Non-overlapping Schwarz Waveform-Relaxation for Nonlinear Advection-Diffusion Equations

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PinT 11 CIRM July 11 – 15, 2022

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Outline

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Nonlinear Advection-Diffusion Equation

$$\partial_t u + \operatorname{div}\big(\mathbf{f}(u) - p(u)\nabla u\big) = 0 \quad \text{in} \quad \Omega \times (0,T)$$

$$u|_{t=0} = u_0 \quad \text{in} \quad \Omega, \qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega \times (0,T)$$

- Bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $0 < T < \infty$
- Diffusion coefficient $p \in C_b^{0,1}(\mathbb{R})$ with $p(v) \geq \overline{p} > 0$ for all $v \in \mathbb{R}$
- Advective flux $\mathbf{f} \in C_b^{0,1}(\mathbb{R}, \mathbb{R}^d)$
- Initial data $u_0 \in H_0^1(\Omega)$

Nonlinear Advection-Diffusion Equation

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$$u|_{t=0} = u_0 \quad \text{in} \quad \Omega, \qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega \times (0,T)$$

Definition (Weak mono-domain solution)

A weak mono-domain solution $u \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ satisfies

$$\int_0^T \langle \partial_t u,\ v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \big(p(u)\nabla u - \mathbf{f}(u),\ \nabla v\big)_{L^2(\Omega)}\, dt = 0$$

for all $v \in L^2(0,T;H_0^1(\Omega))$, and $u|_{t=0} = u_0$ a.e. in Ω.

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$$u|_{t=0} = u_0 \quad \text{in} \quad \Omega, \qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega \times (0,T)$$

Lemma (Existence and uniqueness of a weak mono-domain solution)

Under the stated assumptions, there exists a unique weak mono-domain solution $u \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, *which also satisfies* $\partial_t u \in L^2(0,T;L^2(\Omega))$.

Proof.

Application of [Alt, Luckhaus 1983] for the Kirchhoff transformed solution □

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Non-overlapping Schwarz Waveform-Relaxation Algorithm

Iteratively solve for $k \in \mathbb{N}$ and $i \in \{1,2\}$:

$$\partial_t u_i^k + \operatorname{div}\big(\mathbf{f}(u_i^k) - p(u_i^k)\nabla u_i^k\big) = 0 \quad \text{in} \quad \Omega_i \times (0,T)$$

$$u_i^k|_{t=0} = u_0 \quad \text{in} \quad \Omega_i, \qquad\qquad u_i^k = 0 \quad \text{on} \quad (\partial\Omega_i \setminus \Gamma) \times (0,T)$$

$$\mathfrak{B}_i(u_i^k) = \mathfrak{B}_i(u_{3-i}^{k-1}) \quad \text{on} \quad \Gamma \times (0,T)$$

$$\mathfrak{B}_i(u) = \big(p(u)\nabla u - \mathbf{f}(u)\big)\cdot \mathbf{n}_i + \lambda u \qquad \Leftrightarrow \qquad \mathfrak{B}_i(u_i^k) = 2\lambda u_{3-i}^{k-1} - \mathfrak{B}_{3-i}(u_{3-i}^{k-1})$$

- Non-overlapping, Lipschitz sub-domains Ω_1 and Ω_2, interface $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$
- Transmission parameter $\lambda > 0$
- Initial guesses $\mathfrak{B}_i(u_i^1) = g_i \in L^2(0,T;L^2(\Gamma))$

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Non-overlapping Schwarz Waveform-Relaxation Algorithm

Definition (Weak solution to the SWR algorithm)

