Deep Learning aided solutions for wave equations

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Current state-of-the-art Deep Learning models

Example text: “A pear cut into seven pieces arranged in a ring”
Output images:

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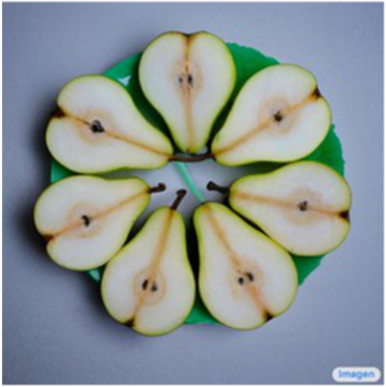

Current state-of-the-art Deep Learning models

- A text-to-image model called Imagen
- consists of a text encoding model 11B parameters, trained on 807GB of text data
- consists of several U-Net models with another 2.5B parameters, trained on 850MB of text-image data

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Reference: Saharia et al. 2022; Ramesh et al. 2022

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Other related models: DALL-E 2

Deep Learning models to PDE

$$\begin{aligned}\mathcal{L}_c[u] &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega\end{aligned}$$

- Solve PDE by minimizing “Physics” loss, e.g. PINN, DeepOnet etc. See Song, Alkhalifah, and Waheed 2021; Jagtap and Karniadakis 2021

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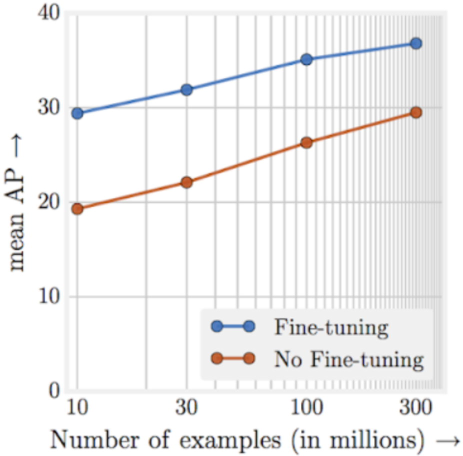
- Augment standard numerical scheme Um et al. 2020; Siahkoohi, Louboutin, and Herrmann 2019; Kochkov et al. 2021

How much data?



Object detection performance when pre-trained on different subsets of JFT-300M from scratch. x-axis is the dataset size in log-scale, y-axis is the detection performance in mAP@[.5,.95] on COCO-minival subset.

Sun et al. 2017; De Hoop et al. 2022

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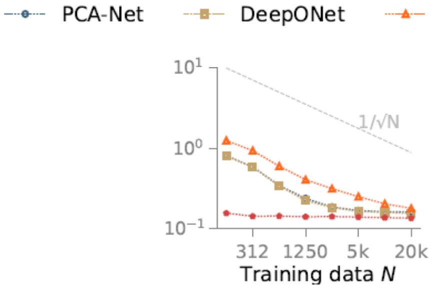

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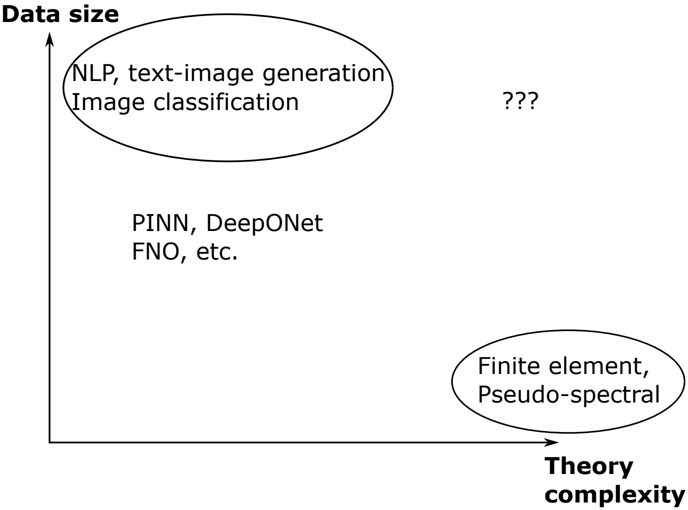
Theory-driven vs data-driven models



Content

- Wave equation

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- Deep learning aided parareal iteration

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- Numerical results

The second-order Wave Equation

Find solution $u = u(x, t)$ such that

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Boundary condition is periodic.

