# Master Class 2021 CEPS

# Mathematical Modeling and numerical simulation for shallow water equations

## PART1 : Generalities

Mehmet Ersoy

2021, 29 January, CIRM, France

Outline of the talk

1. Hydrostatic models, applications and limits
   - Hydrostatic models
   - Application to tsunamis propagation

2. Non-hydrostatic models and applications
   - Historical background and motivations
   - Toward the first dispersive section-averaged model

- Fluids are everywhere ! ! !
  - Atmosphere/land : weather, rain, storms, flooding, water ressources, *etc.*

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  - Underground : sandy beaches, underground networks, sewers, rivers, phreatic (groundwater), erosion, sedimentation *etc.*

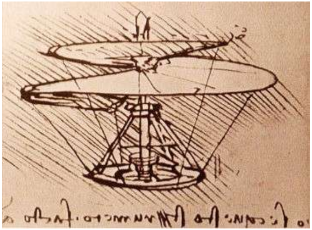
- Fluids are everywhere ! ! !
  - Atmosphere/land
  - Underground
  - Sea/ocean/Channel : maritime, navigation, erosion, sedimentation, tsunamis, breaking waves and even sounds like health *etc.*

Nazare


Tsunami

## Motivations

- Fluids are everywhere ! ! !
  - Atmosphere/land
  - Underground
  - Sea/ocean/Channel
- A topic of investigation/interest old as the world yielding to almost existing branches of applied mathematics, computers sciences, *etc.*



Aerial screw (Vinci, 1487)

Waterwheel (Poncelet, 1825)

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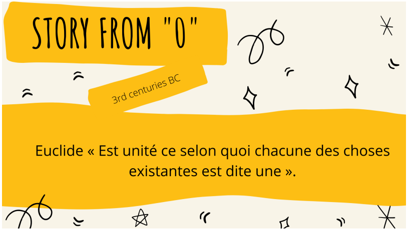

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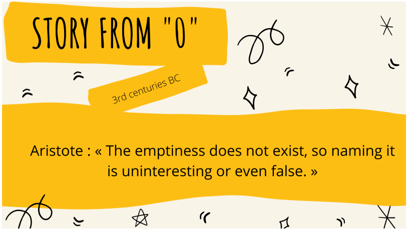

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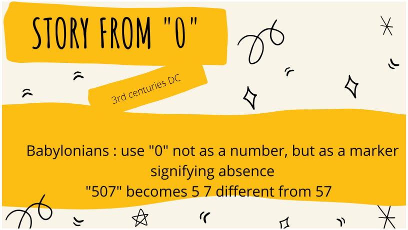

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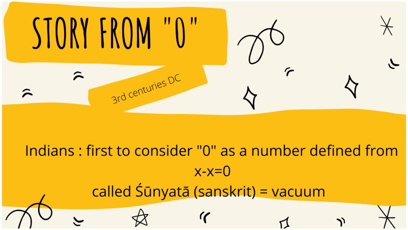

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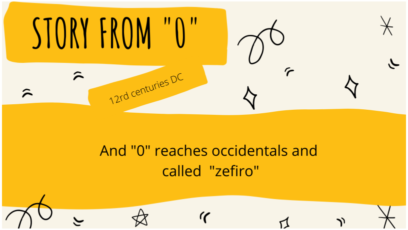

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- A topic of investigation/interest old as the world yielding to almost existing branches of applied mathematics, computers sciences, *etc.*
- Multiple scales, non trivial interactions/coupling yielding to hydrostatic to non hydrostatic phenomenon involving modern applied mathematics

OUTLINE

1 HYDROSTATIC MODELS, APPLICATIONS AND LIMITS
- Hydrostatic models
- Application to tsunamis propagation

2 NON-HYDROSTATIC MODELS AND APPLICATIONS
- Historical background and motivations
- Toward the first dispersive section-averaged model

- Introducing characteristic scales :
  - length $L$
  - width $l$
  - height $H$

- Introducing characteristic scales : $L$, $l$ and $H$
- Introducing aspect ratio numbers :
  - $\varepsilon_z = \dfrac{H}{L}$ following the depth
  - $\varepsilon_y = \dfrac{l}{L}$ following the width

- Introducing characteristic scales : $L$, $l$ and $H$
- Introducing aspect ratio numbers : $\varepsilon_z = \dfrac{H}{L}$ and $\varepsilon_y = \dfrac{l}{L}$
- One can reduce the initial model (Navier-Stokes or Euler equations)
  - 3D-2D depth averaged model reduction if

$$\varepsilon_z \ll 1 \text{ and } \varepsilon_y \approx 1$$

$$u(x,y,z,t;\varepsilon_z) = u(x,y,t;0) + \varepsilon_z \partial_{\varepsilon_z} u(x,y,t;0) \text{ where } u(x,y,z,t;0) = u_0(x,y,t)$$

Asymptotic expansion = Taylor expansion with respect to $\varepsilon$

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  - 3D-1D section averaged model reduction if

$$\varepsilon_z \approx \varepsilon_y \ll 1$$

$$u(x, y, z, t; \varepsilon_z) = u(x, y, t; 0) + \nabla_{\varepsilon_y, \varepsilon_z} u(x, y, t; 0) \text{ where } u(x, y, z, t; 0) = u_0(x, t)$$

Asymptotic expansion = Taylor expansion with respect to $\varepsilon$

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- Opposite to DNS, model reduction $\rightarrow$ to decrease the computational cost

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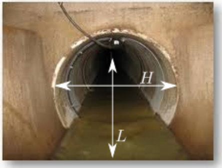
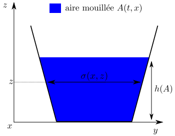
- One can reduce the initial model (Navier-Stokes or Euler equations)
- Opposite to DNS, model reduction $\rightarrow$ to decrease the computational cost
- Some applications :



Outline

- SV equations for closed water pipes/channels/rivers




$$\left\{\begin{array}{l} \partial_t A + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left( \dfrac{Q^2}{A} + g I_1(x,A) \right) = g I_2(x,A) \end{array}\right.$$

with

| | | |
|---|---|---|
| $A(t,x)$, $Q(t,x)$, $g$ | : | wet area, discharge, gravity |
| $I_1(x,A) = \displaystyle\int_d^\eta \sigma(x,z)(\eta - z)\,dz$ | : | hydrostatic pressure |
| $I_2(x,A) = \displaystyle\int_d^\eta \frac{\partial}{\partial x}\sigma(x,z)(\eta - z)\,dz$ | : | hydrostatic pressure source |

C. Bourdarias, **M. Ersoy**, S. Gerbi.
*A kinetic scheme for pressurized flows in non uniform pipes.*
Monografias de la Real Academia de Ciencias, 2009.

