Necessary optimality conditions for optimal control problems with non-control regions

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Keywords: Optimal control, hybrid maximum principle, non-control regions.

Abstract: We consider a smooth control system that is subject to loss of control in the sense that the state space is partitioned into several disjoint regions and, in each region, either the system can be controlled, as usual, in a permanent way (that is, one can change the value of the control at any real time), or, on the contrary, the control has to remain constant from the entry time into the region until the exit time. The latter case corresponds to a non-control region. The goal is to state the necessary optimality conditions for a Mayer optimal control problem in such a setting of loss of control, which will be based on hybrid maximum principle.

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Domain Decomposition Method in Time Direction for Transport Control

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Abstract

We study the Optimized Schwarz method based on a domain decomposition in time direction (described in [2]) for control problems for the transport equation. We consider in particular internal control where the optimization problem can be transformed into a PDE system using Lagrangian (as in [1]) which unconditionally guarantee the controllability. The DD technique in time direction is then applied on this PDE system. Under Fourier analysis, we find optimal parameters for continuous and discrete cases. We illustrate the phenomena by some numerical tests.

References

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*Speaker

Relationships Between the Maximum Principle and Dynamic Programming for Infinite Dimensional Stochastic Control Systems

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Key Words: Pontryagin type maximum principle, dynamic programming, stochastic optimal control, stochastic distributed parameter systems, stochastic verification theorem

Abstract: The Pontryagin type maximum principle and Bellman's dynamic programming principle serve as two of the most important tools in solving optimal control problems. There is a huge literature on the study of relationship between them. In this talk, I will present the latest progress about the relationship between the Pontryagin type maximum principle and the dynamic programming principle for control systems governed by stochastic evolution equations in infinite dimensional space, with the control variables entering into both the drift and the diffusion terms. To do so, we first prove the dynamic programming principle for those systems without employing the martingale solutions. Then we establish the desired relationships in both cases that value function associated is smooth or nonsmooth. For the nonsmooth case, in particular, by employing the relaxed transposition solution, we discover the connection between the superdifferentials and subdifferentials of value function and the first-order and second-order adjoint equations. As an application, we provide a stochastic verification theorem for optimal control problems of stochastic evolution equations.

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Design of arbitrarily shaped acoustic cloaks through PDE-constrained optimization satisfying sonic-metamaterial design requirements

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Abstract

Since the first time the theory of transformation was applied for designing electromagnetic cloaks, a lot of effort has been spent for building devices able to hide objects from different energy fields. We developed an optimization framework for the design of acoustic cloaks, with the aim of overcoming the limitations of usual transformation-based cloaks in terms of microstructure complexity and shape arbitrarity of the obstacle. This is achieved by recasting the acoustic cloaking design as a nonlinear optimal control problem constrained by the Helmholtz wave equation, which is a linear elliptic partial differential equation. Thus, an objective function that weights the scattered field of the obstacle in the surrounding domain is minimized. In this setting, isotropic material properties' distributions realizing the cloak take the form of control functions and a system of first-order optimality conditions is derived accordingly. Such isotropic media can then be obtained in practice with simple hexagonal lattices of inclusions in water. For this reason, the optimization problem is directly formulated to take into account suitable partitions of the control domain.
Then, two families of solid inclusions are introduced, and long-wavelength homogenization is used to define the feasible set of material properties that is embedded in the optimization problem as a further constraint. In this manner, we link the stage of material properties optimization with that of microstructure design, aiming at finding the optimal implementable solution.
The resulting cloak is numerically tested via coupled structural acoustic simulations. As test benchmarks, cloaking of the silhouette of a ship and a concave target are considered, other than the usual axisymmetric cloak. In these setups, we show the versatility of the optimization framework by considering obstacles characterized by different acoustic impedance, and where the incident pressure field is a superposition of waves with different frequencies and directions of incidence.
This study has been published in [1].

