Ideas about a Multi-level Parareal Method

PinT 2022 — 11th Conference on Parallel-in-Time Integration, Marseille

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11-15 July 2022

Motivation

Applications of weather and climate

- problems with many scales many
- **quadratic non-linearity** in fluid flow problems
- fast and slow modes in the problem

Numerical method

- exploit increasing parallel resources of modern computers

→ version of Parareal Method

- exploit properties of the underlying problem when constructing a numerical method
- fast oscillations, averaging to mitigate oscillatory stiffness

→ construction of coarse propagator

The model problem

problem that exhibits scale separation

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\varepsilon}\mathscr{L}\mathbf{u} = \mathscr{N}(\mathbf{u}) + \mathscr{D}\mathbf{u}$$

- ε small parameter
- stiff linear term $\mathscr{L}$, skew Hermitian
  - purely imaginary eigenvalues, temporal oscillations
- quadratic non-linearity $\mathscr{N}$
- diffusive term $\mathscr{D}$

Transformed system given by **modulation equation**

$$\frac{\partial \mathbf{w}}{\partial t} = \exp\left(-\frac{\mathscr{L}}{\varepsilon}t\right)\mathscr{N}\left(\exp\left(\frac{\mathscr{L}}{\varepsilon}t\right)\mathbf{w}\right)$$

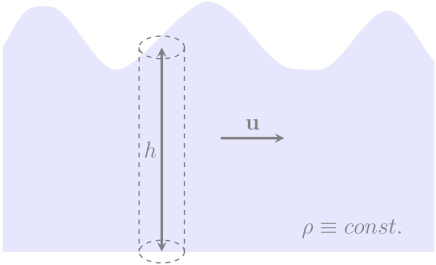
- numerous scientific applications
- includes atmospheric and oceanic simulations

Example: The rotating shallow water equations

- horizontal scale much larger than vertical scale
- momentum and mass equations

$$\frac{\partial \mathbf{u}}{\partial t} + \underline{\mathbf{u} \cdot \nabla \mathbf{u}} + \mathbf{f} \times \mathbf{u} = -g \nabla h$$

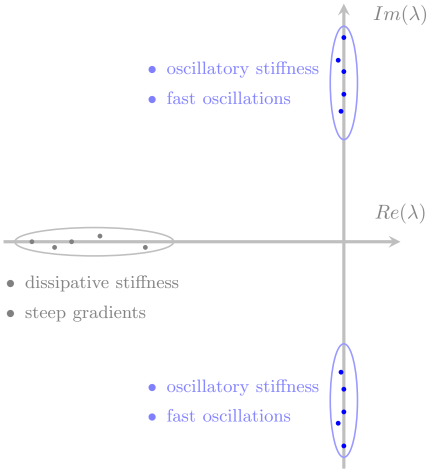
$$\frac{\partial h}{\partial t} + \underline{\mathbf{u} \cdot \nabla h} + \underline{h \nabla \cdot \mathbf{u}} = 0$$



- $\mathbf{u}$ is horizontal velocity, h is total fluid thickness
- g gravitational acceleration, $\mathbf{f}$ vector with Coriolis coefficient
- hyperbolic system
- diffusion terms can be added

Oscillatory stiffness

- skew Hermitian linear term
- → fast oscillations in the solutions
- → oscillatory stiffness
- can apply transformation
- → eliminate linear term
- oscillations are still in solution
- higher order derivatives
- explore **averaging**

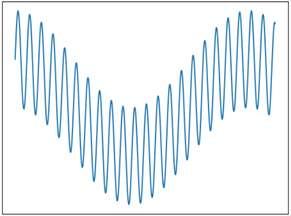


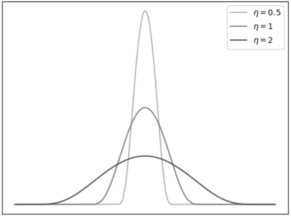
What is temporal averaging?