Parallel-in-time method

- Pareareal iteration

$$u_{n+1}^{k+1} = \mathcal{G}u_n^{k+1} + (\mathcal{F}u_n^k - \mathcal{G}u_n^k)$$

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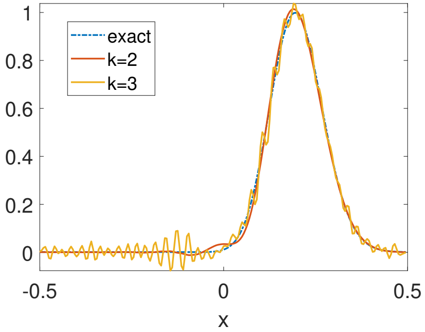
- *Unstable iteration for solving hyperbolic equations*!



DL-aided parareal

- Introduce a neural network $\mathcal{H}_\theta$ to postprocess coarse solution

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- Operate in the energy component

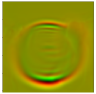
$$\|\nabla u\|_2^2 + \left\| \frac{\partial_t u}{c} \right\|_2^2 = constant$$

JNet

similar to U-Net c.f. Ronneberger, Fischer, and Brox 2015

physical component

$u, \partial_t u$



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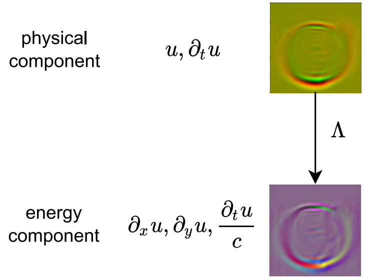

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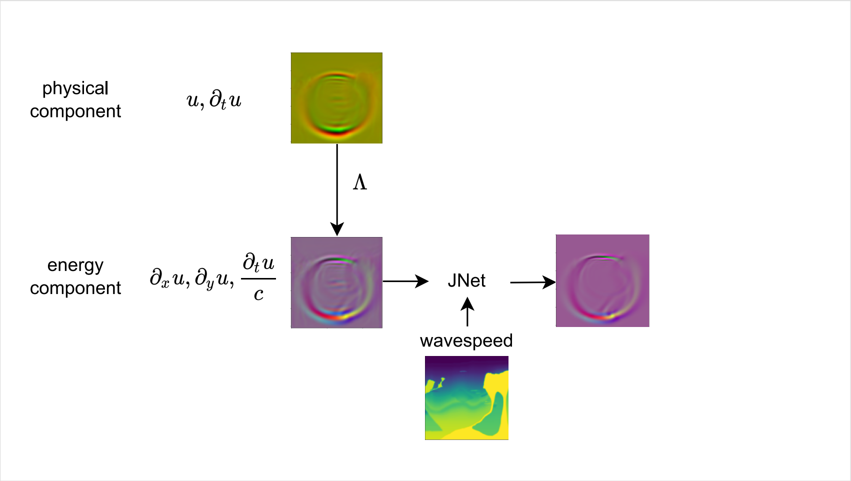

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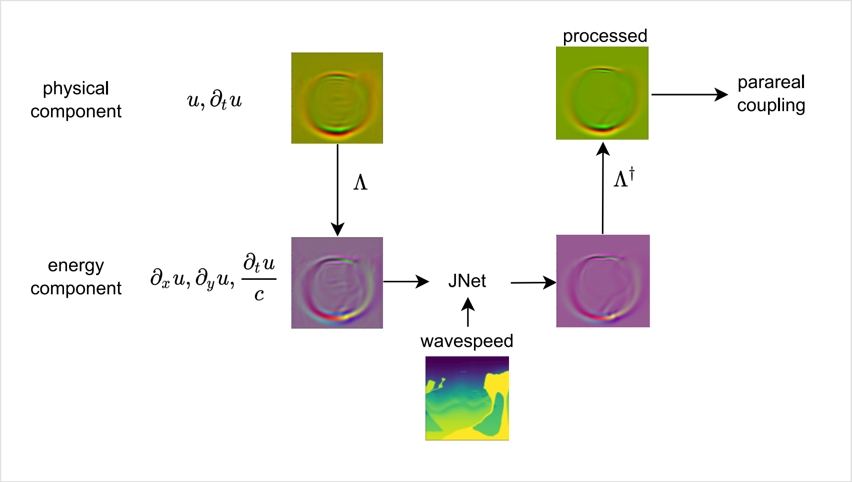
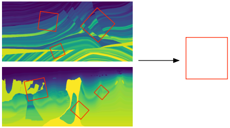