[1] S. Cominelli et Al., Design of arbitrarily shaped acoustic cloaks through partial differential equation-constrained optimization satisfying sonic-metamaterial design requirements, Proceedings of the Royal Society A, 478, 2257 (2022)

An optimal control problem to resolve traction changes on a fracture embedded in an elastic structure based on fictitious domain decomposition method

Olivier Bodart, Valérie Cayol, Farshid DABAGHI and Jonas Koko

An inverse problem based on optimal control optimization, applied on volcanologie is studied. It consists in the determination of the traction applied on volcanic fractures from ground deformations data. The inversion is modeled by minimizing a cost function which involves a misfit between the computed solution to direct problem and the observed data, together with smoothing terms on the traction and its gradient. The direct and adjoint problems in iterative minimization procedure are solved via a fictitious domain decomposition method, using a finite element discretization of XFEM type. 3D synthetic tests confirm the efficiency of our methods. From realistic application point of view, the inverse problem is reformulated to take into account interferometric synthetic aperture radar (InSAR) data and Global Navigation Satellite System (GNSS) data. Finally, an application to the May 2016 “Piton de la Fournaise” eruption is considered.

Schwarz method in time for internal controlled wave problem

Thanh Vuong Dang

Abstract

We study the minimization problem of an internal control under a wave constraint. With the Lagrangian function and optimality condition, we have to deal with a coupled wave system and corresponding initial, final data. Using time domain decomposition method and finite volume approach, we compute the discrete control and determine the order of convergence. Furthermore, we optimize the algorithm by changing computation of transmission condition.

Space-time finite element methods for optimal control problems

Richard Löscher[1], Olaf Steinbach[1]

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Applying space-time methods to optimal control problems allows for solving a large system at once, rather than a forward-backward primal and adjoint problem successively. In this talk we will give a derivation of the space-time optimality system and we will discuss the solvability, stability and discretisation for selected model problems, with L^2- and/or energy stabilisations. Futhermore, we will complement the theoretical findings by numerical examples using the finite element method on (unstructured) space-time simplicial meshes, supporting the results.

This talk is based on joint work with Ulrich Langer and Huidong Yang.

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Non-overlapping domain decomposition methods for parabolic control problems

Martin J. Gander, Liu-Di LU*

Abstract

Non overlapping domain decomposition methods (DDMs) refer to decompositions of the domain into the union of mutually disjoint subdomains. These methods are known to be very well suited to parallel computing, and particularly efficient in many applications, for example when considering heterogeneous problems with jumps in coefficients. Since their emergence and the seminal work of Pierre-Louis Lions, they have received a considerable amount of attention. Their study, whether it is conducted at the continuous level or the discrete level, remains a challenging issue. In this study, we aim at presenting the Dirichlet-Neumann (DN) and Neumann-Neumann (NN) methods applied to optimal control problems arising from parabolic partial differential equations (PDE). This problem reads as: for a given state y governed by a parabolic PDE on the time interval $[0,T]$, we wish to drive the solution of this parabolic PDE to a desired state $\hat{y}$ through a control u. The goal into non-overlapping subdomains is to find the optimal control u^* which minimizes the discrepancy between these states (i.e. original state y and desired state $\hat{y}$). After a semi-discretization in space, we use the Lagrange multiplier approach to derive a coupled forward-backward system. This system can then be solved by using DN and NN methods by separating the time domain into two non-overlapping subdomains. Finally, we provide the convergence analysis for these two methods along with some numerical results.

*Speaker

Convergence Bound for Parareal with Spatial Coarsening

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Co-authors:
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Abstract:
Over the last two decades, the number of processors per computer has greatly increased, which enabled the full exploitation of parallelism when solving partial differential equations. However, nowadays, communication between those processors has become a bottleneck.

Parallel-in-time algorithms enable us to parallelize our problem along the time dimension allowing us to relieve that bottleneck. One such method is Parareal [3]. For performance and stability reasons, we consider the sequential operator on a coarser grid in space (additionally to time) [2].

Following the method of Dobrev, Kolev, Petersson and Schroder [1], we will prove a linear convergence bound for Parareal with coarsening in space of factor two. The bound will be valid for most classical spatial restriction and prolongation operators.

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Research School - Domain Decomposition for Optimal Control Problems

CIRM, Marseille, September 05th–09th, 2022.

Domain Decomposition Methods for Eigenvalue Problems

Abstract:
In Frozen Density Embedding Theory [1], the so called "Freeze & Thaw" iteration method [2] highly resembles the well known Schwarz algorithm. The main idea is to divide a quantum system in an environment, which is fully described by an electronic density function and the embedded species, which is characterized by eigenfunctions of a nonlinear operator. While the environment is hold "frozen", we compute these eigenfunctions and eigenvalues in dependence of the environment density. Then, the latter provide a new density descriptor, which can be frozen again, in order to "thaw" the environment and solve its similar eigenvalue problem. The total density is obtained by addition of the optimized subsystem densities.