For $i \in \{1,2\}$ and $k \in \mathbb{N}$, a weak SWR solution $u_i^k \in H^1(0,T;X_i^*) \cap L^2(0,T;X_i)$ satisfies

$$\int_0^T \left\langle \partial_t u_i^k,\ v \right\rangle_{X_i^*, X_i} + \left(p(u_i^k) \nabla u_i^k - \mathbf{f}(u_i^k),\ \nabla v \right)_{L^2(\Omega_i)} + \left(\lambda u_i^k - \mathfrak{B}_i^k,\ v \right)_{L^2(\Gamma)} dt = 0$$

for all $v \in L^2(0,T;X_i)$, and $u_i^k|_{t=0} = u_0$ a.e. in Ω_i. Here, $\mathfrak{B}_i^1 = g_i$ and

$$\mathfrak{B}_i^{k+1} = 2\lambda u_{3-i}^k - \mathfrak{B}_{3-i}^k \quad \text{in} \quad L^2((0,T);L^2(\Gamma)) \quad \text{for} \quad k \in \mathbb{N}.$$

$$X_i := \left\{ v \in H^1(\Omega_i) \ : \ v|_{\partial\Omega_i \setminus \Gamma} = 0 \right\}$$

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Non-overlapping Schwarz Waveform-Relaxation Algorithm

Theorem (Existence of weak solutions to the SWR algorithm)

Under the stated assumptions, for $i \in \{1,2\}$ *and all* $k \in \mathbb{N}$ *a weak SWR solution* $u_i^k \in H^1(0,T;X_i^*) \cap L^2(0,T;X_i)$ *exists and satisfies*

$$\|u_i^k\|^2_{L^\infty(0,T;L^2(\Omega_i))} + \|u_i^k\|^2_{L^2(0,T;X_i)} + \|\partial_t u_i^k\|^2_{L^2(0,T;X_i^*)} + \|u_i^k\|^2_{L^2(0,T;L^2(\Gamma))} \leq C\left(1 + \|\mathfrak{B}_i^k\|^2_{L^2(0,T;L^2(\Gamma))}\right),$$

where $C = C(|\Omega_i|, T, \lambda, \overline{p}, \|p\|_{C_b^0}, \|\mathbf{f}\|_{C_b^0}, \|u_0\|_{H^1(\Omega_i)})$.

Proof.

- Time-discrete problem has unique solution for Δt small enough
- A-priori estimates for the time-discrete solutions
- Compactness arguments for limit $\Delta t \to 0$ □

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Non-overlapping Schwarz Waveform-Relaxation Algorithm

Theorem (Uniqueness of the SWR iteration)

Under the stated assumptions, weak solutions u_i^k *to the SWR algorithm are unique if they satisfy* $\partial_t u_i^k \in L^2(0,T;L^2(\Omega_i))$ *for* $i \in \{1,2\}$ *and all* $k \in \mathbb{N}$.

Proof.

- Based on Theorem 2.2 in [Alt, Luckhaus 1983]
- Take the difference of the equations
- Test with the regularized positive part of the difference of weak solutions
- The limit of the regularization to zero yields uniqueness in $L^1(\Omega_i \times (0,T))$ □

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Non-overlapping Schwarz Waveform-Relaxation Algorithm

Theorem (Convergence of the SWR iteration)

Under the stated assumptions, if the weak mono-domain solution u satisfies

$$\nabla u \in L^2(0,T;L^\infty(\Omega)) \quad \text{and} \quad p(u)\nabla u \cdot \mathbf{n}_i \in L^2(0,T;L^2(\Gamma)),$$

then the weak solutions of SWR algorithm converge, i.e.

$$u_i^k \to u \quad \text{in} \quad L^\infty(0,T;L^2(\Omega_i)) \cap L^2(0,T;X_i) \quad \text{as} \quad k \to \infty.$$

Proof.

