# Optimized Schwarz Waveform Relaxation Method for 1D Shallow Water equations

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Joint work with Martin Gander

# Outline

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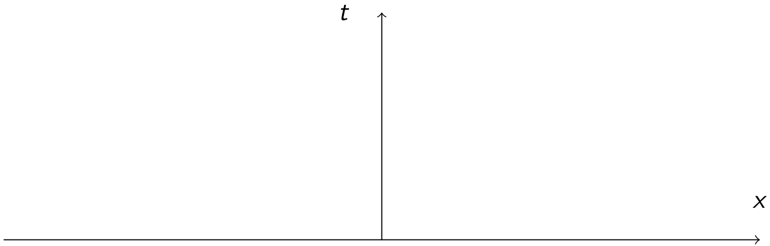
# Introduction

We want to solve the time dependant pde

$$\begin{cases} \mathcal{L}W = f & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ W(\cdot, 0) = W_0 & \text{in } \mathbb{R}, \end{cases}$$

by a Schwarz Waveform Relaxation Method (SWR).



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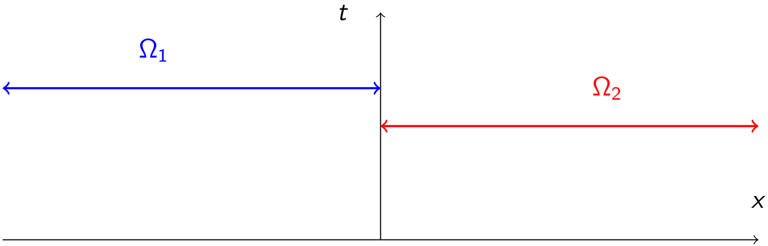

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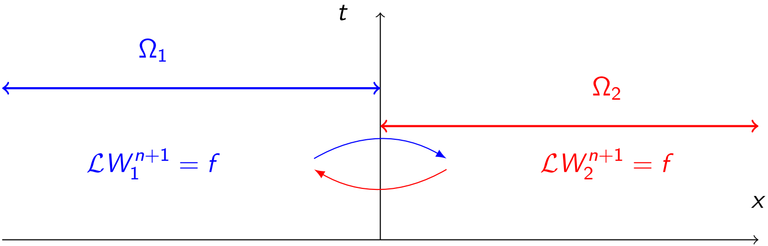
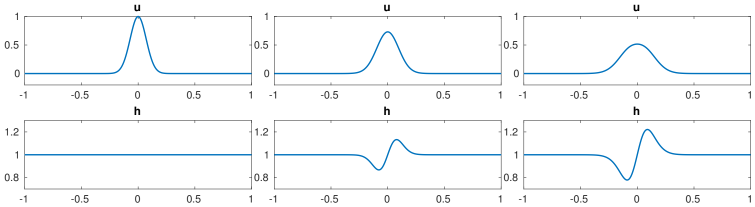


# Linearized Viscous Shallow Water equations

Linearized adimensionalized equations $\mathcal{L}W = F$ are

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{Fr^2}\dfrac{\partial h}{\partial x} - \dfrac{1}{Re}\dfrac{\partial^2 u}{\partial x^2} = f & \text{on } \mathbb{R} \times (0, +\infty) \\ \dfrac{\partial h}{\partial t} + \dfrac{\partial u}{\partial x} = 0 & \text{on } \mathbb{R} \times (0, +\infty) \\ u(\cdot, 0) = u_0, h(\cdot, 0) = h_0 & \text{on } \mathbb{R} \end{cases}$$

where $W = (u, h)$, $Fr = U/c$, $Re = UL/\nu$.