C. Bourdarias, **M. Ersoy**, S. Gerbi.
*A model for unsteady mixed flows in non uniform closed water pipes and a well-balanced finite volume scheme.*
International Journal on Finite Volumes, 2009.

C. Bourdarias, **M. Ersoy**, S. Gerbi.
*A kinetic scheme for transient mixed flows in non uniform closed pipes : a global manner to upwind all the source terms.*
Journal of Scientific Computing, 2011.

C. Bourdarias, **M. Ersoy**, S. Gerbi.
*Unsteady mixed flows in non uniform closed water pipes : a Full Kinetic Appraoch.*
Numerische Mathematik, 2014.

- SV equations for closed water pipes/channels/rivers

$$\left\{\begin{array}{l}\partial_t A + \partial_x Q = 0,\\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + gI_1(x,A)\right) = gI_2(x,A) - gAK(x,Q/A)\end{array}\right.$$

with $\quad K(x,u) = \dfrac{K_0(u)}{A}\displaystyle\int_{\Gamma_b(x,t)} ds$ where

  - $K_0(u) = C_l + C_t\,|u|$
  - $A/\displaystyle\int_{\Gamma_b}(x,t)ds$ is the so-called hydraulic radius

- SV equations for closed water pipes/channels/rivers including friction

**M. Ersoy**.
*Dimension reduction for incompressible pipe and open channel flow including friction*.
Applications of Mathematics, 2015.

**M. Ersoy**.
*Dimension reduction for compressible pipe flows including friction*.
Asymptotic Analysis, 2016.

- SV equations for closed water pipes/channels/rivers
- SV equations for closed water pipes/channels/rivers including friction
- SV equations for urban/overland flows including precipitation and recharge

$$\left\{\begin{array}{l} \partial_t h + \partial_x q = S := R - I, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{A} + g\dfrac{h^2}{2}\right) = -gh\partial_x Z + S\dfrac{q}{h} - \left(k_+(R) + k_-(I) + k_0\left(\dfrac{q}{h}\right)\right)\dfrac{q}{h} \end{array}\right.$$

with

| | | |
|---|---|---|
| $h(t,x)$, $q(t,x)$ | : | water height, discharge |
| $k_\pm$ | : | friction generated from precipitation and infiltration |

where $I$ can be driven by the solution of the Richards' equation.

**M. Ersoy**, O. Lakkis, P. Townsend.
*A Saint-Venant shallow water model for overland flows with precipitation and recharge.*
Mathematical and Computational Applications, Natural Sciences, 2020.

J.-B. Clément, **M. Ersoy**, F. Golay, and D. Sous.
*Discontinuous galerkin method for steady-state richards equation.*
Topical Problems of Fluid Mechanics, 2019

J.-B. Clément, **M. Ersoy**, F. Golay, and D. Sous.
*Adaptive discontinuous galerkin method for richards equation.*
Topical Problems of Fluid Mechanics, 2020

J.-B. Clément, **M. Ersoy**, F. Golay, and D. Sous.
*An adaptive strategy for discontinuous Galerkin simulations of Richards' equation.*
Preprint, 2020

J.-B. Clément, D. Sous, F. Golay, and **M. Ersoy**.
*Wave-driven Ground- water Flows in Sandy Beaches : A Richards Equation-based Model.*
Journal of Coastal Research, 2020

- SV equations for closed water pipes/channels/rivers
- SV equations for closed water pipes/channels/rivers including friction
- SV equations for urban/overland flows including precipitation and recharge
- Existence of an entropy, energetically consistant, Galilean invariant, FV based on Kinetic scheme, accurate compare to exp data

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- Existence of an entropy, energetically consistant, Galilean invariant, FV based on Kinetic scheme, accurate compare to exp data
- Example : applications to Tsunamis propagation

$$\begin{cases} \partial_t h + \mathsf{div}(h\overline{u}) = 0, \\ \partial_t(h\overline{u}) + \mathsf{div}\left(h\overline{u}\otimes\overline{u} + g\dfrac{h^2}{2}I\right) = -gh\nabla Z, \end{cases}$$

with $\overline{u}(t,x) \in \mathbb{R}^2$ : depth averaged velocity

K. Pons, **M. Ersoy**.
*Adaptive mesh refinement method. Part 1 : Automatic thresholding based on a distribution function.*
SEMA SIMAI Springer Series, Partial Differential Equations : Ambitious Mathematics for Real-Life Applications, D. Donatelli and C. Simeoni Editors, 2020

K. Pons, **M. Ersoy** , F. Golay and R. Marcer.
*Adaptive mesh refinement method. Part 2 : Application to tsunamis propagation.*
SEMA SIMAI Springer Series, Partial Differential Equations : Ambitious Mathematics for Real-Life Applications, D. Donatelli and C. Simeoni Editors, 2020

- Tsunamis are water waves that start in the deep ocean : $H$ is huge

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- But, the wavelength $\lambda$ of the tsunami is huge as well (200 km)

  - Change $\lambda$ in $L$ in the derivation $\rightarrow$ shallow water models
  - Dynamics of tsunamis are "essentially" governed by the shallow water equations.
  - Consequence phase speed of propagation $v_\phi \approx \sqrt{gH}$ ($H$ ocean depth), either $v_\phi \approx 600$ km/h for $H = 3$km.
  - Thus, $\lambda$ in $L$ in the derivation $\rightarrow$ shallow water models : justify the use of Saint-Venant equations for some tsunamis.

