

# PinT 2022

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# Two parts

## Two-parts:

1. Convergence of two-level PinT
2. Nonsymmetric PDE solvers for space-time/all-at-once

# Space-time operators

Space-time PDE:

$$u_t + \mathcal{L}(u, \mathbf{x}) = g(\mathbf{x}, t).$$

# Space-time operators

Space-time PDE:

$$u_t + \mathcal{L}(u, \mathbf{x}) = g(\mathbf{x}, t).$$

- Discretize and linearize  $\mathcal{L}$  in space
- $\mathbf{u}(t) = \mathbf{u}_{t_i}$  discrete solution in space at time  $t$

$$\implies \quad \mathbf{u}_t + \mathcal{L}(t)\mathbf{u} = \mathbf{g}(t).$$

# Multigrid reduction in time (MGRiT)

Reduction-based multigrid for matrices

$$A\mathbf{u} := \begin{bmatrix} I & & & & & \\ -\Phi_1 & I & & & & \\ & -\Phi_2 & I & & & \\ & & \ddots & \ddots & & \\ & & & -\Phi_{N-1} & I & \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_{N-1} \end{bmatrix} = \mathbf{g},$$

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- Split time points into C-points/F-points
- Time discretization provides error w.r.t. continuous; want discrete error/residual  $\rightarrow 0$  ( $\mathbf{e} = \hat{\mathbf{u}} - \mathbf{u}$  and  $\mathbf{r} = \mathbf{g} - A\mathbf{u}$ )

# Multigrid reduction in time (MGRiT)

Define  $\Phi_i^j := \Phi_i \Phi_{i-1} \dots \Phi_j$ . Then

$$\begin{bmatrix} I & & & & & \\ -\Phi_1 & I & & & & \\ & -\Phi_2 & I & & & \\ & & \ddots & \ddots & & \\ & & & -\Phi_{N-1} & I & \end{bmatrix}^{-1} = \begin{bmatrix} I & & & & & \\ \Phi_1 & I & & & & \\ \Phi_2^1 & \Phi_2 & I & & & \\ \Phi_3^1 & \Phi_3^2 & \Phi_3 & I & & \\ \vdots & \vdots & & \ddots & \ddots & \\ \Phi_{N-1}^1 & \Phi_{N-1}^2 & \dots & \dots & \Phi_{N-1} & I \end{bmatrix}.$$

Define  $A_\Delta \approx B_\Delta :=$

$$\begin{bmatrix} I & & & & & \\ -\Phi_k^1 & I & & & & \\ & -\Phi_{2k}^{k+1} & I & & & \\ & & \ddots & \ddots & & \\ & & & -\Phi_{(N_c-1)k}^{(N_c-2)k+1} & I & \end{bmatrix} \approx \begin{bmatrix} I & & & & & \\ -\Psi_1 & I & & & & \\ & -\Psi_2 & I & & & \\ & & \ddots & \ddots & & \\ & & & -\Psi_{N_c-1} & I & \end{bmatrix}.$$

# Error and residual propagation

$$\mathcal{E}_F^p := \begin{bmatrix} -A_{ff}^{-1}A_{fc} \\ I \end{bmatrix} \begin{bmatrix} \mathbf{0} & (I - B_{\Delta}^{-1}A_{\Delta})^p \end{bmatrix},$$

$$\mathcal{E}_{FCF}^p := \begin{bmatrix} -A_{ff}^{-1}A_{fc} \\ I \end{bmatrix} \begin{bmatrix} \mathbf{0} & \left( (I - B_{\Delta}^{-1}A_{\Delta})(I - A_{\Delta}) \right)^p \end{bmatrix},$$

$$\mathcal{R}_F^p := \begin{bmatrix} \mathbf{0} \\ (I - A_{\Delta}B_{\Delta}^{-1})^p \end{bmatrix} \begin{bmatrix} -A_{cf}A_{ff}^{-1} & I \end{bmatrix},$$

$$\mathcal{R}_{FCF}^p := \begin{bmatrix} \mathbf{0} \\ \left( (I - A_{\Delta}B_{\Delta}^{-1})(I - A_{\Delta}) \right)^p \end{bmatrix} \begin{bmatrix} -A_{cf}A_{ff}^{-1} & I \end{bmatrix}.$$

