Stochastic models and Evolution

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Ecologie (bio) : dynamique des pop.

Evolution : modification des espèces.

Basic mechanisms of Evolution

- Offsprings acquire the genetic information of their parents:

  Hérédité

- Permanent alteration of DNA:

  Mutations

- Differential survival and reproduction of individuals due to differences in phenotype:

  Sélection naturelle

Individual based-models

Modèles individus-centrés

- Processus stochastiques
- Probabilités de survie

Échelle microscopique

↳ échelle microscopique.

Basés sur les v.a exponentielles:

$T \sim \mathcal{E}(\lambda)$: $\mathbb{P}(T \geq t) = e^{-\lambda t} \quad \forall t \geq 0.$

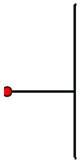
$$\mathbb{P}(T \geq t+s \mid T > s) = \mathbb{P}(T \geq t).$$

Stochastic Birth and death Model

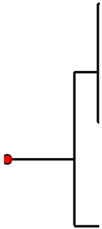
↳ taux de naissance b : $T_b \sim \mathcal{E}(b)$

↳ taux de mort d : $T_d \sim \mathcal{E}(d)$

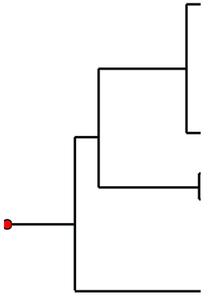
Stochastic Birth and death Model



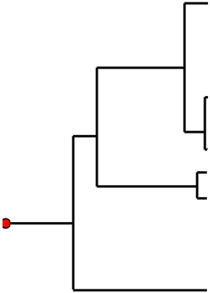
Stochastic Birth and death Model



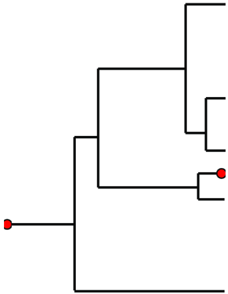
Stochastic Birth and death Model



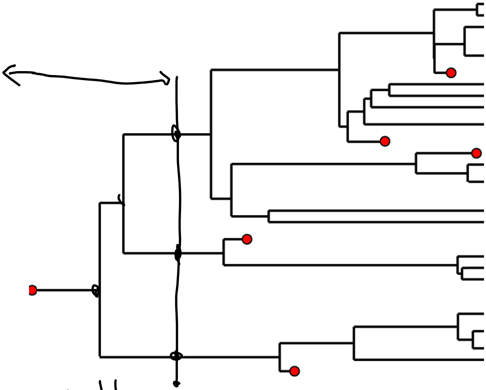
Stochastic Birth and death Model



Stochastic Birth and death Model



Stochastic Birth and death Model



→ taille pop variable.

→ Markovien : dpt uniquement du présent (pas du passé)

Different biological aspects

Birth : $\mathcal{E}(b)$ Death : $\mathcal{E}(d)$

competition : $\mathcal{E}(b)$ $\mathcal{E}(\underbrace{d}_{\text{mort naturelle}} + \underbrace{c N_t}_{\text{compétition}})$

Competition

Ref: Champagnat, Costa, Fournier, Méléard, Tran...

Different biological aspects

Birth : $\mathcal{E}(b)$ Death : $\mathcal{E}(d)$

Age or space structure :

trait génétique u.

$b(u)$ $d(u)$.

Spatial structures

Ref: Champagnat, Costa, Fournier, Méléard, Tran...

Different biological aspects

several species : 1, 2

Birth : $\mathcal{E}(b)$

$b^1(u, N_r^2)$

$b^2(v)$

Death : $\mathcal{E}(d)$

$d^1(u, N_r^1, N_r^2)$.

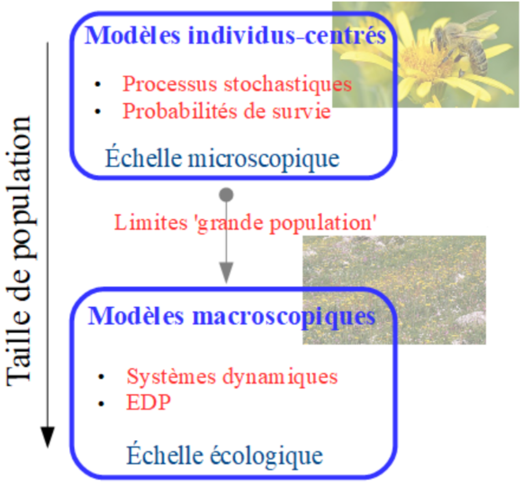
$d^2(v)$.

Cooperation

→ Compromis : difficultés / Modélisation.

Ref: Champagnat, Costa, Fournier, Méléard, Tran...