- Tsunamis are water waves that start in the deep ocean : $H$ is huge
- But, the wavelength $\lambda$ of the tsunami is huge as well (200 km) $\rightarrow$ shallow water models
- Large scale numerical simulation $\rightarrow$ Adaptive strategy : principle.
  - To cluster more grid points in the regions with large solution variations, singularities or oscillations.
  - To get "Optimal mesh" : a mesh on which some physical or computational quantities (gradient, error, *etc.*) are approximately the same on each element (equi-distribution strategy)

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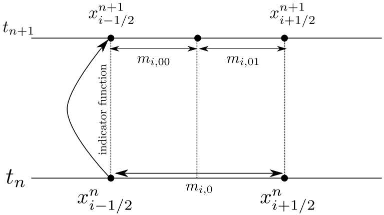
- Large scale numerical simulation $\rightarrow$ Adaptive strategy : methods.
  - h-method (Adaptive Mesh Refinement method) involves automatic refinement or coarsening of the spatial mesh based on a posteriori error estimates, error indicators or heuristic indicators.
  - p-method involves the adaptive enrichment of the polynomial order.




$h$-method

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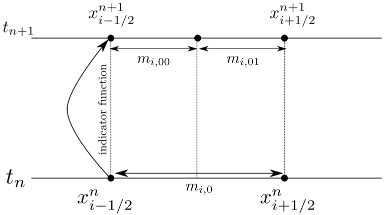
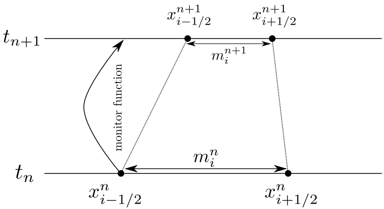
  - p-method involves the adaptive enrichment of the polynomial order.
  - r-method (Moving Mesh Method) relocates grid points in a mesh having a fixed number of nodes.




*h*-method




*r*-method

We focus on general non linear hyperbolic conservation laws

$$\begin{cases} \dfrac{\partial \boldsymbol{w}}{\partial t} + \dfrac{\partial \boldsymbol{f}(\boldsymbol{w})}{\partial x} = 0, \; (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ \boldsymbol{w}(x,0) = \boldsymbol{w}_0(x), \; x \in \mathbb{R} \end{cases}$$

| | | |
|---|---|---|
| $\boldsymbol{w} \in \mathbb{R}^d$ | : | vector state, |
| $\boldsymbol{f}$ | : | flux governing the physical description of the flow. |

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Weak solutions satisfy

$$S = \frac{\partial s(\boldsymbol{w})}{\partial t} + \frac{\partial \psi(\boldsymbol{w})}{\partial x} \begin{cases} = & 0 & \text{for smooth solution} \\ = & 0 & \text{across rarefaction} \\ < & 0 & \text{across shock} \end{cases}$$

where $(s, \psi)$ stands for a convex entropy-entropy flux pair

Entropy inequality $\simeq$ "smoothness indicator"


M. Ersoy, F. Golay, L. Yushchenko.
*Adaptive multi scale scheme based on numerical density of entropy production for conservation laws*
Central European Journal of Mathematics, Springer, 2013

L. Yushchenko, F. Golay, M. Ersoy.
*Entropy production and mesh refinement – Application to wave breaking.*
Mechanics & Industry, EDP Sciences, 2015

F. Golay, M. Ersoy, L. Yushchenko, D. Sous.
*Block-based adaptive mesh refinement scheme using numerical density of entropy production for three-dimensional two-fluid flows.*
International Journal of Computational Fluid Dynamics, 2015.

T. Altazin, M. Ersoy, F. Golay, D. Sous, L. Yushchenko.
*Numerical investigation of BB-AMR scheme using entropy production as refinement criterion.*
International Journal of Computational Fluid Dynamics, 2016.

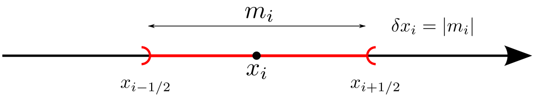


Figure – a cell $m_i$

Finite volume approximation :

$$\boldsymbol{w}_i^{n+1} = \boldsymbol{w}_i^n - \frac{\delta t_n}{\delta x_i}\left(\boldsymbol{F}_{i+1/2}^n - \boldsymbol{F}_{i-1/2}^n\right)$$

with

$$\boldsymbol{w}_i^n \simeq \frac{1}{\delta x_i}\int_{m_i} \boldsymbol{w}\left(x, t_n\right)\, dx \text{ and } \boldsymbol{F}_{i+1/2}^n \approx \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} \boldsymbol{f}(w(x_{i+1/2}, t))\, dx$$

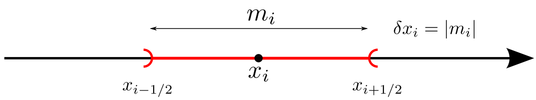


FIGURE – a cell $m_i$

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The numerical density of entropy production :

$$S_i^n = \frac{s_i^{n+1} - s_i^n}{\delta t_n} + \frac{\psi_{i+1/2}^n - \psi_{i-1/2}^n}{\delta x_i} \lessapprox 0$$

Assume that $\boldsymbol{w}_i^n$ is given for all $i$ and $S := |S|$ is a given mesh refinement criterion. Then,

- Compute $S_{i_b}^n$
- Compare to $S_m = \dfrac{1}{|\Omega|} \sum_{i_b} S_{i_b}^n$

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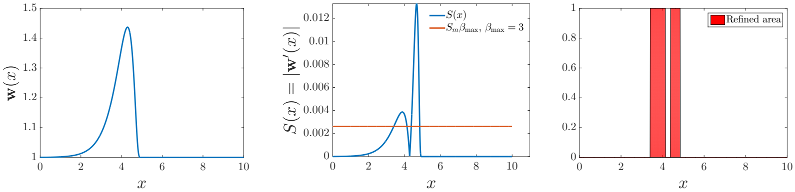
  - if $S_{i_b}^n > \alpha_{\max} = S_m \beta_{\max}$, the cell is refined and split

where $0 < \beta_{\max} \leq 1$ is user calibrated mesh refinement threshold.