# SWR Algorithm with Robin boundary conditions

If $\mathcal{B}(u,h) = -\frac{1}{Re}\frac{\partial u}{\partial x} + \frac{1}{Fr^2}h$ the DD algorithm reads

$$\begin{cases} \mathcal{L}W_1^{n+1} = F & \text{on } \Omega_1 \times (0,+\infty), \\ (\mathcal{B}(u_1^{n+1}, h_1^{n+1}) - \Lambda u_1^{n+1})(0,t) & = (\mathcal{B}(u_2^n, h_2^n) - \Lambda u_2^n)(0,t), \\ u_1^{n+1}(\cdot,0) = u_0,\, h_1^{n+1}(\cdot,0) = h_0 & \end{cases} \qquad (1)$$

$$\begin{cases} \mathcal{L}W_2^{n+1} = F & \text{on } \Omega_2 \times (0,+\infty), \\ (\mathcal{B}(u_2^{n+1}, h_2^{n+1}) + \Lambda u_2^{n+1})(0,t) & = (\mathcal{B}(u_1^n, h_1^n) + \Lambda u_1^n)(0,t), \\ u_2^{n+1}(\cdot,0) = u_0,\, h_2^{n+1}(\cdot,0) = h_0 & \end{cases} \qquad (2)$$

**How to choose $\Lambda$ such that the convergence is fast?**

# SWR Algorithm with Robin boundary conditions

If $\mathcal{B}(u,h) = -\frac{1}{Re}\frac{\partial u}{\partial x} + \frac{1}{Fr^2}h$ the DD algorithm reads

$$\begin{cases} \mathcal{L}W_1^{n+1} = \mathbf{0} & \text{on } ]-\infty,0[\times(0,T), \\ (\mathcal{B}(u_1^{n+1},h_1^{n+1}) - \Lambda u_1^{n+1})(0,t) & = (\mathcal{B}(u_2^n,h_2^n) - \Lambda u_2^n)(0,t), \\ u_1^{n+1}(\cdot,0) = \mathbf{0},\, h_1^{n+1}(\cdot,0) = \mathbf{0} & \end{cases} \quad (1)$$

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If $\mathcal{B}(u,h) = -\frac{1}{Re}\frac{\partial u}{\partial x} + \frac{1}{Fr^2}h$ the DD algorithm reads

$$\begin{cases} \mathcal{L}_{SW} W_1^{n+1} = 0 & \text{on } ]-\infty,0[\times(0,T), \\ (\mathcal{B}(u_1^{n+1},h_1^{n+1}) - \mathbf{p} u_1^{n+1})(0,t) & = (\mathcal{B}(u_2^{n},h_2^{n}) - \mathbf{p} u_2^{n})(0,t), \\ u_1^{n+1}(\cdot,0) = 0,\, h_1^{n+1}(\cdot,0) = 0 & \end{cases} \quad (1)$$

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**In this talk $\Lambda = \mathbf{p}\, Id$ with $p \in \mathbb{R}$ to be chosen.**

# Outline

# Solving in Laplace variables

We use the Laplace transform

$$\hat{u}(s) = \int_0^{+\infty} u(t) e^{-st} dt.$$

The SW equations $\mathcal{L}W = 0$ in Laplace variables are

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{Fr^2}\frac{\partial h}{\partial x} - \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} = 0 \\ \frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} s\hat{u} - (\frac{1}{Re} + \frac{1}{sFr^2})\frac{\partial^2 \hat{u}}{\partial x^2} = 0 \\ \hat{h} = -\frac{1}{s}\frac{\partial \hat{u}}{\partial x} \end{cases}$$

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The solution of the algorithm is

$$\begin{cases} \mathcal{L}(u_1^{n+1}, h_1^{n+1}) = 0 \quad \text{in } \mathbb{R}^- \times \mathbb{R}^+ \\ \mathcal{L}(u_2^{n+1}, h_2^{n+1}) = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \end{cases} \qquad \begin{cases} \hat{u}_1^{n+1}(x,s) = \alpha^{n+1}(s)e^{\mu(s)x} \\ \hat{u}_2^{n+1}(x,s) = \beta^{n+1}(s)e^{-\mu(s)x} \end{cases}$$

$$\text{where } \mu(s) = \frac{sFr\sqrt{Re}}{\sqrt{sFr^2 + Re}}.$$

# Solving in Laplace variables

We have obtained

$$\hat{u}_1^n(x,s) = \alpha^n(s)e^{\mu(s)x} \text{ and } \hat{u}_2^n(x,s) = \beta^n(s)e^{-\mu(s)x}.$$