Restricted to C-points:

$$\begin{aligned} \tilde{\mathcal{E}}_F &:= I - B_{\Delta}^{-1}A_{\Delta}, & \tilde{\mathcal{E}}_{FCF} &:= (I - B_{\Delta}^{-1}A_{\Delta})(I - A_{\Delta}), \\ \tilde{\mathcal{R}}_F &:= I - A_{\Delta}B_{\Delta}^{-1}, & \tilde{\mathcal{R}}_{FCF} &:= (I - A_{\Delta}B_{\Delta}^{-1})(I - A_{\Delta}). \end{aligned}$$



# F-relaxation

Let  $\Phi_* := \Phi_{(N_c-1)k}^{(N_c-2)k+1}$  and define the shift operators and block diagonal matrix

$$I_L = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \\ & & & \end{bmatrix}, I_Z = \begin{bmatrix} I & & & \\ & \ddots & & \\ & & \mathbf{0} & \\ & & & \end{bmatrix}, D = \begin{bmatrix} \Phi_k^1 - \Psi_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \Phi_* - \Psi_{N_c-1} \\ & & & & \mathbf{0} \end{bmatrix}$$

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- $I_L D = (B_\Delta - A_\Delta)$  and  $I_L^T I_L = I_Z$ .

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- $I_L D = (B_\Delta - A_\Delta)$  and  $I_L^T I_L = I_Z$ .
- $\tilde{\mathcal{R}}_F = I - A_\Delta B_\Delta^{-1} = (B_\Delta - A_\Delta) B_\Delta^{-1} = I_L D B_\Delta^{-1}$ .

# F-relaxation

Let  $\Phi_* := \Phi \frac{\binom{N_c-2}{k+1}}{\binom{N_c-1}{k}}$  and define the shift operators and block diagonal matrix

$$I_L = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \\ & & & \ddots \end{bmatrix}, I_Z = \begin{bmatrix} I & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \mathbf{0} \end{bmatrix}, D = \begin{bmatrix} \Phi_k^1 - \Psi_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \Phi_* - \Psi_{N_c-1} \\ & & & & \mathbf{0} \end{bmatrix}$$

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$$\|\mathcal{R}_F\|^2 = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\langle I_L D B_\Delta^{-1} \mathbf{x}, I_L D B_\Delta^{-1} \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\langle I_Z D B_\Delta^{-1} \mathbf{x}, I_Z D B_\Delta^{-1} \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

# F-relaxation

Let  $\Phi_* := \Phi \begin{matrix} (N_c-2)k+1 \\ (N_c-1)k \end{matrix}$  and define the shift operators and block diagonal matrix

$$I_L = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \end{bmatrix}, I_Z = \begin{bmatrix} I & & & \\ & \ddots & & \\ & & \mathbf{0} & \end{bmatrix}, D = \begin{bmatrix} \Phi_k^1 - \Psi_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \Phi_* - \Psi_{N_c-1} \\ & & & & \mathbf{0} \end{bmatrix}$$

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- $I_L D = (B_\Delta - A_\Delta)$  and  $I_L^T I_L = I_z$ .
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$$\|\mathcal{R}_F\|^2 = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\langle I_L D B_\Delta^{-1} \mathbf{x}, I_L D B_\Delta^{-1} \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} = \|I_z D B_\Delta^{-1}\|^2 = \|\tilde{D} \tilde{B}_\Delta^{-1}\|^2.$$