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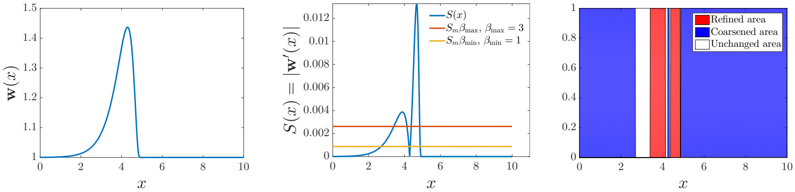
  - if $S_{i_b}^n > \alpha_{\max} = S_m\beta_{\max}$, the cell is refined and split
  - if $S_{i_{b0}}^n < \alpha_{\min} = S_m\beta_{\min}$ and $S_{i_{b1}}^n < \alpha_{\min}$, the cell is coarsened into a cell $m_{i_b}$

where $0 < \beta_{\min} \leq \beta_{\max} \leq 1$ are user calibrated mesh refinement thresholds.



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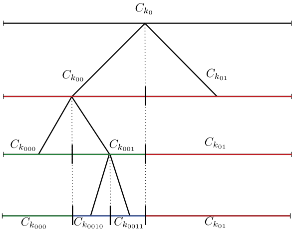
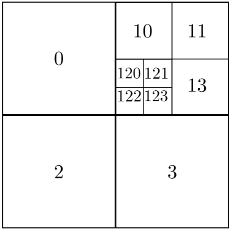
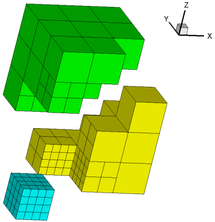



Dyadic tree




quadtree




octree

Assume that $\boldsymbol{w}_i^n$ is given for all $i$ and $S := |S|$ is a given mesh refinement criterion. Then,

- Compute $S_{i_b}^n$
- Compare to $S_m = \dfrac{1}{|\Omega|} \sum_{i_b} S_{i_b}^n$
- $\beta_{\min}$ and $\beta_{\max}$ might be the critical weakness of the AMR methods, or equivalently $\alpha\min$ and $\alpha\max$

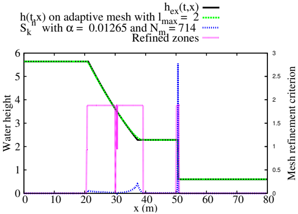
## Principle of AMR methods

Assume that $\boldsymbol{w}_i^n$ is given for all $i$ and $S := |S|$ is a given mesh refinement criterion. Then,

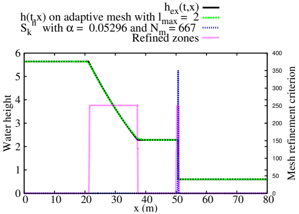
- Compute $S_{i_b}^n$
- Compare to $S_m = \dfrac{1}{|\Omega|} \sum_{i_b} S_{i_b}^n$
- $\beta_{\min}$ and $\beta_{\max}$ might be the critical weakness of the AMR methods, or equivalently $\alpha\min$ and $\alpha\max$

How to overcome such a "major" drawback in $h$-method ? See Pons-Ersoy automatic threshold

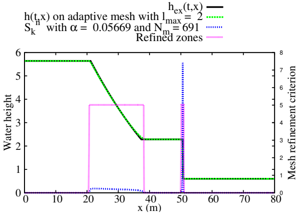
K. Pons, **M. Ersoy** .
*Adaptive mesh refinement method. Part 1 : Automatic thresholding based on a distribution function.*




(a) $|h_i^n - h_{ex}(x_i, t_n)|$



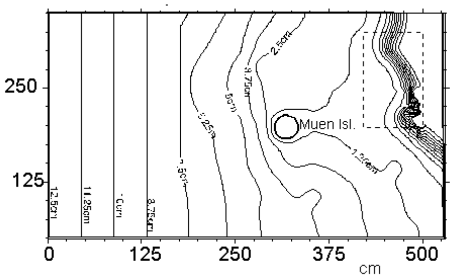

(b) Numerical density of entropy production



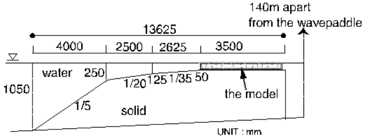

(c) $\left|\frac{h_{i+1}^n - h_i^n}{x_{i+1} - x_i}\right|$

FIGURE – Numerical results for the water height at time $t = 2$ s

# Test case : Tsunami runup onto a complex three dimensional Monai-Valley beach




(a) Top view




(b) Side view

Figure – Settings

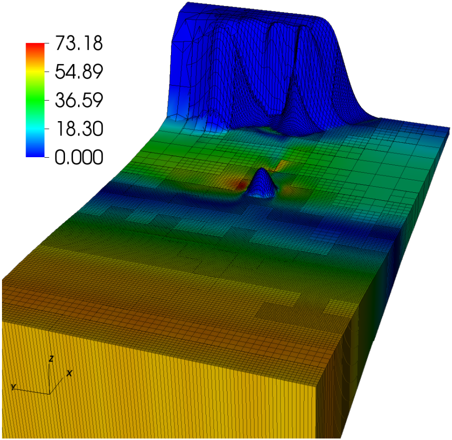
K. Pons, **M. Ersoy** , F. Golay and R. Marcer.
*Adaptive mesh refinement method. Part 2 : Application to tsunamis propagation.*

# Test case : Tsunami runup onto a complex three dimensional Monai-Valley beach

| | Adaptive mesh simulation | Uniform mesh simulation |
|---|---|---|
| Simulation time | 30 s | 30 s |
| Number of blocks | 240 | 240 |
| Number of cells | 8 000-40 000 | 62 000 |
| Re-meshing time step | 0.25 s | not applicable |
| Time order integration | 2 | 2 |
| Space order integration | 1 | 1 |
| CFL | 0.5 | 0.5 |

Table – Numerical parameters

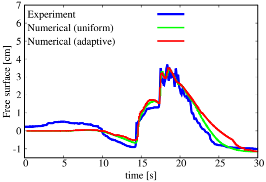
# Test case : Tsunami runup onto a complex three dimensional Monai-Valley beach



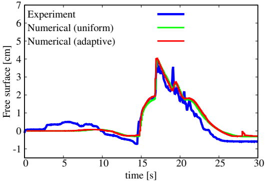

(a) $t = 11.25$ s

Figure – Numerical water height (coloration is issue from the kinetic energy)