- Energy estimates for the errors $e_i^k := u_i^k - u|_{\Omega_i} \in H^1(0,T;X_i^*) \cap L^2(0,T;X_i)$
- Summation over sub-domains and iterations
- Uniformly bounded total error by Gronwall's lemma

□

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Numerical Discretization by the Finite Volume Method

Definition (Cell-centered finite volume method on $\omega \in \{\Omega, \Omega_1, \Omega_2\}$)

Let $N \in \mathbb{N}$ and $\Delta t := T/N$. For $n \in \{0, \ldots, N-1\}$ and each equilateral triangle $\mathscr{T}$ of the conforming triangulation $\mathscr{T}_{\omega,\Delta x}$, the cell-average value at time $t^{n+1} = (n+1)\Delta t$ is

$$u_{\mathscr{T}}^{n+1} = u_{\mathscr{T}}^{n} - \frac{\Delta t \Delta x}{|\mathscr{T}|} \sum_{\sigma \in \mathscr{S}(\mathscr{T})} F_\sigma^n .$$

The numerical flux F_σ^n on edge σ is given by [Eymard et al. 2000; Kurganov, Petrova 2005]

- $F_\sigma^n := -\frac{P(u_{\mathscr{T}'}^n) - P(u_{\mathscr{T}}^n)}{\Delta x/\sqrt{3}} + \left(\frac{a_\sigma^{in} \mathbf{f}(u_{\mathscr{T}'}^n) + a_\sigma^{out} \mathbf{f}(u_{\mathscr{T}}^n)}{a_\sigma^{in} + a_\sigma^{out}} \right) \cdot \mathbf{n}_{\mathscr{T}} - \frac{a_\sigma^{in} a_\sigma^{out}}{a_\sigma^{in} + a_\sigma^{out}} \left(u_{\mathscr{T}'}^n - u_{\mathscr{T}}^n \right)$ for $\sigma \in \mathscr{S}(\mathscr{T}')$,
- the boundary condition for $\sigma \in \partial\omega \cap \partial\Omega$,
- the discrete transmission conditions for $\sigma \in \partial\omega \cap \Gamma$.

The initial cell-average value is $u_{\mathscr{T}}^0 = u_0(\mathbf{x}_{\mathscr{T}})$, where $\mathbf{x}_{\mathscr{T}}$ denotes the barycenter of $\mathscr{T}$.

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Numerical Discretization by the Finite Volume Method

Asymptotic-Preserving Robin Transmission Condition

- Conforming meshes $\mathcal{T}_{\Omega_1,\Delta x} \cup \mathcal{T}_{\Omega_2,\Delta x} = \mathcal{T}_{\Omega,\Delta x}$
- Numerical flux $F_{\sigma,1}^{k,n}$ at edge $\sigma \subset \Gamma$ of the triangle $\mathcal{T} \in \Omega_1$

$$F_{\sigma,1}^{k,n} := -\frac{P(u_{\sigma,1}^{k,n}) - P(u_{\mathcal{T}}^{k,n})}{\Delta x/\sqrt{3}} + \left(\frac{a_\sigma^{in}\mathbf{f}(u_{\sigma,1}^{k,n}) + a_\sigma^{out}\mathbf{f}(u_{\mathcal{T}}^{k,n})}{a_\sigma^{in} + a_\sigma^{out}}\right) \cdot \mathbf{n}_{\mathcal{T}} - \frac{a_\sigma^{in} a_\sigma^{out}}{a_\sigma^{in} + a_\sigma^{out}} \left(u_{\sigma,1}^{k,n} - u_{\mathcal{T}}^{k,n}\right),$$

- Ghost value $u_{\sigma,1}^{k,n}$ given by the transmission condition

$$F_{\sigma,1}^{k,n} = \lambda\left(\beta_1 u_{\mathcal{T}}^{k,n} + (1-\beta_1)\, u_{\sigma,1}^{k,n}\right) - \mathfrak{B}_{\sigma,1}^{k,n}$$

  with weighting parameters $\beta_1, \beta_2 \in [0,1]$
- Analogously for $\mathcal{T}' \in \Omega_2$
- Nonlinear equations for $F_{\sigma,i}^{k,n}$ and $u_{\sigma,i}^{k,n}$ solved locally by a (damped) Newton method

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Numerical Discretization by the Finite Volume Method