At the same time, there exist domain decomposition methods for eigenvalue problems in the work from S. H. Lui [3]. With the assumption that the solution function can be separated as a sum, we need for the coupling of the functionals the concept of fuzzy domain decomposition (FDD) [4].

Our goal is, thus, to find the relation between these three domains. As a result, mathematical attributes such as existence, uniqueness and convergence rates, can be understood in the already used "Freeze & Thaw" method.

This is a joint work between the department of physical chemistry and the department of mathematics at the university of Geneva.

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Numerical analysis of a one dimensional superconductivity model

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Abstract

We analyze numerically a 1D Maxwell-Ginzburg-Landau model to describe the time evolution of the vector potential denoted A and the order parameter denoted ψ when a superconductor receives an pulse of magnetic field. The equations are integrated using finite-differences in space and ODE solvers in time. We obtain an exact solution of the model and this helps us validate the accuracy of the numerical schemes.

LiQuOFETI : a FETI-inspired method for elliptic quadratic optimal control problems

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Abstract

We present a decomposition method for a class of quadratic optimal control problems with a linear elliptic equation constraint. This method, inspired by the FETI method and the direct method for optimal control problems, consists in decomposing directly in the optimization problem the elliptic equation, and to add continuity constraints. These two extra constraints are then handled with, on one hand, a virtual control on the interface, and on the other hand, an augmented lagrangian penalization technique. We analyze in depth this new algorithm, prove its convergence and its theoretical rate of convergence. We conclude the poster with a numerical example to illustrate our method.

Convergence Analysis for Multi-step One-shot Inversion Methods

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Abtract

We analyze the convergence of the gradient descent (GD) method to solve large-scale inverse problems, where the corresponding forward and adjoint problems are solved iteratively by fixed-point iteration methods.

1. Introduction

We study the *linear forward problem:*

$$u = Bu + M\sigma + F$$

where $u \in \mathbb{R}^{n_u}$ the state variable; $\sigma \in \mathbb{R}^{n_\sigma}$ the design variable; B, M, H are real matrices.

Inverse problem: Find σ from $f = Hu(\sigma) \in \mathbb{R}^{n_f}$.

Assumption: $\rho(B) < 1$ and $H(I-B)^{-1}M$ is injective.

Method of least squares with the cost function $J(\sigma) = \frac{1}{2}\|Hu(\sigma) - f\|^2$.

Lagrangian technique to define the adjoint state $p = p(\sigma)$: $p = B^*p + H^*[Hu(\sigma) - f]$.

Usual GD with fixed step $\tau > 0$:

$$\begin{cases} \sigma^{n+1} = \sigma^n - \tau M^* p^n, \\ u^n = Bu^n + M\sigma^n + F, \\ p^n = B^* p^n + H^*(Hu^n - f). \end{cases}$$

Shifted GD with fixed step $\tau > 0$:

$$\begin{cases} \sigma^{n+1} = \sigma^n - \tau M^* p^n, \\ u^{n+1} = Bu^{n+1} + M\sigma^n + F, \\ p^{n+1} = B^* p^{n+1} + H^*(Hu^{n+1} - f). \end{cases}$$

One-shot inversion methods: solve the forward and adjoint problems iteratively and *iterate at the same time* on u, p and σ.

Multi-step one-shot inversion methods: do k fixed-point interations on u and p. We study two variants of them (see section 2).