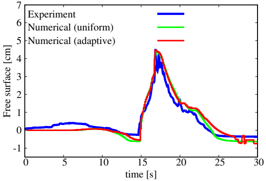
# Test case : Tsunami runup onto a complex three dimensional Monai-Valley beach




(a) Gauge 1



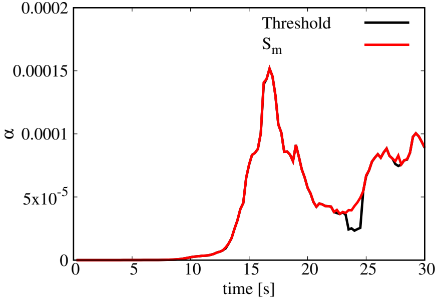

(b) Gauge 2



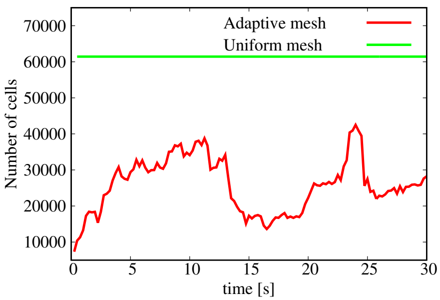

(c) Gauge 3

Figure – Free surface results at different positions : experimental data versus numerical simulation with and without mesh adaptivity

# Test case : Tsunami runup onto a complex three dimensional Monai-Valley beach




(a) Threshold




(b) Number of cells

Figure – Time evolution of the mesh refinement threshold and the number of cells : speed up the computation by 3 time

## Coming back to the modelling problem : "SVE for certain tsunamis"

- Are the SVE are pertinent for all Tsunamis ?

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  - Earthquake generated tsunamis, if the magnitude is large enough, hydrostatic models are accurate. Monai-Walley beach flooding is an example (Hokkaido-Nansei-Oki tsunami, 1993, Mw 7,7).

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  - Land-slide/subaerial landslide generated tsunamis (depending on landslide thickness, water depth) cannot be represented by hydrostatic models ! [a]
    $\rightarrow$ dispersion effects are expected.

Parisot and Ersoy's experimental wave generator
(Malaga, NumHyp 2019)

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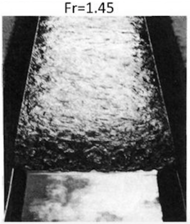


"Strong" bore

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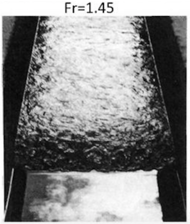


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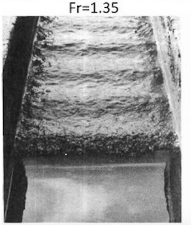


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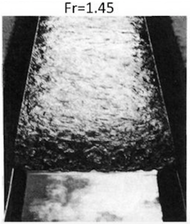

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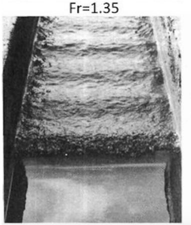

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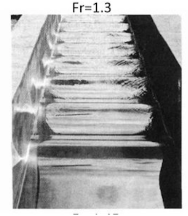

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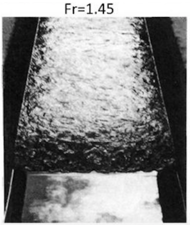

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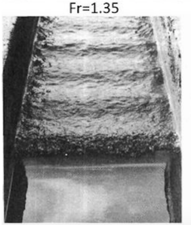

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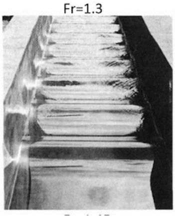

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- Dispersive wave model are also required

## Coming back to the modelling problem : "SVE for certain tsunamis"

- Are the SVE are pertinent for all Tsunamis ? No !
- Dispersive wave model are also required
- Of course, Navier-Stokes equation can deal for both but too costly !

1. Hydrostatic models, applications and limits
   - Hydrostatic models
   - Application to tsunamis propagation

2. Non-hydrostatic models and applications
   - Historical background and motivations
   - Toward the first dispersive section-averaged model

Let $\omega = \frac{2\pi}{T}$ be the angular frequency (pulsation) and $k = \frac{2\pi}{\lambda}$ wavenumber.

- A wave $\phi(kx - \omega t)$ is characterised by two different characteristic speeds
  - phase velocity $C_p = \frac{\omega}{k}$ which corresponds to the displacement of the wave fronts
  - group velocity $C_g = \frac{\partial \omega}{\partial k}$ which corresponds to the displacement of the wave's envelope
  - dispersion relation is given by $\omega = C_p k$

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  - dispersion relation is given by $\omega = C_p k$
- If $C_p$ is constant then the wave is not dispersive.

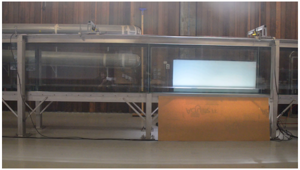
Dispersive wave

Non dispersive wave

- Everything starts with Russell's "Wave of translation"

*"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion ; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".* **John Scott Russell**

- Everything starts with Russell's "Wave of translation"
- Observation of Soliton



Russell's experiments “like” in 1834

- Everything starts with Russell's "Wave of translation"
- Heuristic and innovative proof of the stability of the solitary wave given by Boussinesq [a] in 1872 through a 1D model on a flat bottom assuming $\varepsilon = O(\mu) \ll 1.$ These equations can be written as follows

$$\begin{cases} \dfrac{\partial}{\partial t}\xi + \dfrac{\partial}{\partial x}(hu) & = & O(\mu^2) \\ \dfrac{\partial}{\partial t}u + \varepsilon u \dfrac{\partial}{\partial x}u + \nabla \xi + \mu \mathcal{D}(u) & = & O(\mu^2) \end{cases}$$

with

| | | |
|---|---|---|
| $\varepsilon = \dfrac{a}{H},\ \mu = \left(\dfrac{H}{\lambda}\right)^2$ | : | non-linear parameter, dispersive parameter |
| $H,\ \xi,\ u$ | : | water depth, free surface elevation, averaged speed |
| $\mathcal{D}$ | : | dispersive term |

a. "All engineers know the beautiful experiments of J. Scott Russell and M. Basin on the generation and propagation of solitary waves" Joseph Valentin Boussinesq