Asymptotic-Preserving Robin Transmission Condition

(A) $$\beta_i = \begin{cases} \frac{1}{2} & a_\sigma^{out} \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$ or (B) $$\beta_i = \frac{1}{2}$$

asymmetric upwind [GLR 2022]

Classical centered

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Numerical Discretization by the Finite Volume Method

Asymptotic-Preserving Robin Transmission Condition

(A) $\beta_i = \begin{cases} \frac{1}{2} & a_\sigma^{out} \geq 0, \\ 0 & \text{otherwise,} \end{cases}$ or (B) $\beta_i = \frac{1}{2}$

asymmetric upwind [GLR 2022] — Classical centered

- Continuous 1D problem: $a u_x - \nu u_{xx} = f$ in $(-1,1)$, $u(-1) = 0$, $\nu u(1) = 0$
- Cell-centered FVM with discrete Robin transmission operators

$$B_i^h(\mathbf{u}^h) := (-1)^i \left(a u_{-1} - \frac{\nu}{h}(u_0 - u_{-1}) \right) + \lambda \left((1-\beta_i) u_{-1} + \beta_i u_0 \right)$$

- Consistency, convergence and AP property

$$\beta_1 \geq \beta_2, \quad \nu/\beta_1 = o(1) \quad \wedge \quad \beta_2 = \mathcal{O}(\nu) \quad \text{as} \quad \nu \to 0, \quad \rho = \left| \frac{\lambda - a}{\lambda + a} \right| \left| \frac{\nu}{\nu + a h \beta_1} \right| + \mathcal{O}\left(\left(\frac{\nu}{\nu + a h} \right)^{N-1} \right)$$

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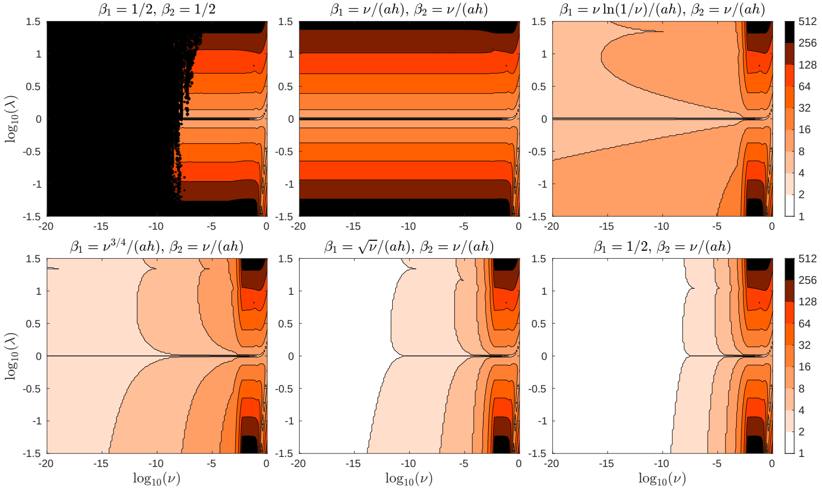
Numerical Discretization by the Finite Volume Method



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Numerical Results: Simplified Two-phase Flow

- Simplified two-phase flow in a porous medium

$$\partial_t u + \operatorname{div}\left(\frac{u^2}{u^2+(1-u)^2}\mathbf{v} + \kappa \nabla p_c(u) \right) = 0 \qquad \text{in} \quad \Omega \times (0,T) = (-1,1)^2 \times (0,1)$$

- Parameters

$$\mathbf{v} = \binom{v_1}{0}, \qquad v_1 > 0, \qquad p_c(u) = 1 - u,$$

$$u_0 \equiv 0, \qquad u(\mathbf{x},t)\Big|_{\partial\Omega} = \begin{cases} 1 - x_2^2 & \text{for } x_1 = -1 \\ 0 & \text{otherwise} \end{cases}$$