Data generation



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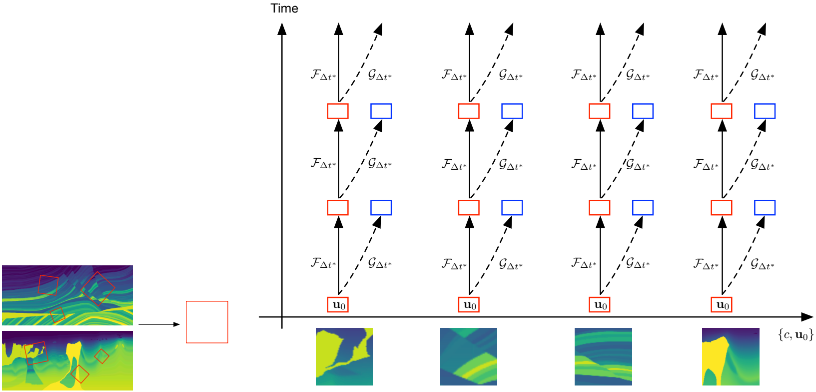

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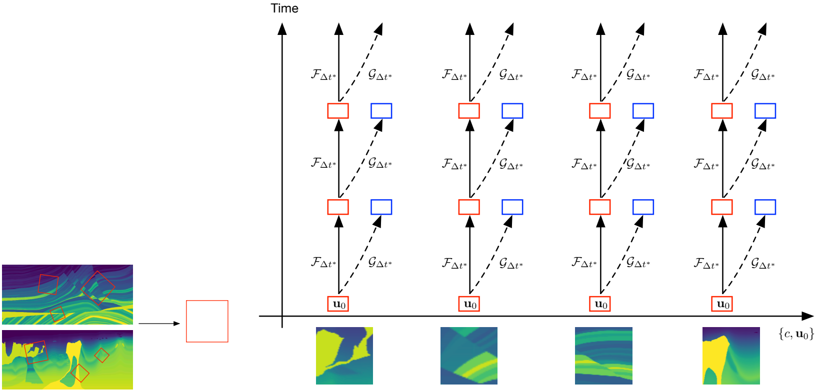
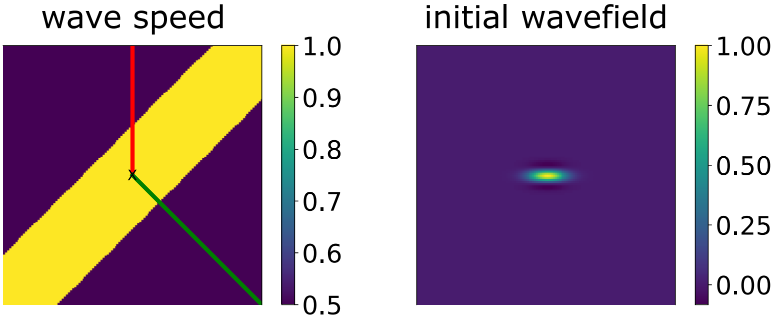
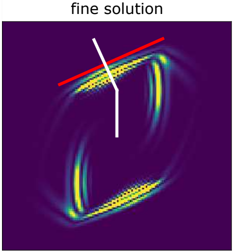


Using the Procrustes parareal c.f. Nguyen and Tsai 2020 to generate data

Numerical results - Refraction



Numerical results - Refraction sequential correction



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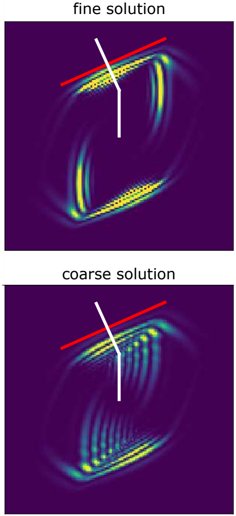

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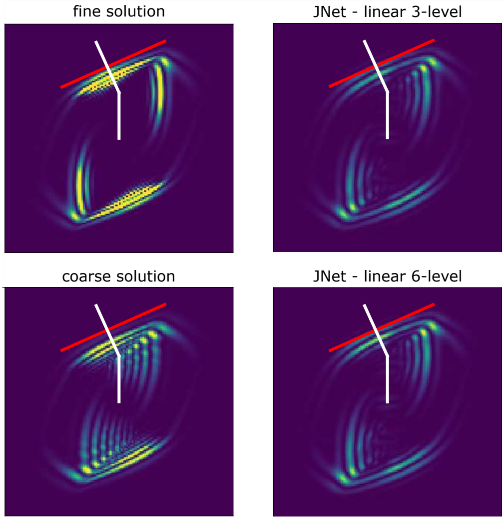