First assumption



Hypothèse :
Grande population

~

Loi des gds nombres.

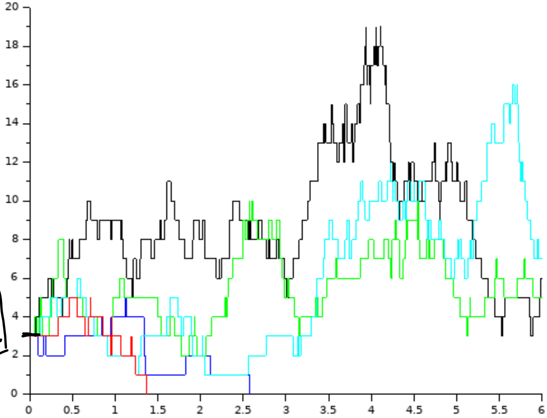
Simulations : birth-death-competition model

Birth : $\mathcal{E}(\underset{\smile}{b})$ Death : $\mathcal{E}\left(\underset{\sim}{d} + \underbrace{\frac{c}{K} N_t^K}\right)$

$K = 1$

$\boxed{K \to +\infty}$

$\boxed{3 \times K}$

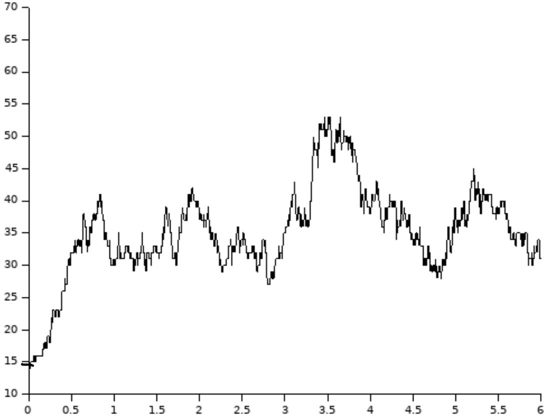


Simulations : birth-death-competition model

Birth : $\mathcal{E}(b)$ Death : $\mathcal{E}\left(d + \frac{c}{K} N_t^K\right)$

$K = 5$

$K \to +\infty.$

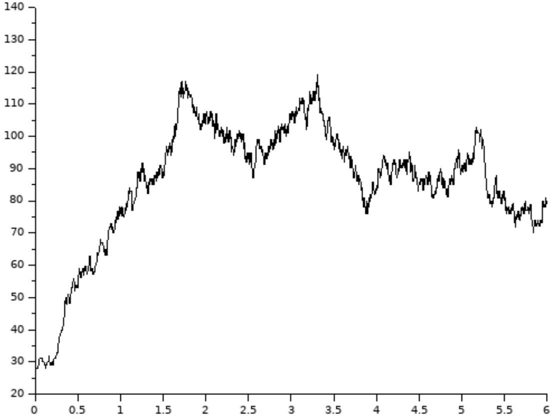


$3K$

Simulations : birth-death-competition model

Birth : $\mathcal{E}(b)$ Death : $\mathcal{E}\left(d + \frac{c}{K} N_t^K\right)$

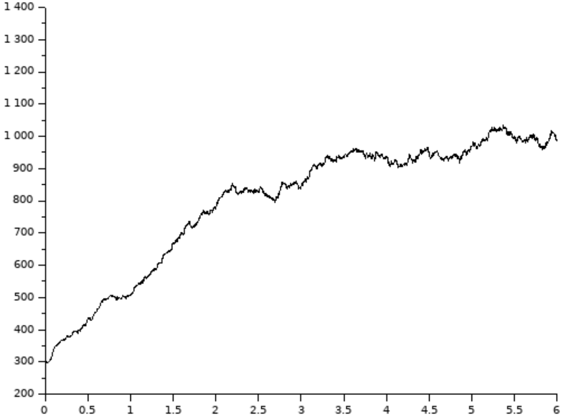
$K = 10$



Simulations : birth-death-competition model

Birth : $\mathcal{E}(b)$ Death : $\mathcal{E}\left(d + \frac{c}{K} N_t^K\right)$

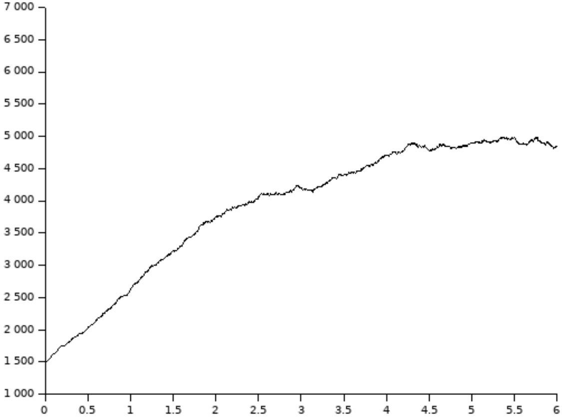
$$K = 100$$



Simulations : birth-death-competition model

Birth : $\mathcal{E}(b)$ Death : $\mathcal{E}\left(d + \frac{c}{K} N_t^K\right)$

$$K = 500$$

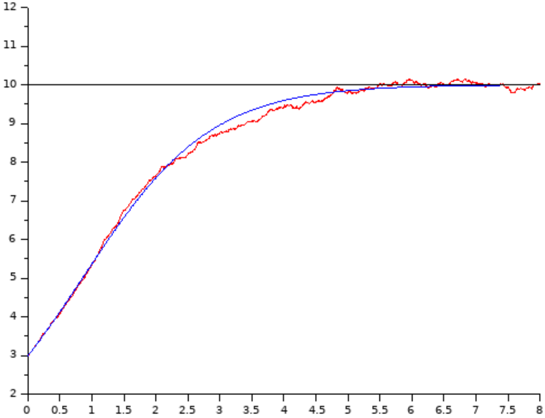


Simulations : birth-death-competition model

Birth : $\mathcal{E}(b)$ Death : $\mathcal{E}\left(d + \frac{c}{K} N_t^K\right)$

$K = 1000$

$\frac{N_r}{K}$