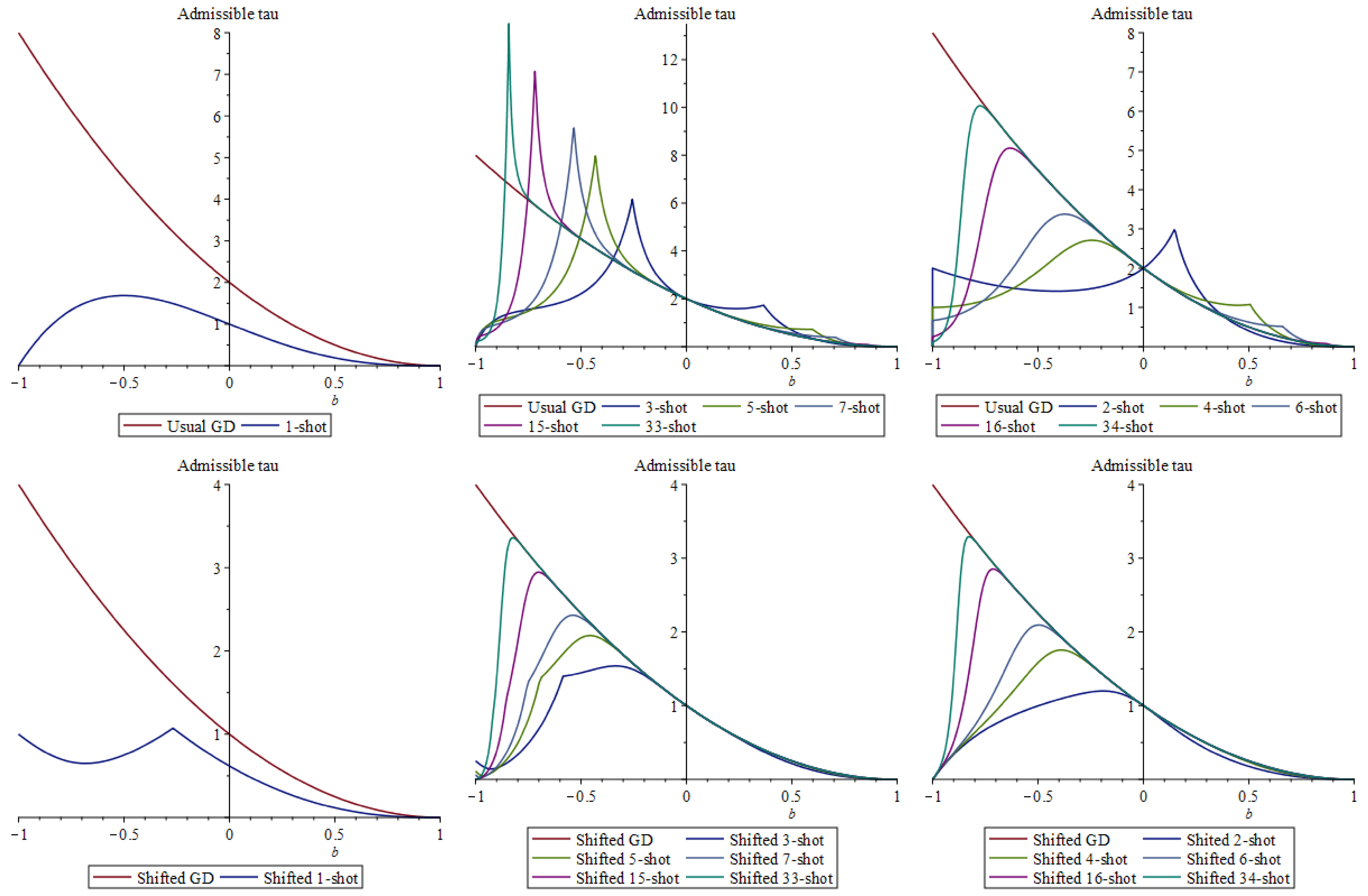
2. Multi-step one-shot algorithms

k-step one-shot	Shifted k-step one-shot
$\begin{cases} \sigma^{n+1} = \sigma^n - \tau M^* p^n, \\ u_0^{n+1} = u^n, p_0^{n+1} = p^n, \\ \quad u_{\ell+1}^{n+1} = Bu_\ell^{n+1} + M\sigma^{n+1} + F, \\ \quad p_{\ell+1}^{n+1} = B^* p_\ell^{n+1} + H^*(Hu_\ell^{n+1} - f), \\ u^{n+1} = u_k^{n+1}, p^{n+1} = p_k^{n+1} \end{cases}$	$\begin{cases} \sigma^{n+1} = \sigma^n - \tau M^* p^n, \\ u_0^{n+1} = u^n, p_0^{n+1} = p^n, \\ \quad u_{\ell+1}^{n+1} = Bu_\ell^{n+1} + M\sigma^{n} + F, \\ \quad p_{\ell+1}^{n+1} = B^* p_\ell^{n+1} + H^*(Hu_\ell^{n+1} - f), \\ u^{n+1} = u_k^{n+1}, p^{n+1} = p_k^{n+1} \end{cases}$
Converge to the usual GD as $k \to \infty$	Converge to the shifted GD as $k \to \infty$
Wait for σ before updating u, p	Update σ, u, p at the same time