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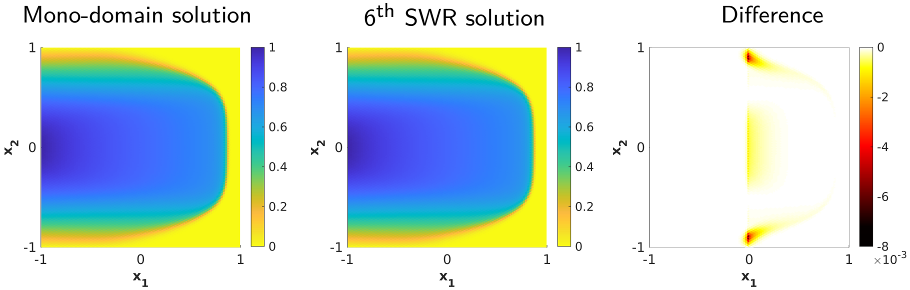
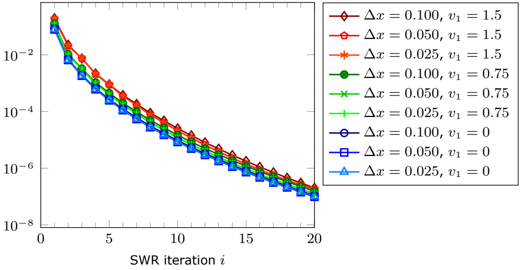


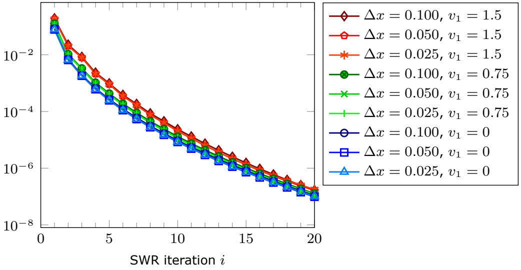
Numerical Results: Simplified Two-phase Flow

Diffusion-dominated regime ($\kappa = 1$)

L^2-error $\mathcal{E}^i_{\Delta x}$ for choice (A)



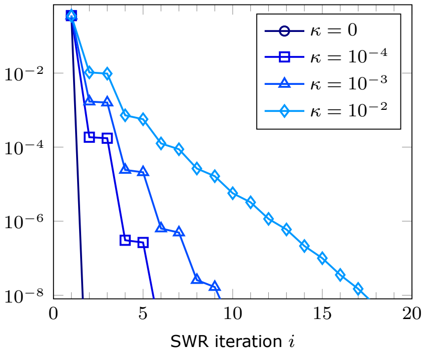
L^2-error $\mathcal{E}^i_{\Delta x}$ for choice (B)



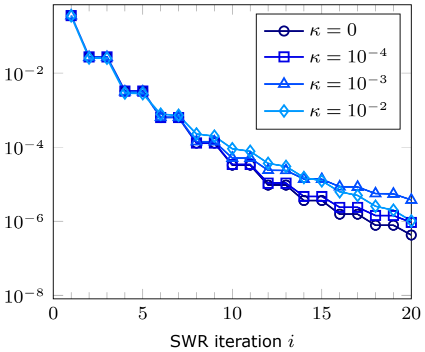
Numerical Results: Simplified Two-phase Flow

Advection-dominated regime ($v_1 = 1.5$)

L^2-error $\mathcal{E}^i_{\Delta x}$ for choice (A)



L^2-error $\mathcal{E}^i_{\Delta x}$ for choice (B)



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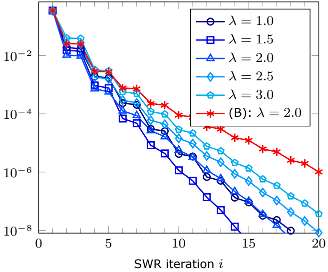
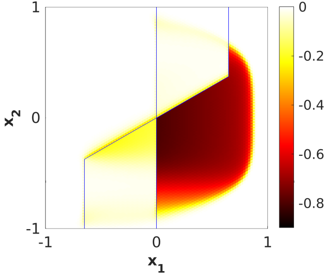
Numerical Results: Simplified Two-phase Flow