The transmission conditions at $\{0\} \times (0,+\infty)$ are

$$\begin{cases} -\frac{1}{Re}\partial_x u_1^{n+1} + \frac{1}{Fr^2}h_1^{n+1} - pu_1^{n+1} & = & -\frac{1}{Re}\partial_x u_2^n + \frac{1}{Fr^2}h_2^n - pu_2^n \\ -\frac{1}{Re}\partial_x u_2^n + \frac{1}{Fr^2}h_2^n + pu_2^n & = & -\frac{1}{Re}\partial_x u_1^{n-1} + \frac{1}{Fr^2}h_1^{n-1} + pu_1^{n-1} \end{cases}$$

In Laplace variables they become

$$\begin{aligned} (\frac{1}{Re} + \frac{1}{Fr^2 s})\partial_x \hat{u}_1^{n+1} + p\hat{u}_1^{n+1} &= (\frac{1}{Re} + \frac{1}{Fr^2 s})\partial_x \hat{u}_2^n + p\hat{u}_2^n \\ -(\frac{1}{Re} + \frac{1}{Fr^2 s})\partial_x \hat{u}_2^n + p\hat{u}_2^n &= -(\frac{1}{Re} + \frac{1}{Fr^2 s})\partial_x \hat{u}_1^{n-1} + p\hat{u}_1^{n-1} \end{aligned}$$

# Solving in Laplace variables

## Obtaining the convergence factor

We have the relation

$$\hat{u}_1^{n+1}(0,s) := \rho(p,s)\hat{u}_1^{n-1}(0,s)$$

with

$$\rho(\mathrm{p},\omega) = \left|\frac{\sqrt{i\omega + A} - \mathrm{p}}{\sqrt{i\omega + A} + \mathrm{p}}\right|^2 = \frac{\mathrm{p}^2 - \sqrt{2}\mathrm{p}\sqrt{A + \sqrt{A^2+\omega^2}} + \sqrt{A^2+\omega^2}}{\mathrm{p}^2 + \sqrt{2}\mathrm{p}\sqrt{A + \sqrt{A^2+\omega^2}} + \sqrt{A^2+\omega^2}}$$

where $A = Re/Fr^2$ et $\mathrm{p} = \sqrt{Re}p$.

## Optimizing the convergence factor

$$\min_{p\in\mathbb{R}} \max_{\omega\in[\omega_{min},\omega_{max}]} \rho(p,\omega).$$

# Remark on a limit case

We consider the heat equation

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = 0.$$

## Theorem (M.J. Gander, L. Halpern, 2003)

*The solution of the min-max problem involving the convergence factor (case $A = 0$)*

$$\rho_{heat}(p, \omega) = \left| \frac{\sqrt{i\omega} - p}{\sqrt{i\omega} + p} \right|^2$$

*is $p^\star = (\omega_{min}\omega_{max})^{1/4}$ and $\rho_{heat}(p^\star, \omega)$ equi-oscillates between $\omega_{min}$ and $\omega_{max}$. We have the asymptotic result*

$$\rho_{heat}(p^\star, \omega) = 1 - C\Delta t^{1/4} + o(\Delta^{1/4}).$$

# Optimizing the convergence factor

Physical and numerical data: $\omega_{min} = \frac{\pi}{2}$, $\omega_{max} = \frac{\pi}{dt} \simeq 2011$, $A = 1$.

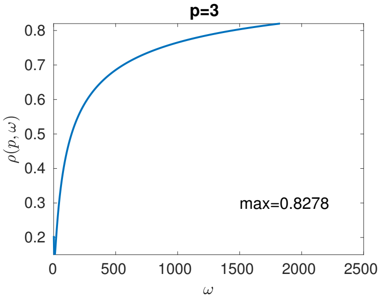


Figure: $\omega \to \rho(p, \omega)$

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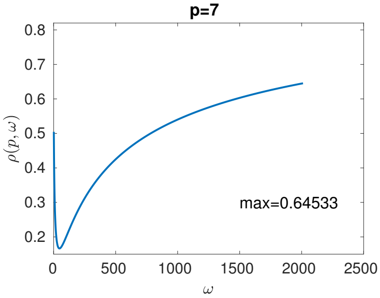


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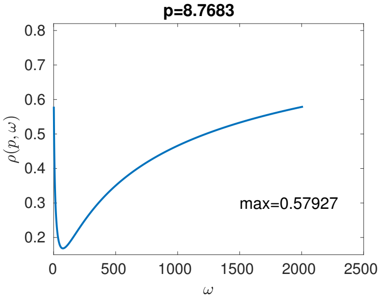