- basic idea: original problem is replaced by smoother problem
- mitigate oscillatory stiffness of f through a filter function

$$\bar{f}(t) = \frac{1}{\eta} \int_{-\eta/2}^{\eta/2} \rho\left(\frac{s}{\eta}\right) f(t+s)\,\mathrm{d}s\,,$$

with averaging window η , filter function ρ





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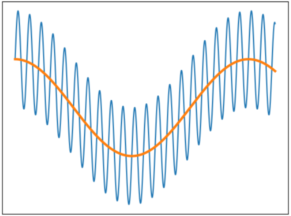
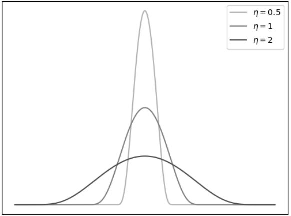




Given problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y),$$

- fast oscillations in solutions, small time steps

New problem

- Idea: apply averaging to RHS of the problem
- see also HMM

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}t} = \bar{f}(t, \bar{y})$$

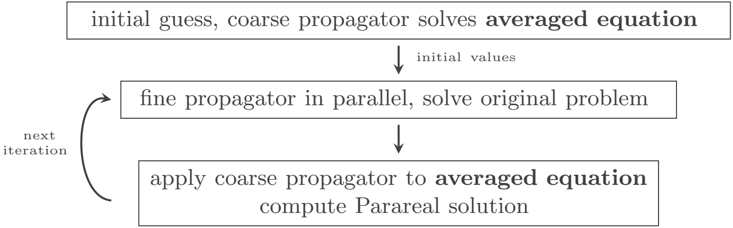
- filter fast oscillations, bigger time steps
- apply this strategy to build a **coarse propagator** for **Parareal**

Two-level Parareal combined with averaging

$$Y_{n+1}^{k+1} = \overline{\phi}(Y_n^{k+1}) + \varphi(Y_n^k) - \overline{\phi}(Y_n^k)$$

- $\overline{\phi}$ coarse propagator, solves **averaged system**
- φ fine propagator, solves original system

→ solution to original system

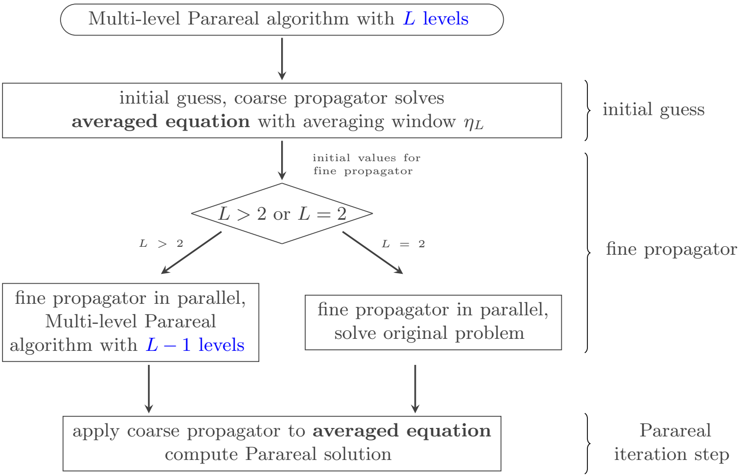


An Asymptotic Parallel-in-Time Method for Highly Oscillatory PDEs, Terry S. Haut and Beth A. Wingate, SIAM J. Sci. Comput., 2014, 36

Peddle, A. G., Haut, T., Wingate, B., *Parareal Convergence for Oscillatory PDEs with Finite Time-Scale Separation*, SIAM Journal on Scientific Computing, 2019

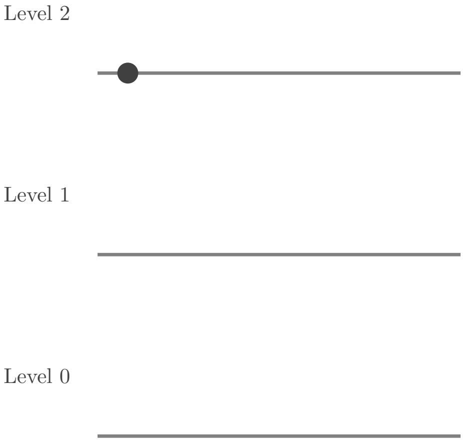
Multi-level scheme

- Idea: the fine propagator is a Multi-level Parareal algorithm
- has one level less