# F-relaxation

Altogether:

$$\|\tilde{\mathcal{R}}_F\| = \sigma_{\max}(\tilde{D}\tilde{B}_\Delta^{-1}) = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|\tilde{D}\mathbf{x}\|}{\|\tilde{B}_\Delta\mathbf{x}\|} = \frac{1}{\sigma_{\min}(\tilde{B}_\Delta\tilde{D}^{-1})},$$

where  $\tilde{B}_\Delta\tilde{D}^{-1} :=$

$$\begin{bmatrix} I & & & & \\ -\Psi_1 & I & & & \\ & \ddots & \ddots & & \\ & & & -\Psi_{N_c-2} & I \end{bmatrix} \begin{bmatrix} (\Phi_k^1 - \Psi_1)^{-1} & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & & (\Phi_* - \Psi_{N_c-1})^{-1} \end{bmatrix}.$$

(similar results for  $\tilde{\mathcal{E}}_F$ ,  $\tilde{\mathcal{E}}_{FCF}$ , and  $\tilde{\mathcal{R}}_{FCF}$ ).

# Diagonalizable operators

## Theorem 1.

Let  $\Phi, \Psi$  be independent of time and simultaneously diagonalizable with eigenvectors  $U$ , eigenvalues  $\{\lambda, \mu\}$ , coarsening factor  $k$ , and  $N_c$  coarse-grid time points. Then,

$$\sup_i \frac{|\mu_i - \lambda_i^k|}{\sqrt{(1 - |\mu_i|)^2 + \frac{\pi^2 |\mu_i|}{N_c^2}}} \leq \|\mathcal{R}_F\|_{(UU^*)^{-1}} \leq \sup_i \frac{|\mu_i - \lambda_i^k|}{\sqrt{(1 - |\mu_i|)^2 + \frac{\pi^2 |\mu_i|}{6N_c^2}}},$$
$$\sup_i \frac{|\lambda_i^k| |\mu_i - \lambda_i^k|}{\sqrt{(1 - |\mu_i|)^2 + \frac{\pi^2 |\mu_i|}{N_c^2}}} \leq \|\mathcal{R}_{FCF}\|_{(UU^*)^{-1}} \leq \sup_i \frac{|\lambda_i^k| |\mu_i - \lambda_i^k|}{\sqrt{(1 - |\mu_i|)^2 + \frac{\pi^2 |\mu_i|}{6N_c^2}}}.$$



# Time-independent operators

## Definition 2 (Temporal approximation property).

Let  $\Phi \sim$  fine-grid and  $\Psi \sim$  coarse-grid time-stepping, independent of time, and with coarsening factor  $k$ .  $\Phi$  satisfies an F-relaxation temporal approximation property (F-TAP) with respect to  $\Psi$  with constant  $\varphi_F$ , if, for all  $\mathbf{v}$ ,

$$\|(\Psi - \Phi^k)\mathbf{v}\| \leq \varphi_F \left[ \min_{x \in [0, 2\pi]} \|(I - e^{ix}\Psi)\mathbf{v}\| \right]. \quad (1)$$

Similarly,  $\Phi$  satisfies an FCF-TAP with respect to  $\Psi$  with constant  $\varphi_{FCF}$ , if, for all  $\mathbf{v}$ ,

$$\|(\Psi - \Phi^k)\mathbf{v}\| \leq \varphi_{FCF} \left[ \min_{x \in [0, 2\pi]} \|(\Phi^{-k}(I - e^{ix}\Psi))\mathbf{v}\| \right]. \quad (2)$$

# Time-independent operators

## Theorem 2 (Necessary and sufficient conditions).

Suppose  $\Phi$  and  $\Psi$  are linear, stable ( $\|\Phi^p\|, \|\Psi^p\| < 1$  for some  $p$ ), and independent of time; and that  $(\Psi - \Phi^k)$  is invertible. Suppose  $\Phi$  satisfies an F-TAP w.r.t.  $\Psi$  with constant  $\varphi_F$ , and  $\Phi$  satisfies an FCF-TAP w.r.t.  $\Psi$  with constant  $\varphi_{FCF}$ . Then, worst-case convergence of residual is exactly bounded by

$$\frac{\varphi_F}{1 + O(1/\sqrt{N_c})} \leq \frac{\|\mathbf{r}_{i+1}^{(F)}\|}{\|\mathbf{r}_i^{(F)}\|} < \varphi_F,$$
$$\frac{\varphi_{FCF}}{1 + O(1/\sqrt{N_c})} \leq \frac{\|\mathbf{r}_{i+1}^{(FCF)}\|}{\|\mathbf{r}_i^{(FCF)}\|} < \varphi_{FCF}$$

for iterations  $i > 1$  (i.e., not the first iteration).