Effect of the transmission parameter and multiple sub-domains $(\kappa = 0.01,\ v_1 = 1.5)$

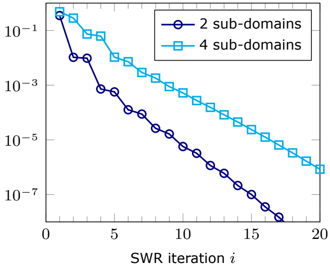
L^2-error $\mathcal{E}^i_{\Delta x}$ $(\Delta x = 0.025)$



L^2-error $\mathscr{E}^2_{0.025}$



L^2-error $\mathscr{E}^i_{\Delta x}(\Delta x = 0.025)$



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Numerical Results: Two-phase Flow with Nonlinear p_c

- Two-phase flow in a porous medium with Brooks–Corey parameterization

$$\partial_t u + \operatorname{div}\left(\frac{u^2}{u^2+(1-u)^2}\mathbf{v} + \frac{u^2(1-u)^2}{u^2+(1-u)^2}\nabla p_c(u)\right) = 0 \qquad \text{in} \quad \Omega \times (0,T) = (-1,1)^2 \times (0,1)$$

- Parameters

$$\mathbf{v} = \binom{1.5}{0}, \qquad p_c(u) = u^{-\frac{1}{\lambda_{BC}}}, \qquad \lambda_{BC} = 3,$$

$$u_0 = u\Big|_{x_1=+1} \equiv 0.1, \quad u(\mathbf{x},t)\Big|_{x_1=-1} = 0.1 + 0.8(1-x_2^2)^2, \quad \nabla u(\mathbf{x},t)\cdot\mathbf{n}\Big|_{x_2=\pm 1} = 0$$

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Numerical Results: Two-phase Flow with Nonlinear p_c

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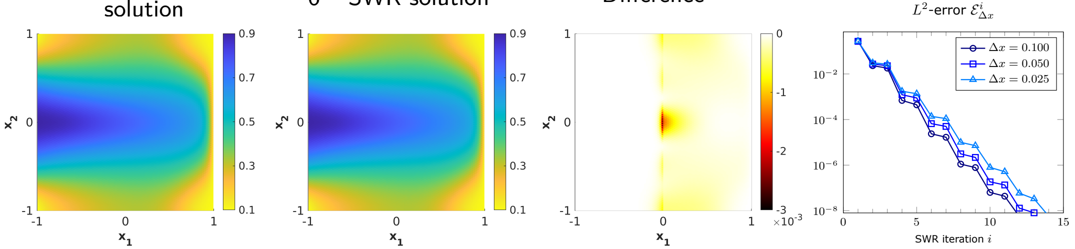
Mono-domain solution

6^{th} SWR solution

Difference



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Conclusion and Outlook

- Non-overlapping SWR algorithm with nonlinear Robin transmission conditions for nonlinear advection-diffusion equations
- Proofs for the existence (and uniqueness) of SWR iterates and for their convergence
- Asymptotic-preserving finite volume scheme (robust in the hyperbolic limit)
- Several numerical examples
  - Linear convergence towards the discrete mono-domain solution
  - Two-step convergence in the hyperbolic limit

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- Non-overlapping SWR algorithm with nonlinear Robin transmission conditions for nonlinear advection-diffusion equations
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- Several numerical examples
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- ▷ Asymptotic-preserving property for nonlinear, time-dependent problems
- ▷ Convergence proof for (asymptotic-preserving) finite volume schemes
- ▷ Transmission parameter optimization
- ▷ Reaction terms and unbounded/degenerate equations

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Thank you

Research Foundation Flanders
Opening new horizons

Swiss National Science Foundation

Deutsche Forschungsgemeinschaft
German Research Foundation

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