- Everything starts with Russell's "Wave of translation"
- Heuristic and innovative proof of the stability of the solitary wave given by Boussinesq in 1872 through a 1D model on a flat bottom assuming $\varepsilon = O(\mu) \ll 1$.
- An other proof of the stability of the solitary wave given by introduced by Boussinesq (1877)/Korteweg and Gustav de Vries (1895) through a 1D scalar equation, a perfect equilibrium between non-linearities and the dispersion term,

$$u_t + 6uu_x + u_{xxx} = 0$$

- On the basis of this work, several models have been proposed :
  - 1967 : a first 2D formulation for non flat weakly dispersive and weakly non linear model of Boussinesq type was proposed by Peregrine

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  - 1976 : Green and Naghdi derived the famous 2D fully nonlinear dispersive equations for uneven bottom (1D below)

$$\begin{cases} \dfrac{\partial}{\partial t} h + \dfrac{\partial}{\partial x}(hu) = 0 \\ \dfrac{\partial}{\partial t}(hu) + \dfrac{\partial}{\partial x}\left(hu^2 + \dfrac{h^2}{2F_r^2}\right) + \mu \dfrac{\partial}{\partial x}\left(\dfrac{h^3}{3}\mathcal{D}(u)\right) = \end{cases}$$

where $\mathcal{D}(u) = \left(\frac{\partial}{\partial x}u\right)^2 - \frac{\partial}{\partial t}\frac{\partial}{\partial x}u - u\frac{\partial}{\partial x}\frac{\partial}{\partial x}u$

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  - 2021 : Debyaoui and Ersoy introduce the first section-averaged non-linear weakly dispersive equations for "arbitrary geometry"
  - Nowadays : Lannes, Bonneton, Cienfuegos, Dutykh, Gavrilyuk, Richard, Sainte-Marie, . . . proposed several improvements

- Waves may penetrate through rivers/channel much faster inland than the coastal inundation reaches over the ground, and may lead flooding in low-lying areas located several km away from the coastline !

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- How to model ?
  - 2D models for rivers/channels can be used but costly in the large scale simulation
  - Non-hydrostatic 1D section-averaged have not yet been derived $\rightarrow$ toward the first full non-linear and weakly dispersive model

Outline

1. Hydrostatic models, applications and limits
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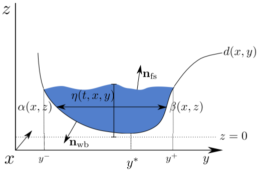
Incompressible and irrotational Euler equations

$$\begin{aligned} \mathrm{div}(\rho_0 \boldsymbol{u}) &= 0, \\ \frac{\partial}{\partial t}(\rho_0 \boldsymbol{u}) + \mathrm{div}(\rho_0 \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \rho_0 \boldsymbol{F} &= 0 \end{aligned}$$



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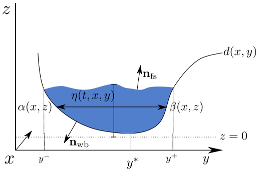
with

| | | |
|---|---|---|
| $\boldsymbol{u} = (u, v, w)$ | : | velocity field |
| $\rho_0$ | : | density |
| $\boldsymbol{F} = (0, 0, -g)$ | : | external force |
| $p$ | : | pressure |



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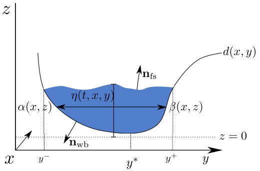


completed with the irrotational relations

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}.$$

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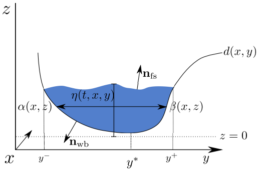



- free surface kinematic boundary condition,

$$\boldsymbol{u} \cdot \boldsymbol{n}_{\mathrm{fs}} = \frac{\partial}{\partial t}\boldsymbol{m} \cdot \boldsymbol{n}_{\mathrm{fs}} \text{ and } p = p_0, \ \forall \boldsymbol{m}(t,x,y) = (x,y,\eta(t,x,y)) \in \Gamma_{\mathrm{fs}}(t,x)$$

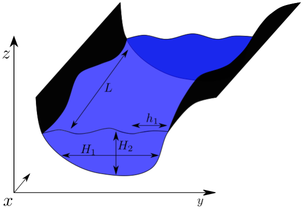
- no-penetration condition on the wet boundary

$$\boldsymbol{u} \cdot \boldsymbol{n}_{\mathrm{wb}} = 0, \ \forall \boldsymbol{m}(x,y) = (x,y,d(x,y)) \in \Gamma_{\mathrm{wb}}(x)$$

Let us define the dispersive parameters

- $\mu_1 = \dfrac{h_1^2}{L^2}$

- $\mu_2 = \dfrac{H_2^2}{L^2}$,




such that

$$h_1 < H_1 = H_2 \ll L, \text{i.e. } \mu_1 < \mu_2^2$$

where

| | | |
|---|---|---|
| $H_1$ | : | characteristic scale of channel width |
| $h_1$ | : | characteristic wave-length in the transversal direction |
| $H_2$ | : | characteristic water depth |
| $F_r = \dfrac{U}{\sqrt{gH_2}}$ | : | Froude's number |
| $T = \dfrac{L}{U}$ | : | characteristic time |
| $\mathcal{P} = U^2$ | : | characteristic pressure |
| $X$ | : | characteristic length of x |

Then, define the dimensionless variables

$$\widetilde{x} = \frac{x}{L}, \quad \widetilde{P} = \frac{P}{\mathcal{P}}, \quad \widetilde{\varphi} = \frac{\varphi}{h_1},$$

$$\widetilde{y} = \frac{y}{h_1}, \quad \widetilde{u} = \frac{u}{U}, \quad \widetilde{d} = \frac{d}{H_2},$$

$$\widetilde{z} = \frac{z}{H_2}, \quad \widetilde{v} = \frac{v}{V} = \frac{v}{\sqrt{\mu_1}U}, \quad \widetilde{\eta} = \frac{\eta}{H_2} \ .$$

$$\widetilde{t} = \frac{t}{T}, \quad \widetilde{w} = \frac{w}{W} = \frac{w}{\sqrt{\mu_2}U} \ .$$