# Related work

## Lemma 3.

Suppose  $\Psi$  is real-valued. Then,

$$\min_{x \in [0, 2\pi]} \|(I - e^{ix}\Psi)\mathbf{v}\|^2 = \|\mathbf{v}\|^2 + \|\Psi\mathbf{v}\|^2 - 2|\langle \Psi\mathbf{v}, \mathbf{v} \rangle|.$$

**B. S. Southworth.** *Necessary conditions and tight two-level convergence bounds for parareal and multigrid reduction in time.*

**B. S. Southworth et al.** *Tight two-level convergence of Linear Parareal and MGRIT: Extensions and implications in practice.*

**S. Friedhoff and B. S. Southworth.** *On “Optimal”  $h$ -independent convergence of Parareal and multigrid-reduction-in-time using Runge-Kutta time integration.*

# Algebraic reduction (and advection)

Two main points:

- Most PinT methods struggle with strong advection (as do  $p$ -multigrid, and many block preconditioning methods!)
-

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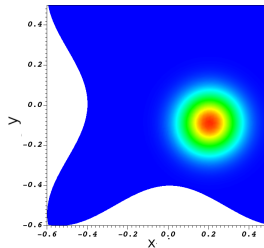
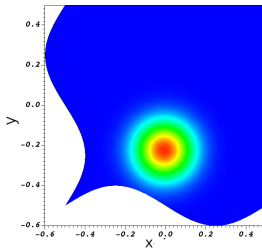
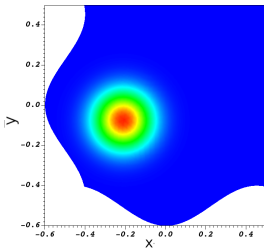
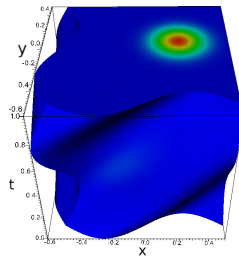
AMG and space-time advection-diffusion

# Advection-diffusion

$$\partial_t u + \mathbf{a} \cdot \nabla u - \nu \nabla^2 u = f$$



$$\hat{\mathbf{a}} \cdot \hat{\nabla} u - \nu \nabla^2 u = f$$



# Conceptual basis for AIR

Partition (discontinuous) elements into C-elements and F-elements.  
Then in matrix form,

$$\begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{ff}^{-1}A_{fc} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{ff}^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{cf}A_{ff}^{-1} & I \end{bmatrix}.$$

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AIR preconditioner  $M^{-1}$  looks like:

$$M^{-1} = \begin{bmatrix} I & \widehat{W} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta_F & 0 \\ 0 & \mathcal{K}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix}.$$

Where  $\mathcal{K} = RAP$ . Want  $\Delta_F \simeq A_{ff}^{-1}$ ,  $Z \simeq -A_{cf}A_{ff}^{-1}$ , etc.



# Conceptual basis for AIR

Partition (discontinuous) elements into C-elements and F-elements.  
Then in matrix form,

$$\begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{ff}^{-1}A_{fc} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{ff}^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{cf}A_{ff}^{-1} & I \end{bmatrix}.$$