Cycles

- three levels
- one iteration on coarsest and second coarsest levels



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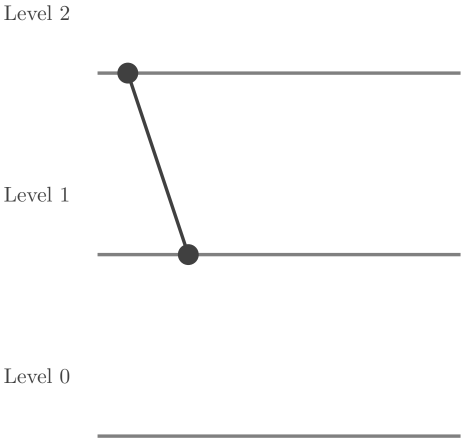

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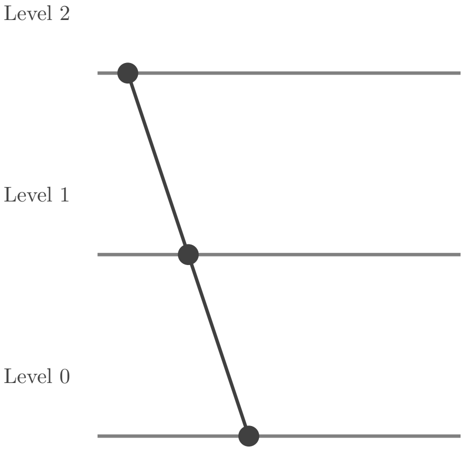

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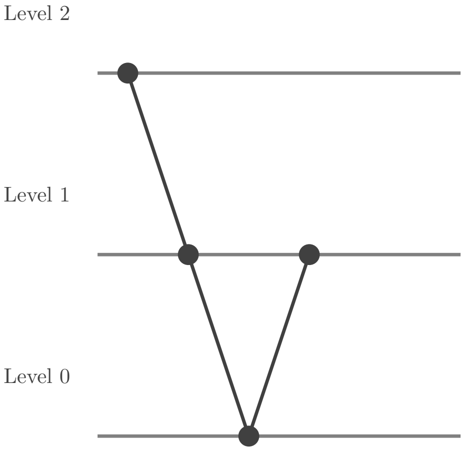

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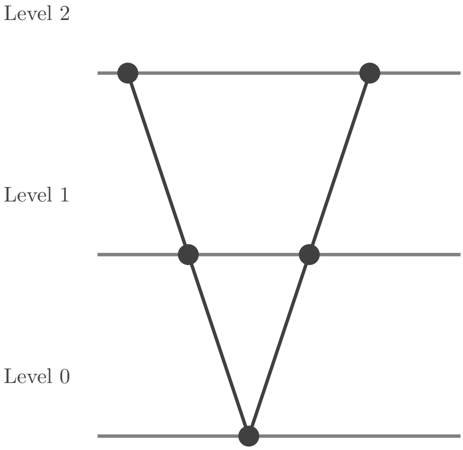
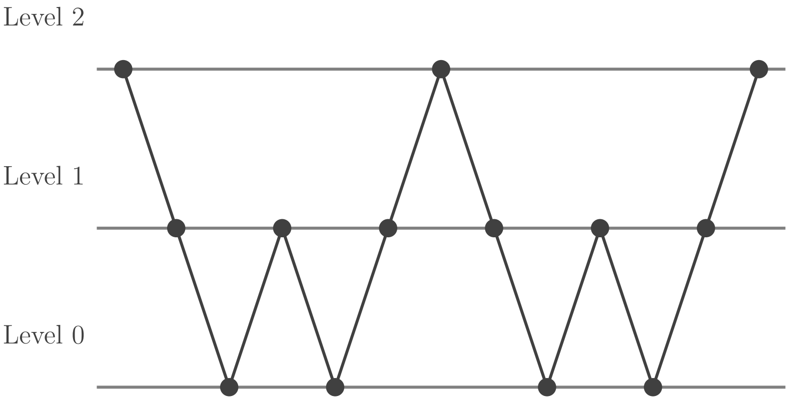


