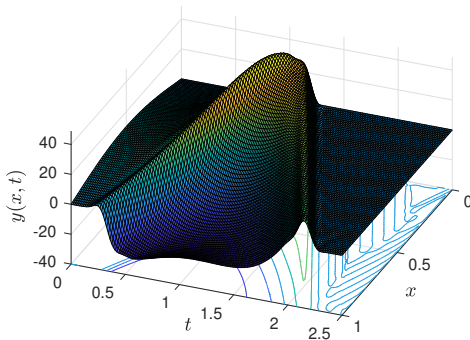
$$v = \frac{\partial \varphi}{\partial \nu} \text{ in } (0, T) \times \Gamma_0 \text{ and } y = L^* \varphi \text{ in } Q_T.$$

$$\left\{ \begin{array}{l} \text{Minimize } J^*(\varphi) = \frac{1}{2} \iint_{Q_T} |L\varphi|^2 dx dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt \\ \quad + \langle (\varphi(\cdot, 0), \varphi_t(\cdot, 0)), (y_0, y_1) \rangle \\ \text{Subject to } \varphi \in \Phi \end{array} \right. \quad (39)$$

$\Phi := \left\{ \varphi : \varphi \in C^0(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), L\varphi \in L^2(Q_T) \right\}$ is endowed with the inner product, $\langle \varphi, \bar{\varphi} \rangle_{\Phi} := \langle \frac{\partial \varphi}{\partial \nu}, \frac{\partial \bar{\varphi}}{\partial \nu} \rangle_{L^2(\Gamma_T)} + \langle L\varphi, L\bar{\varphi} \rangle_{L^2(Q_T)}, \quad \forall \varphi, \bar{\varphi} \in \Phi.$

$$\begin{cases} \partial_{tt}y - \Delta y - 3y(1 + \ln^{3/2}(2 + |y|)) = 0 & \text{in } (0, 1) \times (0, 2.5), \\ y(0, t) = 0, y(1, t) = v(t) & \text{on } (0, 2.5), \\ (y(\cdot, 0), \partial_t y(\cdot, 0)) = (10\sin(\pi x), 0) & \text{in } (0, 1), \end{cases} \quad (40)$$

$$J(y, v) = s \int_0^T \rho_1(s, t)^2 v^2(t) dt + s \int_{Q_T} \rho_s^2(x, t) y^2(t) dt$$



Controlled solution in $(0, 1) \times (0, T)$

Similarly arguments apply for the heat equation

Part 1

- Lemoine, Ervedoza, M. *Exact controllability of semilinear heat equations through a constructive approach*, AIMS EECT, 2023
- Lemoine, M. *Constructive exact control of semilinear 1D heat equations*, AIMS MCRF, 2022
- Gayte-Marin Lemoine, M. *Approximation of null controls for semilinear heat equations using a least-squares approach*, ESAIM COCV 2021
- Bhandari, Lemoine, M. *Constructive exact control of semilinear 1D heat equations*, arxiv

Part 2

- De Souza, Fernandez-Cara, Lemoine, M. *On the numerical controllability of the two-dimensional heat, Stokes and Navier-Stokes equations*, J. Scientific computing, 2017
- De Souza, M. *A mixed formulation for the direct approximation of the control of minimal L^2 -weighted norm for the linear heat equation*, Advances in Computational Mathematics, 2016
- Fernandez-Cara, M. *Numerical null controllability of the 1D heat equation: Duality and Carleman weights*, JOTA 2013
- Fernandez-Cara, M. *Strong convergent approximations of null controls for the heat equation*, SEMA 2013

Numerical illustration for the heat eq. : Part 1 + Part 2: $y_{k+1} = \Lambda_s(y_k)$

$$\begin{cases} \partial_t y - \Delta y - 5y(1 + \ln^{3/2}(2 + |y|)) = v \mathbf{1}_{(0.2, 0.8)} & \text{in } (0, 1) \times (0, 0.5), \\ y = 0 & \text{on } \{0, 1\} \times (0, 0.5), \\ y(\cdot, 0) = 10 \sin(\pi x) & \text{in } \Omega, \end{cases} \quad (41)$$

$$J(y, v) = s \int_0^T \rho_1(s, t)^2 v^2(t) dt + s \int_{Q_T} \rho_s^2(x, t) y^2(t) dt$$