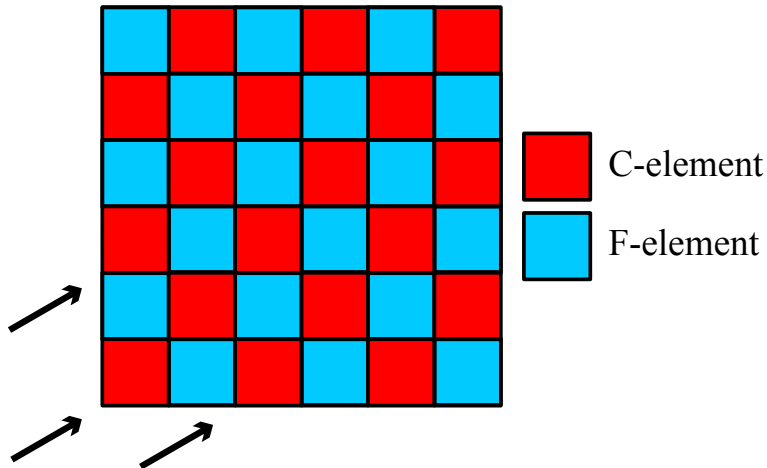
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Where  $\mathcal{K} = RAP$ . Want  $\Delta_F \simeq A_{ff}^{-1}$ ,  $Z \simeq -A_{cf}A_{ff}^{-1}$ , etc.

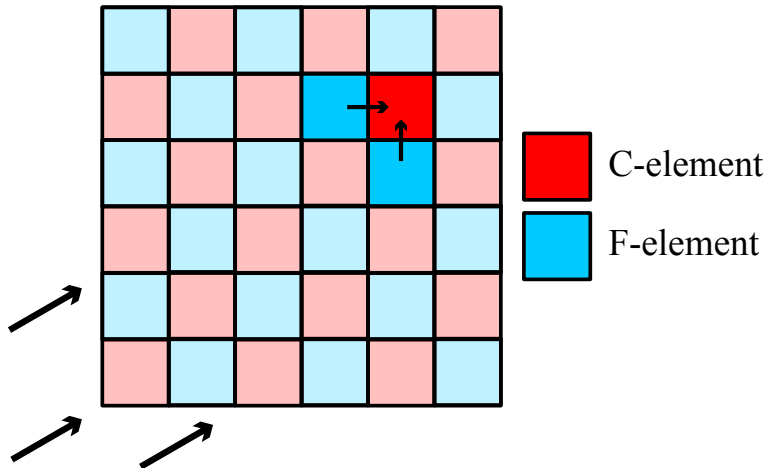
$\implies$  Can we approximate  $A_{ff}^{-1}$  well?

# Conceptual basis for AIR



Consider transport on structured 2d grid and partition elements into C-elements and F-elements.