Cycles

- three levels
- two iterations on coarsest and second coarsest levels



Convergence result

- error bound can be found in *Nonlinear Convergence Analysis for the Parareal Algorithm*, M. J. Gander, E. Hairer
- fine propagator is exact

$$\|y(t_n) - Y_n^k\| \le C_1\,(\Delta T)^{p_1(k_1+1)},$$

  where $C_1 \equiv C_1(k_1, T_n, T_{k+1})$, p_1 accuracy order of coarse propagator, k_1 number of iterations

- assume fine propagator is not exact

$$\|y(t_n) - Y_n^k\| \le C_1\,(\Delta T)^{p_1(k_1+1)} + A_1 C_0 \Delta t^{p_0}$$

- inductive argument, multi-level result

$$\|y(t_n) - Y_n^k\| \le \sum_{l=1}^{L} E_l \prod_{j=l+1}^{L} A_j + \delta_0 \prod_{j=1}^{L} A_j$$

- $E_l = C_l \Delta T_l^{p_l(k_l+1)}$, $\delta_0 = C_0 \Delta T_0^{p_0}$, A_j amplification factor

Convergence result with averaging

- result in Peddle/Haut/Wingate (2019)
- fine propagator is exact

$$\|y(t_n) - Y_n^k\| \leq C_1(\varepsilon\eta + \kappa\Delta T_1^{p_1})\left(\frac{\varepsilon\eta}{\Delta T_1} + \kappa\Delta T_1^{p_1}\right)^{k_1}$$

- assume fine propagator is not exact
- inductive argument
- multi-level result with averaging

$$\|y(t_n) - Y_n^k\| \leq \sum_{l=1}^{L} \bar{E}_l \prod_{j=l+1}^{L} \bar{A}_j + \delta_0 \prod_{j=1}^{L} \bar{A}_j$$

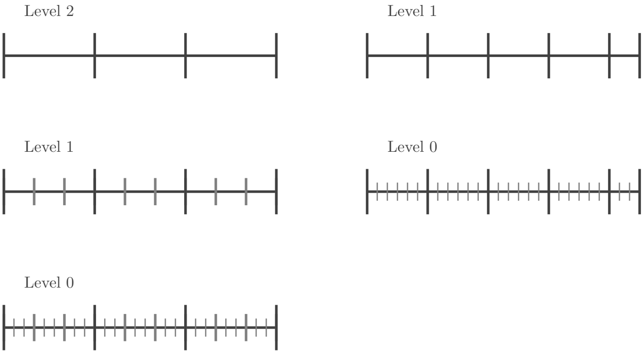
- $\bar{E}_l = C_l(\tilde{C}_l\eta_l\varepsilon + \hat{C}_l\Delta T_l^{p_l})(\bar{C}_l\eta_l\varepsilon + \hat{C}_l\Delta T_l^{p_l})^k$,
- $\delta_0 = C_0\Delta T_0^{p_0}$
- $\bar{A}_j$ amplification factor

Increased parallelism

- consider problem with three scales

→ introduce a level for each scale

- apply 3-level scheme
- increased parallelism for 3-level scheme compared to 2-level scheme



A very first numerical example - Equation

- simple model problem with quadratic non-linearity
- can be used to explain issues due to bi-linear term arising in fluid modelling
  - see *'Atmospheric and Oceanic Fluid Dynamics'*, G. Vallis

$$\begin{pmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \\ \frac{\partial u_3}{\partial t} \end{pmatrix} + \underbrace{\begin{pmatrix} i\omega_1 & 0 & 0 \\ 0 & i\omega_2 & 0 \\ 0 & 0 & i\omega_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}}_{\text{linear term}} + \underbrace{\begin{pmatrix} u_1 u_1 \\ u_2 u_2 \\ u_3 u_3 \end{pmatrix}}_{\text{quadratic non-linearity}} = 0$$