Figure: $\omega \to \rho(p, \omega)$

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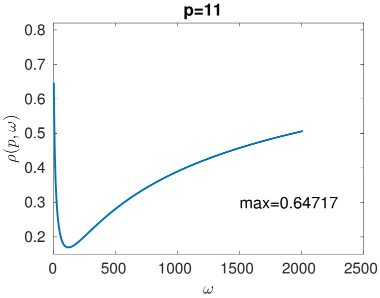



Figure: $\omega \to \rho(p, \omega)$

# Optimizing the convergence factor

Physical and numerical data: $\omega_{min} = \frac{\pi}{2}$, $\omega_{max} = \frac{\pi}{dt} \simeq 2011$, $A = 1000$.

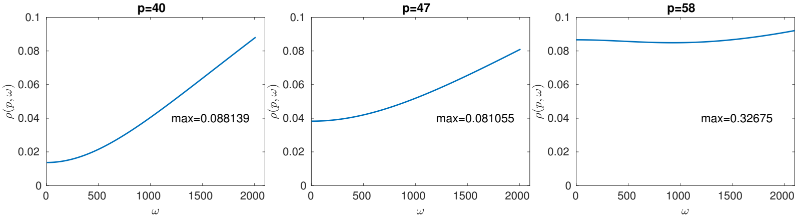



Figure: $\omega \to \rho(\boldsymbol{p}, \omega)$

# Optimization of the convergence factor

Let $q_1$ et $q_2$ defined by

$$q_1 = (A^2 + \omega_{max}^2)^{1/4}$$
$$q_2 = \left( A + \sqrt{A + \sqrt{A^2 + \omega_{min}^2}} \sqrt{A + \sqrt{A^2 + \omega_{max}^2}} \right)^{1/2}$$

## Theorem (M. J. Gander, V.M. 2022)

Optimizing the convergence factor has a unique solution:
$p^\star = \min(q_1, q_2)$,
If $q_1 < q_2$ then the optimized solution satisfies

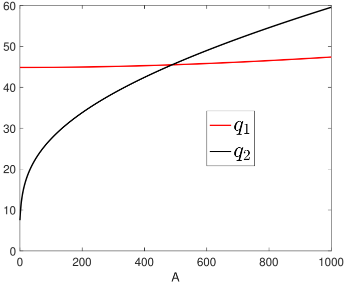
$$\min_{p_0 \in \mathbb{R}} \max_{\omega \in [\omega_{min}, \omega_{max}]} \rho(p_0, \omega) = \rho(q_1, \omega_{max}).$$

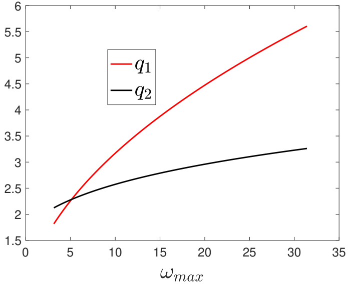
If $q_1 > q_2$ then the optimized solution satisfies

$$\min_{p \in \mathbb{R}} \max_{\omega \in [\omega_{min}, \omega_{max}]} \rho(p, \omega) = \rho(q_2, \omega_{min}) = \rho(q_2, \omega_{max}).$$

# Dependance of $q_1$ and $q_2$ w.r. to the parameters

If $q_1 > q_2$ $p^\star = q_2$ (equi-oscillation)
If $q_1 < q_2$ $p^\star = q_1$ (no equi-oscillation)




$q_1$, $q_2$ as functions of $A$

$q_1$, $q_2$ as functions of $\omega_{max} = \dfrac{\pi}{\Delta t}$

# Outline

# Case $\Delta t \to 0$

## Corollary 1

If $\Delta t \to 0$ then the optimized parameter behaves like

$$p^\star \simeq C_p \Delta t^{-1/4}, C_p = \frac{\pi^{1/4}}{\sqrt{ReFr}}(Re + \sqrt{Re^2 + \omega_{min}^2 Fr^4})^{1/4}$$

and the convergence factor like

$$\min_{p\in[0,+\infty[} \max_{\omega\in[\omega_{min},\omega_{max}]} \rho(p,\omega) = 1 - 2\sqrt{2Re}\frac{C_p}{\sqrt{\pi}}\Delta t^{1/4} + o(\Delta t^{1/4})$$

Proof:

$q_1 \simeq \omega_{max}^{1/2}$

$q_2 \simeq C\omega_{max}^{1/4}$

Case $q_1 > q_2$.