3. Convergence analysis in 1D

Necessary and sufficient condition for the convergence			
Usual GD	Shifted GD	k-step one-shot	Shifted k-step one-shot
$\tau < \frac{2(1-b)^2}{h^2m^2}$	$\tau < \frac{(1-b)^2}{h^2m^2}$	$\tau < \frac{\eta(k,b)}{h^2m^2}$	$\tau < \frac{\kappa(k,b)}{h^2m^2}$

where κ and η are plotted below ($m = h = 1$):



*Note: B, M, H become $b, m, h \in \mathbb{R}$.

4. Main results

Theorem 1. *$\exists \tau > 0$ such that k-step one-shot converges. If $0 \le \|B\| < 1$, take*

$$\tau < \frac{\psi(k, \|B\|)}{\|H\|^2 \|M\|^2}, \quad \psi \text{ is an explicit function.}$$

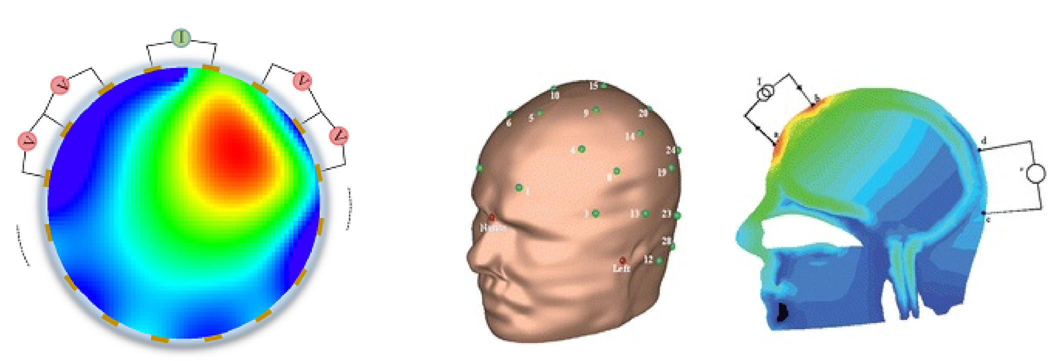
Theorem 2. *$\exists \tau > 0$ such that shifted k-step one-shot converges. If $0 \le \|B\| < 1$, take*

$$\tau < \frac{\chi(k, \|B\|)}{\|H\|^2 \|M\|^2}, \quad \chi \text{ is an explicit function.}$$

5. Numerical results for a toy problem

Linearized conductivity inverse problem

Medical application to EIT (Electrical Impedance Tomography).



Forward problem ($\delta > 0$):

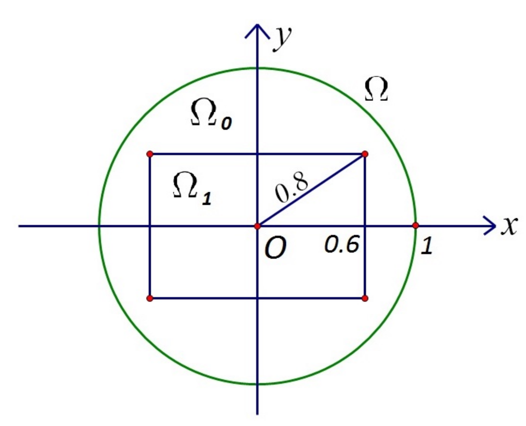
$$\begin{cases} -(1+\delta)\,\mathbf{div}(\sigma_0 \nabla u) + u = -\mathbf{div}(\sigma \nabla u_0) \text{ in } \Omega, \\ \sigma_0 \dfrac{\partial u}{\partial \nu} = 0, \sigma = 0 \text{ on } \partial\Omega \end{cases}$$