We get

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} &= 0 \\
\mu_1 \left( \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) + \frac{\partial P}{\partial y} &= 0 \\
\mu_2 \left( \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) + \frac{\partial P}{\partial z} &= -\frac{1}{{F_r}^2}
\end{aligned}
$$

and

$$
\frac{\partial u}{\partial y} = \mu_1 \frac{\partial v}{\partial x}, \;\; \mu_1 \frac{\partial v}{\partial z} = \mu_2 \frac{\partial w}{\partial y}, \;\; \frac{\partial u}{\partial z} = \mu_2 \frac{\partial w}{\partial x} \; .
$$

## The new model : generalization of the SGN and Free surface flows equations

$$\begin{cases} \dfrac{\partial}{\partial t}A + \dfrac{\partial}{\partial x}Q = 0 \\ \dfrac{\partial}{\partial t}Q + \dfrac{\partial}{\partial x}\left(\dfrac{Q^2}{A} + I_1(x,A)\right) + \mu_2 \dfrac{\partial}{\partial x}(\mathcal{D}(u)G(A,x)) = I_2(x,A) \\ +\mu_2 \mathcal{G}(u,S,\sigma) + O(\mu_2^2) \end{cases}$$

where

$A = \displaystyle\int_{\Omega(t,x)} dy\,dz$ : wet area

$Q = A(t,x)u(t,x)$ : discharge

$I_1 = \displaystyle\int_{\Omega(t,x)} \frac{\eta(t,x) - z}{F_r^2}\sigma(x,z)\,dy\,dz$ : hydro. press.

$I_2 = -\displaystyle\int_{y^-(t,x)}^{y^+(t,x)} \frac{h(t,x)}{{F_r}^2}\frac{\partial}{\partial x}d(x,y)\,dy$ : hydro. press. source

# The new model : generalization of the SGN and Free surface flows equations

$$
\left\{
\begin{array}{l}
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\dfrac{\partial}{\partial t} Q + \dfrac{\partial}{\partial x}\left(\dfrac{Q^2}{A} + I_1(x, A)\right) + \mu_2 \dfrac{\partial}{\partial x}(\mathcal{D}(u) G(A, x)) = I_2(x, A) \\
+\mu_2 \mathcal{G}(u, S, \sigma) + O(\mu_2^2)
\end{array}
\right.
$$

where

$$
\mathcal{D}(u) = \left(\frac{\partial}{\partial x} u\right)^2 - \frac{\partial}{\partial t}\frac{\partial}{\partial x} u - u \frac{\partial}{\partial x}\frac{\partial}{\partial x} u
$$

and

$$
G(A, x) = \int_{d^*(x)}^{\eta} \sigma(x, z) \int_z^{\eta} \frac{S(x, s)}{\sigma(x, s)} \, ds \, dz
$$

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial t}A+\dfrac{\partial}{\partial x}Q=0\\ \dfrac{\partial}{\partial t}Q+\dfrac{\partial}{\partial x}\left(\dfrac{Q^2}{A}+I_1(x,A)\right)+\mu_2\dfrac{\partial}{\partial x}(\mathcal{D}(u)G(A,x))=I_2(x,A)\\ +\mu_2\mathcal{G}(u,S,\sigma)+O(\mu_2^2)\end{array}\right.$$

where

$$\begin{aligned}\mathcal{G}(u,S,\sigma) &= \int_z^\eta \frac{u^2}{\sigma(x,s)}\left(\frac{\frac{\partial}{\partial x}S(x,s)\frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)}-\frac{\partial}{\partial x}\frac{\partial}{\partial x}S(x,s)\right)\\ &\quad+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\frac{S(x,s)\frac{\partial}{\partial x}\sigma(x,s)}{\sigma(x,s)^2}\\ &\quad-\left(\frac{\partial}{\partial t}u+u\frac{\partial}{\partial x}u\right)\frac{\frac{\partial}{\partial x}S(x,s)}{\sigma(x,s)}\,ds\end{aligned}.$$

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t}A + \frac{\partial}{\partial x}Q = 0 \\ \frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + I_1(x,A)\right) + \mu_2 \frac{\partial}{\partial x}(\mathcal{D}(u)G(A,x)) = I_2(x,A) \\ +\mu_2 \mathcal{G}(u,S,\sigma) + O(\mu_2^2) \end{array} \right.$$

Setting $\sigma = 1$, $d = 1$,

- $A = h$
- $S(x,z) \equiv S(z) \Rightarrow \mathcal{G} = 0$ and $I_2 = 0$
- $G = \frac{h^3}{3}$
- $I_1 = \frac{h^2}{2F_r^2}$

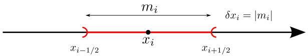
$$\begin{cases} \frac{\partial}{\partial t}A + \frac{\partial}{\partial x}Q = 0 \\ \frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + I_1(x,A)\right) + \mu_2 \frac{\partial}{\partial x}(\mathcal{D}(u)G(A,x)) = I_2(x,A) \\ +\mu_2 \mathcal{G}(u,S,\sigma) + O(\mu_2^2) \end{cases}$$

we recover the classical SGN equations on flat bottom

$$\begin{cases} \frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hu) = 0 \\ \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2 + \frac{h^2}{2F_r^2}\right) + \mu_2 \frac{\partial}{\partial x}\left(\frac{h^3}{3}\mathcal{D}(u)\right) = O(\mu_2^2) \end{cases}$$

where

$$\mathcal{D}(u) = \left(\frac{\partial}{\partial x}u\right)^2 - \frac{\partial}{\partial t}\frac{\partial}{\partial x}u - u\frac{\partial}{\partial x}\frac{\partial}{\partial x}u$$



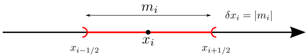

We consider a classical Finite Volume scheme, $\boldsymbol{U} = (A, Q)$

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\delta t^n}{\delta x}\left(\boldsymbol{F}_{i+1/2}(\boldsymbol{U}_i^n, \boldsymbol{U}_{i+1}^n) - \boldsymbol{F}_{i-1/2}(\boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n)\right)$$

where $\boldsymbol{F}_{i\pm1/2} \approx \dfrac{1}{\delta t^n}\displaystyle\int_{m_i} \boldsymbol{F}(\boldsymbol{U}(t, x_{i+1/2}))\, dx$ is a Finite volume solver,

with

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} Au \\ Au^2 + \dfrac{\kappa - 1}{\kappa}\left(I_1 -'' \displaystyle\int I_2''\right) \end{pmatrix}$$