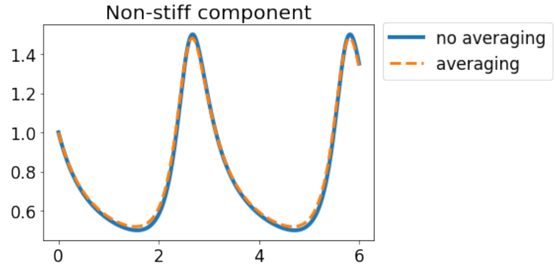
- dynamics on three time scales
- advantages: know exact solution
- can derive expression for averaged equations

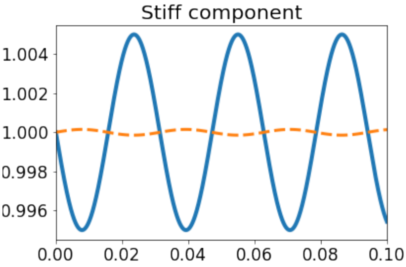
Solutions to averaged and unaveraged problems

- averaging can filter oscillations

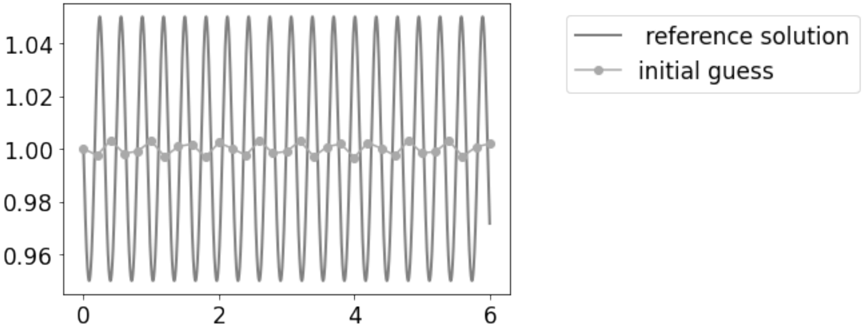
→ reduces oscillatory stiffness

- almost no effect when averaging window η is smaller the wavelength
- strong smoothing effect when averaging window η larger than wavelength





Numerical solution



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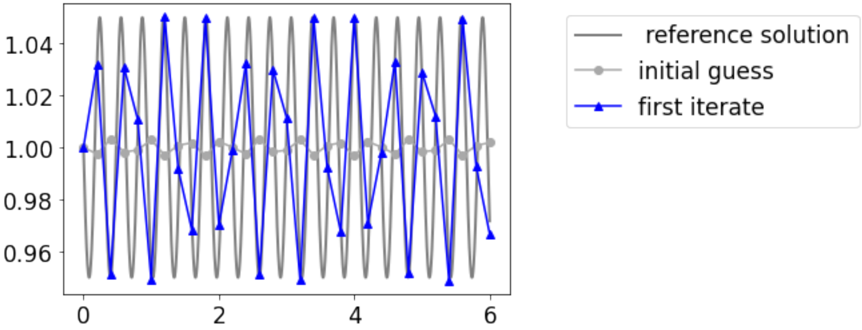

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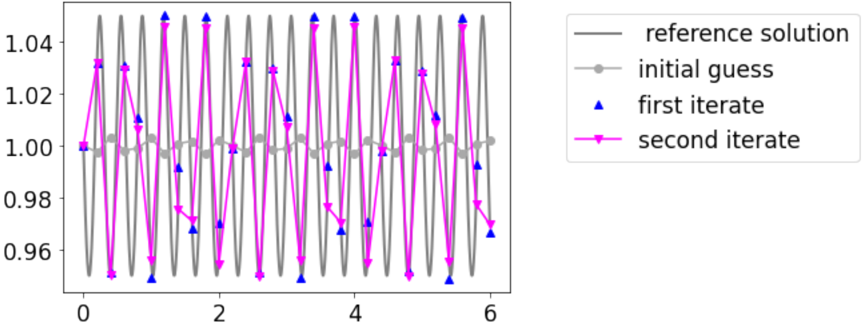