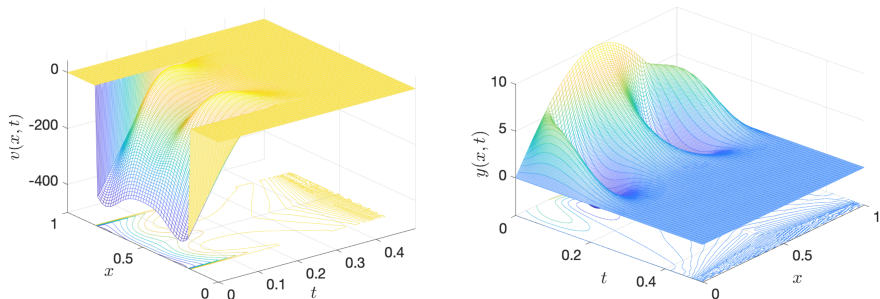


Figure: The control v (Left) and controlled state y (right) in Q_T .

Question to end : Can we use (space-time) DDM to approximate null controls ????

$$\left\{ \begin{array}{l} \text{Minimize } J^*(\varphi) = \frac{1}{2} \iint_{Q_T} |L\varphi|^2 dx dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt \\ \quad + \langle (\varphi(\cdot, 0), \varphi_t(\cdot, 0)), (y_0, y_1) \rangle > \\ \text{Subject to } \varphi \in \Phi \end{array} \right. \quad (42)$$

The corresponding VF is: find $\varphi \in \Phi$ such that

$$\int_{Q_T} L\varphi L\bar{\varphi} + \int_{0^T} \int_{\Gamma_0} \frac{\partial \varphi}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} = \langle (\varphi(\cdot, 0), \varphi_t(\cdot, 0)), (y_0, y_1) \rangle, \quad \forall \bar{\varphi} \in \Phi$$

The corresponding boundary value problem is ($L := \partial_{tt} - \partial_{xx}$, $\Omega = (0, 1)$)

$$\left\{ \begin{array}{ll} L(L\varphi) = 0, & Q_T, \\ \varphi(0, t) = 0, L\varphi(0, t) = 0, & (0, T) \\ \varphi(1, t) = 0, L\varphi(1, t) + \varphi_x(1, t) = 0, & (0, T) \\ L\varphi(x, 0) = y_0, (L\varphi(x, 0))_t = y_1 & (0, 1) \\ L\varphi(x, T) = 0, (L\varphi(x, T))_t = 0 & (0, 1) \end{array} \right. \quad (43)$$

Question to end : Can we use (space-time) DDM to approximate controls ????

The corresponding boundary value problem is ($L := \partial_{tt} - \partial_{xx}$)

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or (equivalently)

$$\begin{cases} Ly = 0, \quad L\varphi = y, & Q_T, \\ \varphi(0, t) = 0, y(0, t) = 0, & (0, T) \\ \varphi(1, t) = 0, y(1, t) - \varphi_x(1, t) = 0, & (0, T) \\ y(x, 0) = y_0, y_t(x, 0) = y_1 & (0, 1) \\ y(x, T) = 0, y_t(x, T) = 0 & (0, 1) \end{cases} \quad (45)$$

Thank you for your attention !!!

Question to end : Can we use (space-time) DDM to approximate controls ????

The corresponding boundary value problem is ($L := \partial_{tt} - \partial_{xx}$)

$$\begin{cases} L(L\varphi) = 0, & Q_T, \\ \varphi(0, t) = 0, L\varphi(0, t) = 0, & (0, T) \\ \varphi(1, t) = 0, L\varphi(1, t) + \varphi_x(1, t) = 0, & (0, T) \\ L\varphi(x, 0) = y_0, (L\varphi(x, 0))_t = y_1 & (0, 1) \\ L\varphi(x, T) = 0, (L\varphi(x, T))_t = 0 & (0, 1) \end{cases} \quad (44)$$

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Thank you for your attention !!!