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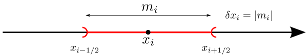
where $\boldsymbol{F}_{i\pm 1/2} \approx \dfrac{1}{\delta t^n}\displaystyle\int_{m_i} \boldsymbol{F}(\boldsymbol{U}(t, x_{i+1/2}))\, dx$ is a Finite volume solver, for instance, with upwind technique to deal with source term

$$\boldsymbol{F}_{i\pm 1/2} = \frac{\boldsymbol{F}(\boldsymbol{U}) + \boldsymbol{F}(\boldsymbol{V})}{2} - \frac{s_i^n}{2}(\boldsymbol{V} - \boldsymbol{U})$$

with

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} Au \\ Au^2 + \dfrac{\kappa - 1}{\kappa}\left(I_1 -'' \displaystyle\int I_2''\right) \end{pmatrix}$$

- Bourdarias, Ersoy, Gerbi. Journal of Scientific Computing, 2011

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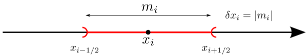
$$-\frac{\delta t^n}{\delta x}\left(\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1} \boldsymbol{D}^n\right)_i$$

with

$$(\boldsymbol{D}^n)_i = \boldsymbol{D}_{i+1/2}(\boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n, \boldsymbol{U}_{i+1}^n) - \boldsymbol{D}_{i-1/2}(\boldsymbol{U}_{i-2}^n, \boldsymbol{U}_{i-1}^n, \boldsymbol{U}_i^n)$$

where $\boldsymbol{D}_{i\pm 1/2}$ and $\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1}$ are the centred approximation of

$$\mathcal{D} = \frac{1}{\kappa}\left(\frac{\partial}{\partial x} I_1 - I_2\right) + \mu_2 A \mathcal{Q} \text{ and } \left[(I_d - \mu_2 \mathbb{L})\right]^{-1}$$

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$$-\frac{\delta t^n}{\delta x}\left(\left[(I_d - \mu_2 \mathbb{L})^n\right]^{-1} \boldsymbol{D}^n\right)_i$$

## Theorem

The numerical scheme is stable under the classical CFL condition,

$$\max_{\lambda \in \mathrm{Sp}(D_{\boldsymbol{U}} \boldsymbol{F}(\boldsymbol{U}))} |\lambda| \frac{\delta t^n}{\delta x} \leqslant 1 \ .$$

- Debyaoui, Ersoy. NumHyp, 2020

- Comparison with the NLSW and the exact solution

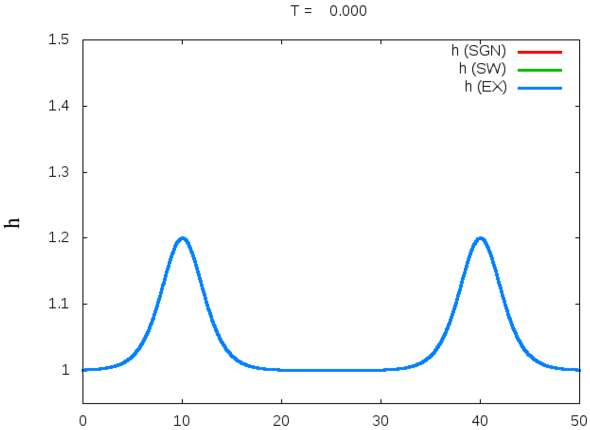
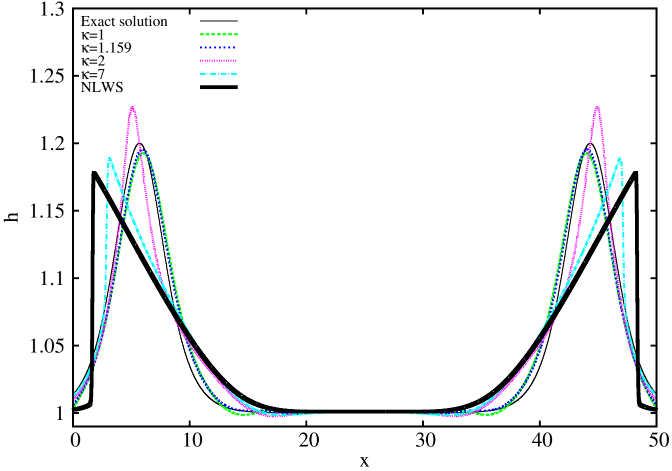


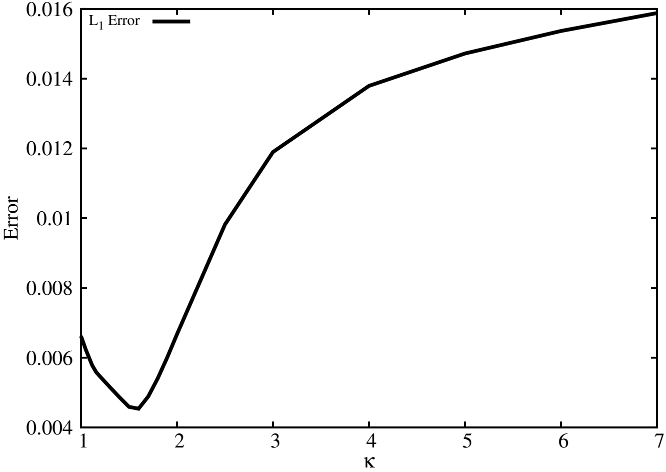

Figure – $\sigma = 1$, $d = 1$, $N = 1000$, $CFL = 0.95$, $T_f = 10$ and $\kappa = 1.159$

- Comparison with the NLSW and the exact solution
- Influence of $\kappa$




(b) Solutions at time $T_f = 10$

- Comparison with the NLSW and the exact solution
- Influence of $\kappa$




(d) $\| h_{ex} - h_{\kappa} \|_1$

To

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