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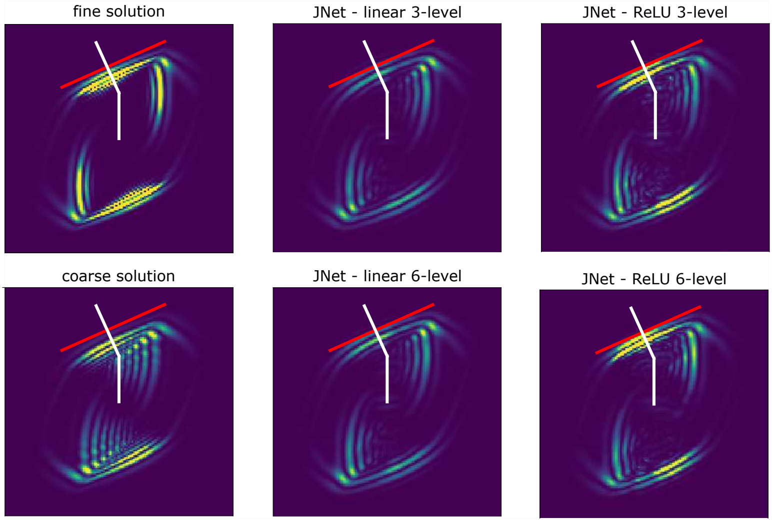

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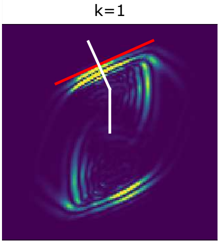

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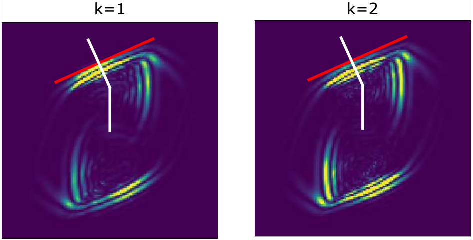

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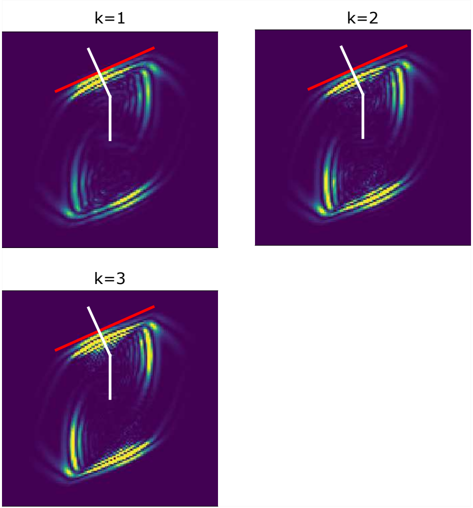

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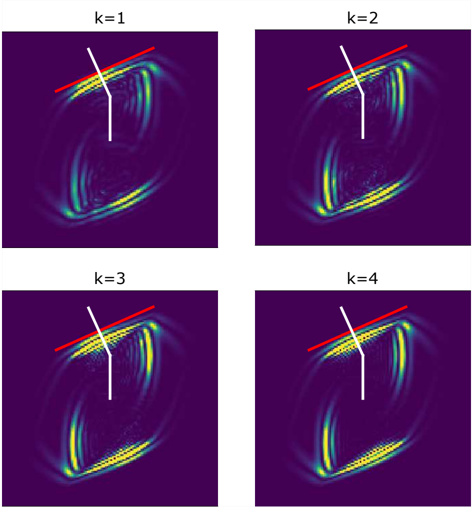
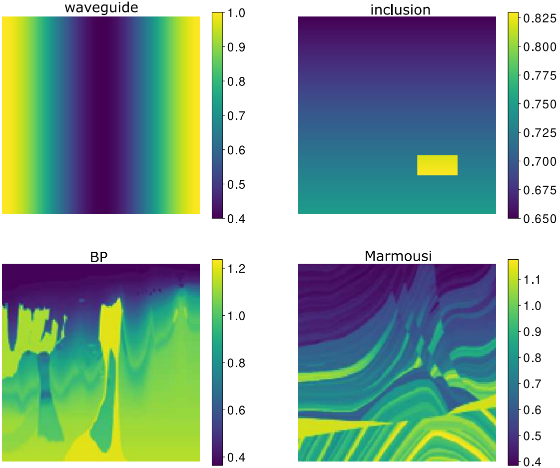