# Case $Re \to +\infty$

## Corollary 2

If $Re \to +\infty$ then the optimized parameter behaves like

$$p^\star \simeq \frac{1}{Fr}$$

and the convergence factor behaves like

$$\min_{p\in[0,+\infty[} \max_{\omega\in[\omega_{min},\omega_{max}]} \rho(p,\omega) \simeq \frac{\omega_{max}^2 Fr^4}{16 Re^2}.$$

Proof:

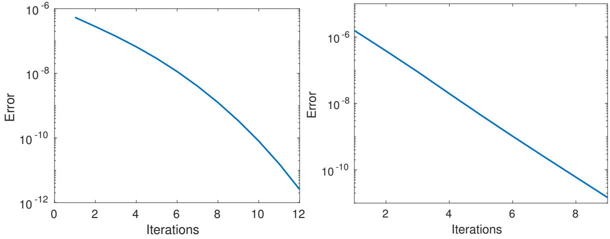
$q_1 \simeq \sqrt{A}$
$q_2 \simeq \sqrt{3A}$ Case $q_2 > q_1$.

# Outline

# Influence of the time interval

## Physical and numerical data

$\Delta x = 1.56 \cdot 10^{-4}, \Delta t = 5 \cdot 10^{-3}$
$u(x,t) = 0, \ h(x,t) = 0$
$u_2^0(0,\cdot) = \text{rand}(-1,1)$




$T = 0.2$ $T = 2$

Error versus iterations

# How sharp is the optimized parameter?

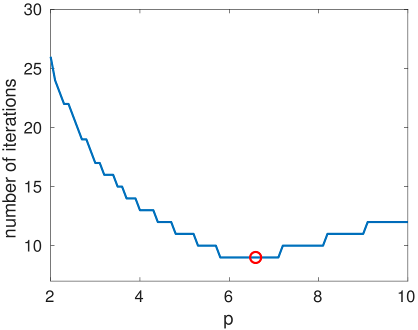
Physical and numerical data

$T = 2$

$\Delta x = 1.56 \cdot 10^{-4}, \Delta t = 5 \cdot 10^{-3}$

$u(x,t) = 0,\ h(x,t) = 0$

$u_2^0(0,\cdot) = \text{rand}(-1,1)$

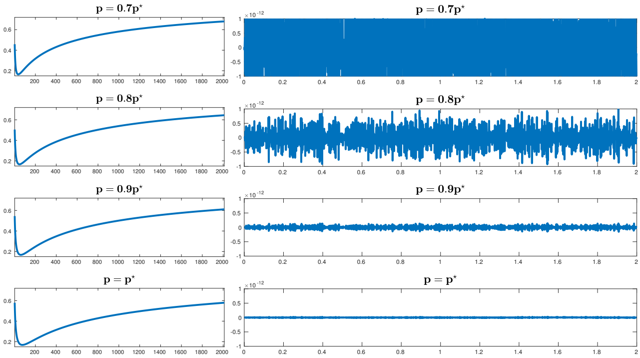



Number of iterations needed to reach an error of $10^{-10}$.

# Effect of $p$ for a sum of frequencies

Physical and numerical data

$T = 2$
$\Delta x = 1.56 \cdot 10^{-4}, \Delta t = 1.6 \cdot 10^{-3}$
$u_2^0(0, \cdot) = \text{rand}(-1, 1)$




Convergence Factor

Error $u^{20}(0, t)$ w.r.t. $t$

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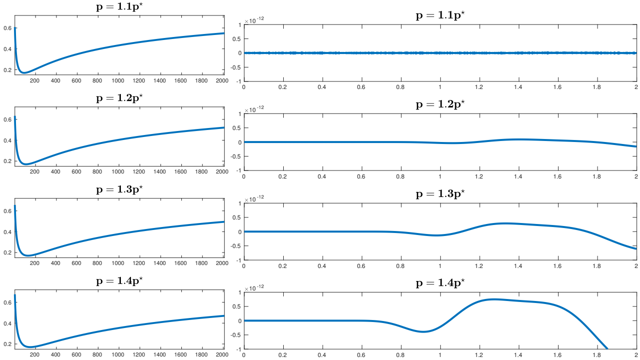