Large population approximation: when $K \to \infty$

Initially: $3K$ individuals.

Théorème (Ethier-Kurtz, 1986)

$$\underbrace{\left(\frac{N^K(t)}{K}\right)_{t\in[0,T]}}_{\textit{Scaled stochastic process}} \xrightarrow[K\to+\infty]{} \underbrace{n(t)}_{\textit{deterministic limit}}$$

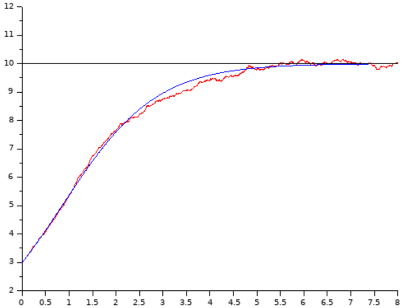
with

$$\begin{cases} n'(t) = n(t)(b - d - cn(t)) \\ n(0) = 3 \end{cases}$$

Averaged behavior

K=1000

$\frac{b-d}{c}$



Limit population behavior :

$$\lim_{t \to +\infty} n(t) = \frac{b-d}{c}$$

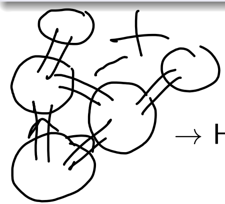
Motivation

Definition

Species: *largest group of organisms in which two individuals are capable of reproducing fertile offspring.*

Definition

Speciation: *evolutionary process by which reproductively isolated biological populations evolve to become distinct species.*



→ How does **reproductive isolation** appear?

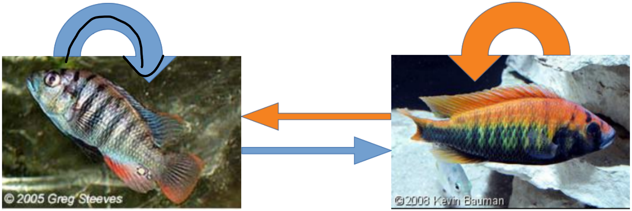
→ joint work with Coron, Costa, Smadi.

Problematic

Préférence de reproduction :

Definition

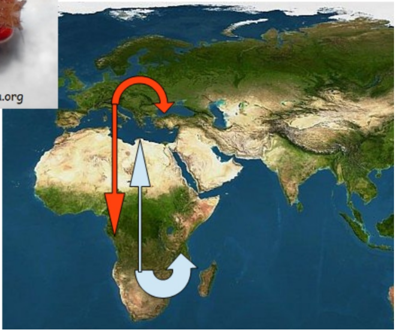
Homogamy/assortative mating: *individuals with similar phenotypes have a higher reproductive success*



Pundamilia pundamilia

Pundamilia nyererei

Drosophila Melonaster

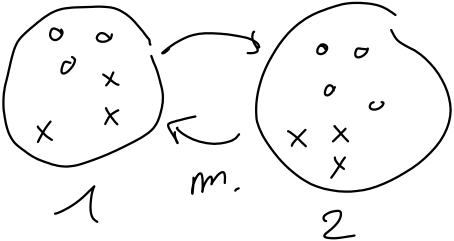


→ Can homogamy entail **reproductive isolation** ?