Numerical results - Parareal simulations



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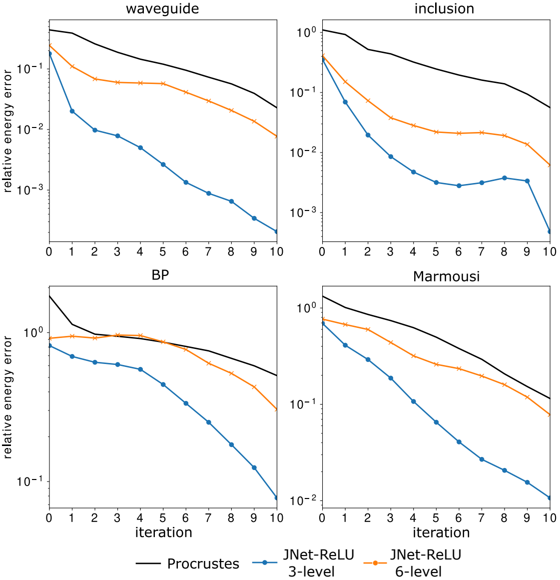
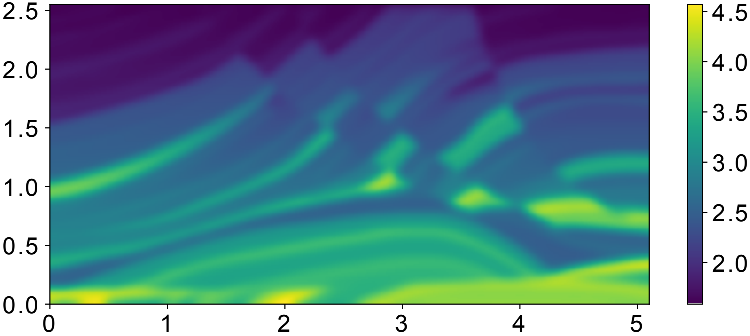


Numerical results - Parareal simulations on Marmousi model



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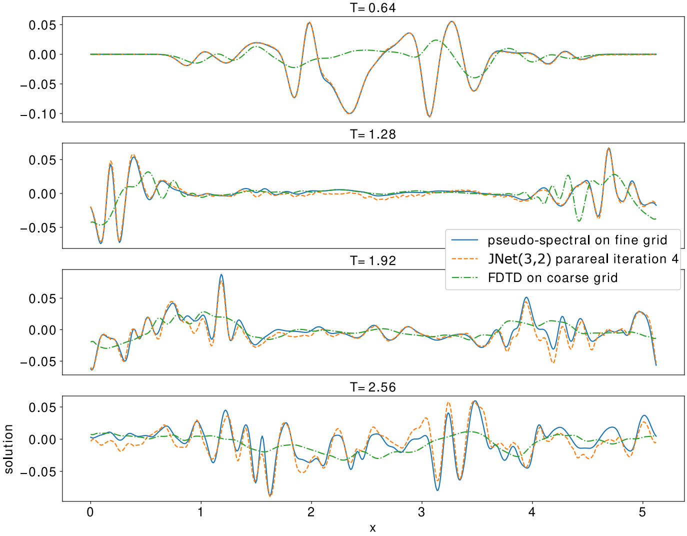

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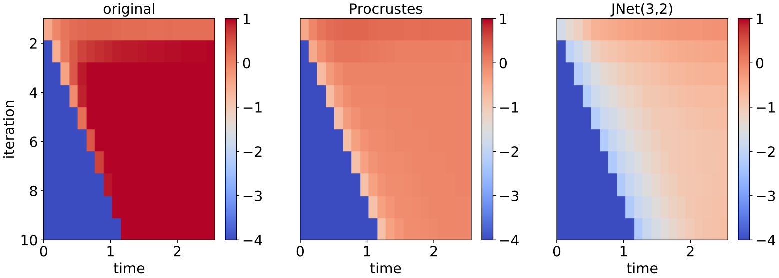
Energy error of solutions in log-10 scale



To sum up

- A deep neural network can stabilize parareal iteration for the wave equation
- Data must consist of stable solutions to train accurate models
- Agnostic to chosen fine/coarse numerical schemes
- Solution accuracy is gained thanks to the parareal coupling

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