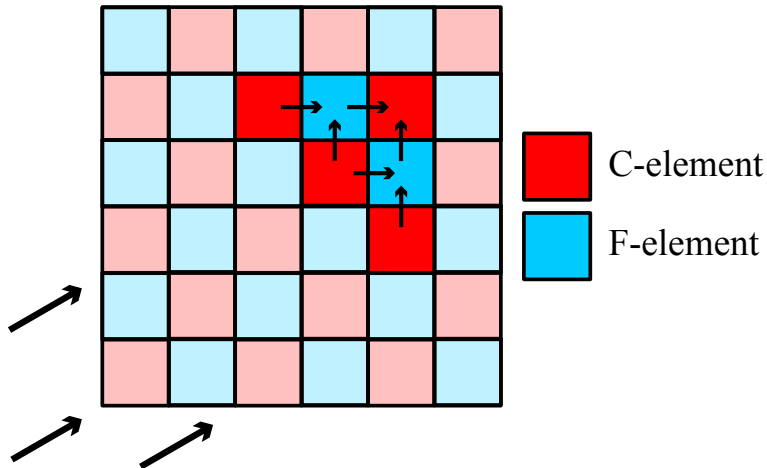
# Conceptual basis for AIR



Notice that there are no C-C or F-F connections

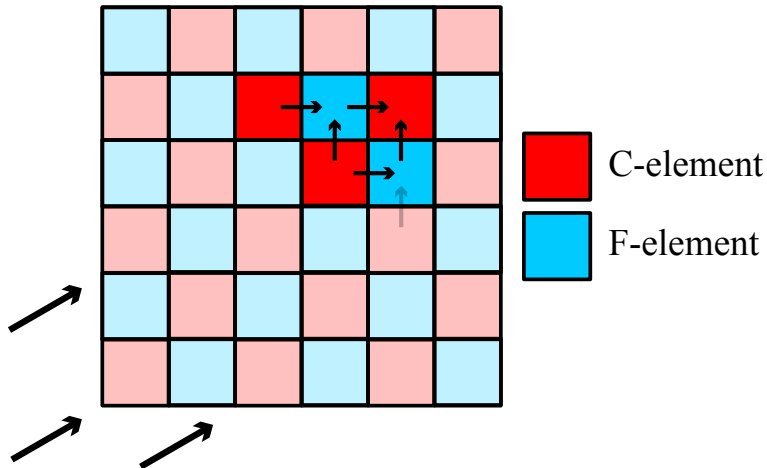
$$\implies A_{ff} = A_{cc} = I.$$

# Conceptual basis for AIR



If  $A_{ff} = I$ , AMG coarse grid given by  $A_{cc} - A_{cf}A_{fc} \iff$  all C-F-C connections.

# Conceptual basis for AIR



If  $A_{ff} = I$ , AMG coarse grid given by  $A_{cc} - A_{cf}A_{fc} \iff$  all C-F-C connections. **One of these connections is weak!**

# Space-time coarsening

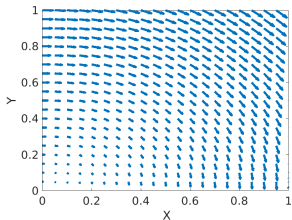


Fig.: Vel. field

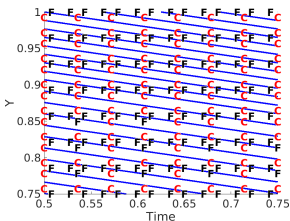


Fig.:  $x = 0.1861$

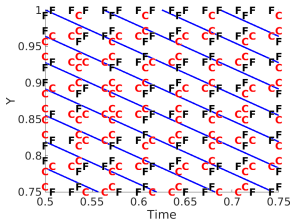


Fig.:  $x = 0.5790$

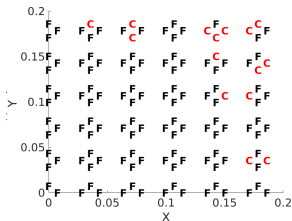


Fig.:  $t = 0.3853$

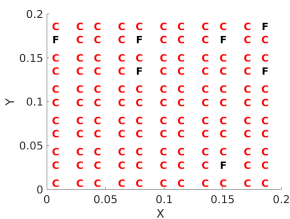


Fig.:  $t = 0.3928$

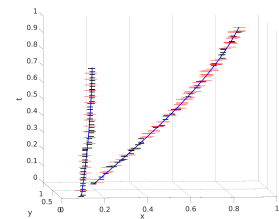


Fig.: Space-time char.

# Space-time AMR

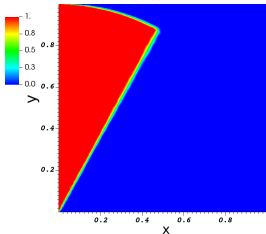
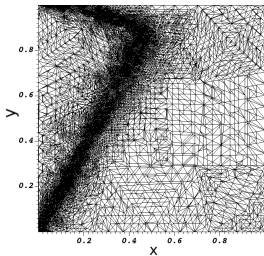


Fig.: Mesh/solution at  $t = 0.5$

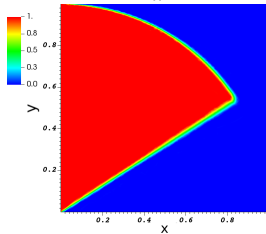
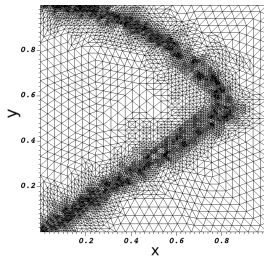


Fig.: Mesh/solution at  $t = 1$

# Results

- AIR-AMG applied to hybridized DG (EDG and HDG)
- BiCGSTAB iterations to relative residual  $10^{-12}$ .