where u_0 satisfies

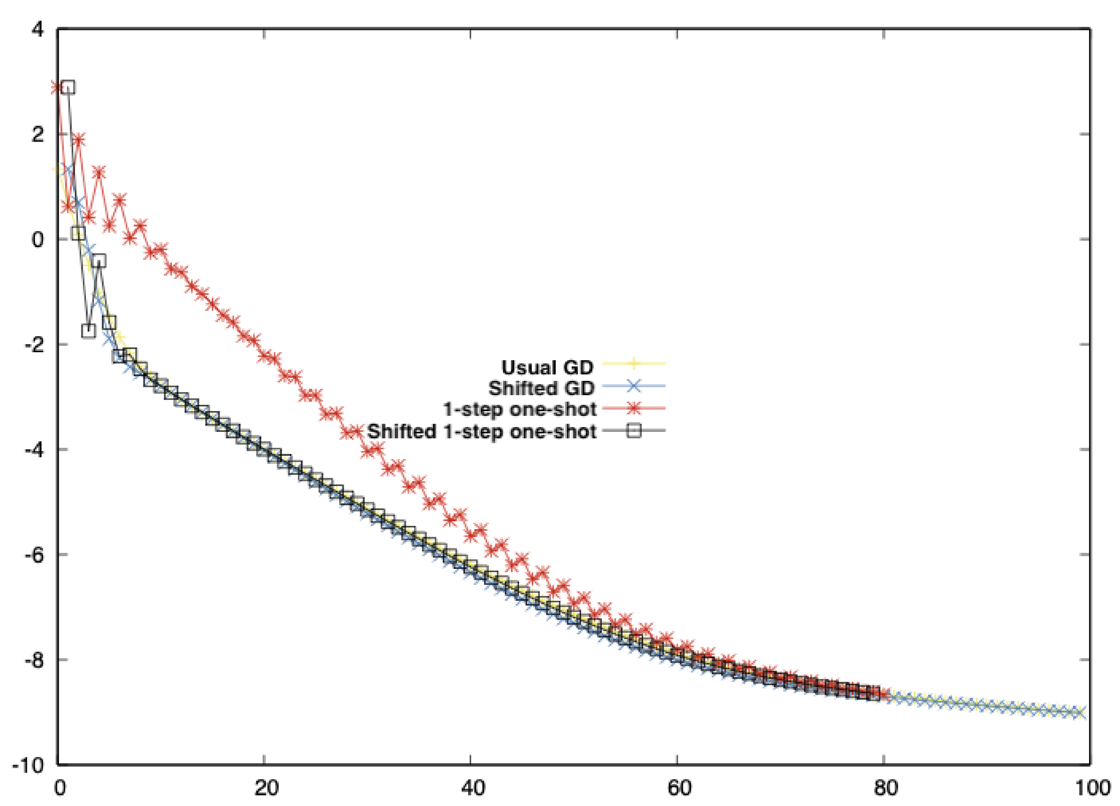
$$\begin{cases} -(1+\delta)\,\mathbf{div}(\sigma_0 \nabla u_0) + u_0 = 0 \text{ in } \Omega, \\ \sigma_0 \dfrac{\partial u_0}{\partial \nu} = g \text{ on } \partial\Omega. \end{cases}$$

Inverse problem: Recover σ from the measurement $f = Hu(\sigma) := u(\sigma)|_{\partial\Omega}$.

- $\delta = 0.5$
- Exact $\sigma^\star = 10 \cdot \mathbb{1}_{\Omega_1}$
- Multiple data $g_m = \cos(m\phi), 1 \le m \le 5$
- Initial guess $\sigma^0 = 15 \cdot \mathbb{1}_{\Omega_1}$
- $\tau = 1$



Log-plot of the cost function for each methods at different iterations:



6. Perspectives

1) Study the convergence when the iterative solver comes from DDM (domain decomposition methods)

2) Extend the analysis to non-linear inverse problems

7. References

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Minisymposium MS13: Optimized Schwarz Menthods for the Brinkman Equations with discontinuous coefficients

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Abstract

We will present non-overlapping Optimized Schwarz domain decomposition methods for solving a low-Reynolds-number flow in porous media described by the Brinkman equations. The permeability coefficient in the Brinkman equations is high-contrast such that the flow is dominated by Darcy in some domains and by Stokes in others. Thus, we wish to decompose the domain into non-overlapping subdomains and solve iteratively. The goal is to find optimized transmission conditions which are also numerically tractable. After a Fourier transform, we first derive optimal transmission conditions for the resulting Brinkman equations. These transmission conditions are similar to Dirichlet-Robin boundary conditions, but contain non-local pseudo-differential operators when using an inverse Fourier transform, which would not be numerically tractable. Thus a series of local approximations for these non-local conditions (2×2-matrices) is proposed, and we study the corresponding min-max problems. Finally, we also provide a convergence analysis for these local approximations.

*Speaker