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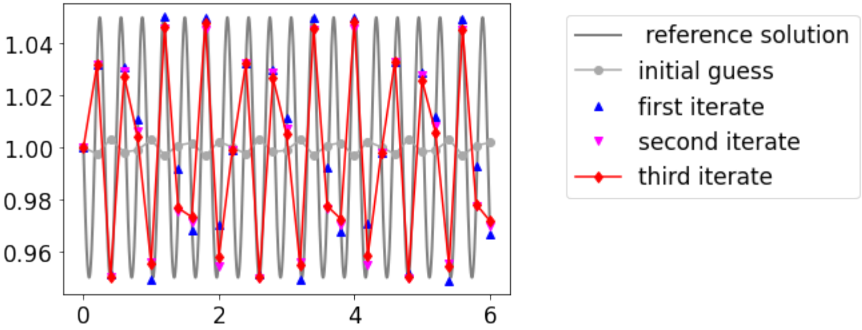
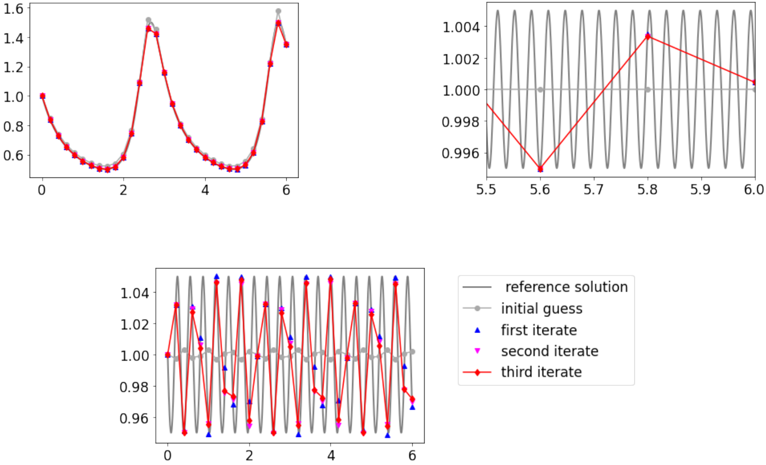
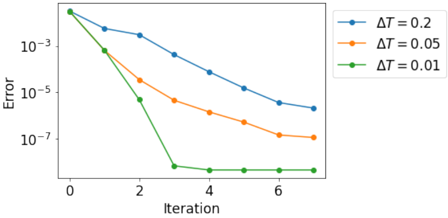



1.6
1.4
1.2
1.0
0.8
0.6
0
2
4
6
1.004
1.002
1.000
0.998
0.996
5.5
5.6
5.7
5.8
5.9
6.0
1.04
1.02
1.00
0.98
0.96
0
2
4
6
reference solution
initial guess
first iterate
second iterate
third iterate

Numerical results

- RK method of second oder
- averaging windows $\eta_1 = 0.1$ and $\eta_2 = 1$
- coarsening factor of 10, adapted to different scales
- number of iterations on Level 1 was chosen as 3



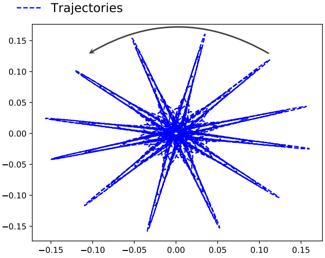
- accuracy bounded by fine propagator

→ higher accuracy can be obtained for smaller step sizes

Swinging Spring (Elastic Pendulum)

- reduction of equations of motion for weather and climate
- resonant interactions

$$\dot{x}_1 = x_2 \quad x_2 = -\omega_r x_1 + \lambda x_1 z_1$$
$$\dot{y}_1 = y_2 \quad y_2 = -\omega_r y_1 + \lambda y_1 z_1$$
$$\dot{z}_1 = z_2 \quad z_2 = -\omega_z z_1 + \frac{1}{2}\lambda\left(x_1^2 + y_1^2\right)$$



Rotating Shallow Water Equations

- important test for weather and climate applications
- PDE with hyperbolic character

Conclusion and Outlook

Conclusion

- Multi-level Parareal method with averaging
- problems with oscillatory stiffness
  - important for weather and climate applications
- account for multi-scale behaviour
- key: definition of the coarse propagators
  - averaging windows adapted to level and problem
- asymptotic error estimate
  - error on the level, amplification factor
- first numerical studies

Outlook

- continue study of numerical examples
- more considerations about efficiency