**$p = 1$**

n	$\nu$				
	$10^{-6}$	$10^{-4}$	$10^{-3}$	$10^{-2}$	$10^{-1}$
136K	8	7	8	10	17
1M	8	8	10	13	54
8.5M	8	9	12	18	×

**$p = 2$**

n	$\nu$				
	$10^{-6}$	$10^{-4}$	$10^{-3}$	$10^{-2}$	$10^{-1}$
272K	8	9	10	14	30
2.1M	9	11	13	18	46
17M	9	14	15	30	83

**$p = 3$**

n	$\nu$				
	$10^{-6}$	$10^{-4}$	$10^{-3}$	$10^{-2}$	$10^{-1}$
454K	9	11	12	18	38
3.6M	9	13	15	25	73
28M	10	17	18	46	144



# Results

- AIR-AMG applied to hybridized DG (EDG and HDG)
- BiCGSTAB iterations to relative residual  $10^{-12}$ .

$p = 1$

n	$\nu$				
	$10^{-6}$	$10^{-4}$	$10^{-3}$	$10^{-2}$	$10^{-1}$
136K	8	7	8	10	17
1M	8	8	10	13	54
<b>245K</b>	<b>17</b>	<b>17</b>	<b>16</b>	<b>15</b>	<b>12</b>

$p = 2$

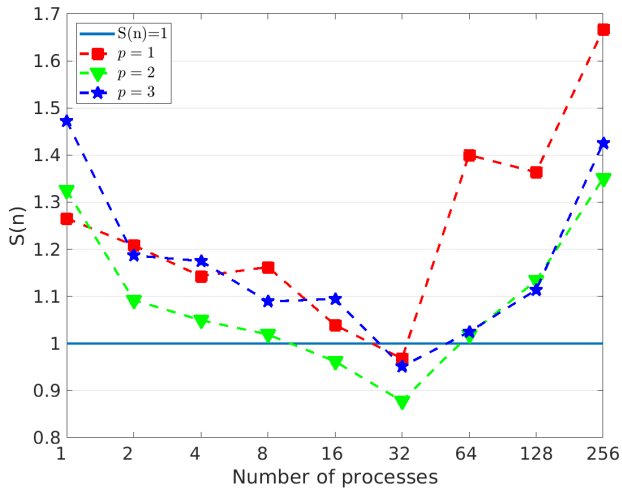
n	$\nu$				
	$10^{-6}$	$10^{-4}$	$10^{-3}$	$10^{-2}$	$10^{-1}$
272K	8	9	10	14	30
2.1M	9	11	13	18	46
<b>1.9M</b>	<b>17</b>	<b>16</b>	<b>19</b>	<b>21</b>	<b>30</b>

$p = 3$

n	$\nu$				
	$10^{-6}$	$10^{-4}$	$10^{-3}$	$10^{-2}$	$10^{-1}$
454K	9	11	12	18	38
3.6M	9	13	15	25	73
<b>6.5M</b>	<b>19</b>	<b>17</b>	<b>14</b>	<b>16</b>	<b>41</b>

**EDG**

# Speedup over sequential



# Navier Stokes

# Navier Stokes

- Working on block AIR for incompressible NS. Initial results for steady decent up to Reynolds  $10^5$ .

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- Working on block AIR for incompressible NS. Initial results for steady decent up to Reynolds  $10^5$ .
- Working on AIR for shallow water/compressible NS/Euler. Haven't tested yet.

# Navier Stokes

- Working on block AIR for incompressible NS. Initial results for steady decent up to Reynolds  $10^5$ .
- Working on AIR for shallow water/compressible NS/Euler. Haven't tested yet.
- $\implies$  Add time to discretization, solve all at once!

# Thank you!

## Papers:

T. A. Manteuffel, J. Ruge, and B. S. Southworth. *Nonsymmetric Algebraic Multigrid Based on Local Approximate Ideal Restriction ( $\ell$ AIR)*.

A. A. Sivas, B. S. Southworth, and S. Rhebergen. *AIR algebraic multigrid for a space-time hybridizable discontinuous Galerkin discretization of advection (-diffusion)*.