→ Wu, Hollocher, Begun, Aquadro, Xu, Wu (1995)

Model description

- Two patches,
- Haploid population,
- two alleles: a and A,



Notation (population):

$$\left(N_t^{A,1}, N_t^{a,1}, N_t^{A,2}, N_t^{a,2}\right).$$

Preferential Mendelian reproduction

Any a-individual :

- meets with an individual of its patch at rate B:
  - ▷ mates with probability: $\frac{b}{B} \rightarrow A$, $\frac{b \times \beta}{B} \rightarrow a$. $\beta > 1$.
- Offspring types follow Mendelian rules

$$(a,a) \rightarrow a \qquad (a,A) \xrightarrow{1/2} a, \quad (a,A) \xrightarrow{1/2} A.$$

Total production rate of $(a, 1)$

$$b N^{a,1} \left[\frac{\beta N^{a,1} + N^{A,1}}{N^{a,1} + N^{A,1}} \right]$$

Migration and death

- Migration:
  - ▷ mating success dependent migration,

$$m N^{a,1}\left[\frac{N^{A,1}}{N^{a,1}+N^{A,1}}\right]$$

- Death:
  - ▷ natural and due to intra-patch competition,

$$d N^{a,1} + c N^{a,1}\left(N^{a,1}+N^{A,1}\right)$$

Large population approximation

Lemma (Ethier, Kurtz 1986)

There is a convergence in probability to $(\mathbf{n}(t), t \in [0,T])$*, solution to*

$$\begin{cases} \frac{d}{dt} n_{A,1}(t) = n_{A,1}\left[b\frac{\beta n_{A,1}+n_{a,1}}{n_{A,1}+n_{a,1}} - d - c(n_{A,1}+n_{a,1}) - m\frac{n_{a,1}}{n_{A,1}+n_{a,1}}\right] + m\frac{n_{A,2}n_{a,2}}{n_{A,2}+n_{a,2}} \\ \frac{d}{dt} n_{a,1}(t) = n_{a,1}\left[b\frac{\beta_a n_{a,1}+n_{A,1}}{n_{A,1}+n_{a,1}} - d - c(n_{A,1}+n_{a,1}) - m\frac{n_{A,1}}{n_{A,1}+n_{a,1}}\right] + m\frac{n_{A,2}n_{a,2}}{n_{A,2}+n_{a,2}} \\ \frac{d}{dt} n_{A,2}(t) = n_{A,2}\left[b\frac{\beta_A n_{A,2}+n_{a,2}}{n_{A,2}+n_{a,2}} - d - c(n_{A,2}+n_{a,2}) - m\frac{n_{a,2}}{n_{A,2}+n_{a,2}}\right] + m\frac{n_{A,1}n_{a,1}}{n_{A,1}+n_{a,1}} \\ \frac{d}{dt} n_{a,2}(t) = n_{a,2}\left[b\frac{\beta_a n_{a,2}+n_{A,2}}{n_{A,2}+n_{a,2}} - d - c(n_{A,2}+n_{a,2}) - m\frac{n_{A,2}}{n_{A,2}+n_{a,2}}\right] + m\frac{n_{A,1}n_{a,1}}{n_{A,1}+n_{a,1}}. \end{cases}$$

→ Comprendre du système.

Dynamics

Theorem

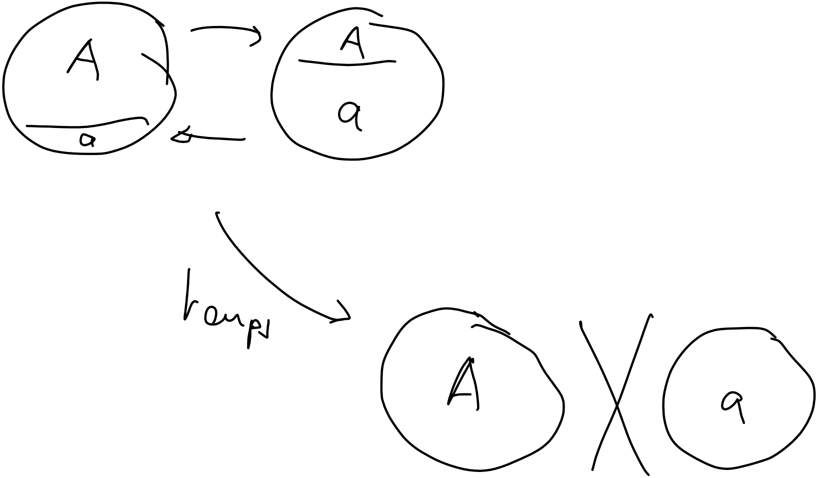
Almost all trajectories converge to one of the four stable equilibria

$$(\zeta_A, 0, \zeta_A, 0), \quad (0, \zeta_a, 0, \zeta_a)$$

$$(\zeta_A, 0, 0, \zeta_a), \quad (0, \zeta_a, \zeta_A, 0)$$

A A | A X a

Dynamics



Questions ?