Convergence Factor

Error $u^{20}(0, t)$ w.r.t. $t$

# Outline

# 2D Linear Viscous Shallow Water equations

The equations with $(u, v)$ the velocity field, $\epsilon = U/fL$ the Rossby number

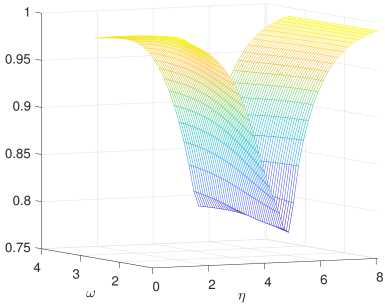
$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{Fr^2}\dfrac{\partial h}{\partial x} - \dfrac{1}{Re}\triangle u - \dfrac{1}{\epsilon}v = F_1, \\ \dfrac{\partial v}{\partial t} + \dfrac{1}{Fr^2}\dfrac{\partial h}{\partial y} - \dfrac{1}{Re}\triangle v + \dfrac{1}{\epsilon}u = F_2, \\ \dfrac{\partial h}{\partial t} + \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0. \end{cases}$$

The transmission boundary conditions

$$\mathcal{B}(u, v, h) - \Lambda(u, v, h) = \begin{pmatrix} -\dfrac{1}{Re}\dfrac{\partial u}{\partial x} + \dfrac{1}{Fr^2}h \\ -\dfrac{1}{Re}\dfrac{\partial v}{\partial x} \end{pmatrix} - \begin{pmatrix} \dfrac{1}{Fr}u + \dfrac{Fr}{2Re}\dfrac{\partial u}{\partial t} \\ qv \end{pmatrix}$$

**with $q$ to be chosen**

# Optimizing the convergence factor




Laplace in time $t \to \omega$
Fourier in space $y \to \eta$

# Optimizing the convergence factor

**Lemma (M.J. Gander, V.M. 2022)**

*The convergence factor for the algorithm with* $q = \frac{\kappa_q}{\Delta y^\alpha}$, $\alpha > 0$, *behaves at* $\eta = \eta_{min}$ *like*

$$\rho(q, \eta_{min}, 0) = 1 - \frac{8|\eta_{min}|}{Re}\frac{\Delta y^\alpha}{\kappa_q} + \mathcal{O}(\Delta y^{2\alpha}).$$

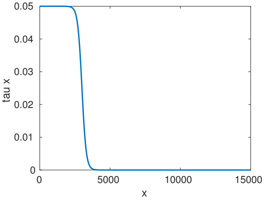
**Lemma (M.J. Gander, V.M. 2022)**

*The convergence factor of the algorithm with* $q = \kappa_q/\Delta y^\alpha$, $\alpha > 0$, *behaves at* $\eta = \eta_{max} = \frac{\pi}{\Delta y}$ *like*

$$\rho(q, \frac{\pi}{\Delta y}, 0) = 1 - \frac{8}{3}\frac{Re}{\pi Fr}\Delta y + \cdots$$

**Choose** $\mathbf{q} = \frac{|\eta_{\min}|\pi \mathbf{Fr}}{\mathbf{Re}^2}\frac{\mathbf{1}}{\boldsymbol{\Delta} \mathbf{y}}$

# Numerical Results




## Physical and numerical data

$\Omega = [0, 15000\,\text{km}] \times [-1500\,\text{km}, 1500\,\text{km}]$
$F = (2.5 \cdot 10^{-2}(1 + \tanh((3000 - x)/300)\,\text{N}/\text{m}^2, 0)$
$\nu = 500\, m^2/s^{-1},\ f = 2 \cdot 10^{-11}\text{m}^{-1}s^{-1}$

Iteration 1
Iteration 3
Iteration 10

# Outline

# Conclusion

- 1D equations
  A good optimized parameter can be designed for large time $T$.
- 2D equations
  We need to mix physical informations and optimization results.