Space-Time Finite Element Methods in Moving Domains

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SCIENCE PASSION TECHNOLOGY

Outline

Introduction

- Consider bounded reference domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with Lipschitz boundary $\partial\Omega = \Gamma$
- For $T \in (0, \infty)$ consider moving domain

$$\Omega(t) := \Big\{ \boldsymbol{y} = \boldsymbol{\varphi}(t, \boldsymbol{x}) \ \text{ for } \ \boldsymbol{x} \in \Omega \Big\}, \qquad \text{for } t \in (0, T).$$

- Deformation φ bijective for all t and sufficiently regular, satisfying $\boldsymbol{\varphi}(0, \boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \Omega$.

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- Deformation φ bijective for all t and sufficiently regular, satisfying $\boldsymbol{\varphi}(0, \boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \Omega$.
- For fixed $\boldsymbol{x} \in \Omega$ consider trajectory $\boldsymbol{y}(t) = \boldsymbol{\varphi}(t, \boldsymbol{x})$ with velocity $\boldsymbol{v}(t, \boldsymbol{y}) = \frac{d}{dt}\boldsymbol{y}(t)$.
- Space-time domain

$$Q := \Big\{ (\boldsymbol{y}, t) \in \mathbb{R}^{n+1} \ : \ \boldsymbol{y} = \boldsymbol{\varphi}(t, \boldsymbol{x}) \in \Omega(t), \ t \in (0, T), \ x \in \Omega \Big\}.$$

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Heat equation

- Heat equation

$$\varrho(t,\boldsymbol{y})\left[\frac{\partial}{\partial t}u(t,\boldsymbol{y})+\boldsymbol{v}(t,\boldsymbol{y})\cdot\nabla_y u(t,\boldsymbol{y})\right]-\mathrm{div}_y[\Lambda(t,\boldsymbol{y})\nabla_y u(t,\boldsymbol{y})]=f(t,\boldsymbol{y})$$

for $(\boldsymbol{y},t)\in Q$, where $\varrho(t,\boldsymbol{y})$ heat capacity, diffusion $\Lambda(t,\boldsymbol{y})$ positive definite and uniformly elliptic.

- Homogeneous initial and Dirichlet boundary conditions:

$$u(0,\boldsymbol{x})=0 \quad \text{for} \quad \boldsymbol{x}\in\Omega,$$
$$u(t,\boldsymbol{y})=0 \quad \text{for} \quad \boldsymbol{y}\in\partial\Omega(t),\ t\in(0,T).$$

Bochner spaces

- Introduce Bochner spaces:

$$Y := L^2(0,T; H_0^1(\Omega(t))),$$

$$X := \left\{ u \in Y \ : \ \frac{d}{dt} u \in Y^*, \ u(0,\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \in \Omega \right\},$$

with norms

$$\|u\|_Y^2 = \int_0^T \int_{\Omega(t)} [\Lambda(t,\boldsymbol{y}) \nabla_y u(t,\boldsymbol{y})] \cdot \nabla_y u(t,\boldsymbol{y}) d\boldsymbol{y} dt,$$

$$\|u\|_X^2 = \|u\|_Y^2 + \|\varrho \frac{d}{dt} u\|_{Y^*}^2,$$

$$\|\varrho \frac{d}{dt} u\|_{Y^*} = \sup_{0 \neq z \in Y} \frac{\langle \varrho \frac{d}{dt} u, z \rangle_Q}{\|z\|_Y}.$$

Eddy current problem

- Eddy current problem

$$\sigma_{\Omega(t)} \frac{d}{dt} u(t, \boldsymbol{y}) - \text{div}\,(\nu_{\Omega(t)}(|\nabla u|)\nabla_y u(t, \boldsymbol{y})) = j_z + M^{\perp} \quad \text{for } (t, \boldsymbol{y}) \in Q,$$

with different conductivities σ and reluctivities ν.

- Homogeneous initial and Dirichlet boundary conditions:

$$u(0, \boldsymbol{x}) = 0 \quad \text{for} \quad \boldsymbol{x} \in \Omega,$$
$$u(t, \boldsymbol{y}) = 0 \quad \text{for} \quad \boldsymbol{y} \in \partial\Omega(t),\ t \in (0, T).$$

Eddy current problem

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$$\sigma_{\Omega(t)} \frac{d}{dt} u(t, \boldsymbol{y}) - \text{div}\, (\nu_{\Omega(t)}(|\nabla u|) \nabla_y u(t, \boldsymbol{y})) = j_z + M^{\perp} \quad \text{for } (t, \boldsymbol{y}) \in Q,$$
with different conductivities σ and reluctivities ν.

- Homogeneous initial and Dirichlet boundary conditions:
$$\begin{aligned} u(0, \boldsymbol{x}) &= 0 \quad \text{for} \quad \boldsymbol{x} \in \Omega, \\ u(t, \boldsymbol{y}) &= 0 \quad \text{for} \quad \boldsymbol{y} \in \partial\Omega(t),\ t \in (0, T). \end{aligned}$$

- Bochner spaces X and Y as before, with norms
$$\|u\|_Y^2 = \int_0^T \int_{\Omega(t)} [\nu_{\Omega(t)} \nabla_y u(t, \boldsymbol{y})] \cdot \nabla_y u(t, \boldsymbol{y}) d\boldsymbol{y} dt,$$
$$\|u\|_X^2 = \|u\|_Y^2 + \|\sigma_{\Omega(t)} \frac{d}{dt} u\|_{Y^*}^2, \quad \|\sigma_{\Omega(t)} \frac{d}{dt} u\|_{Y^*} = \sup_{0 \neq z \in Y} \frac{\langle \sigma_{\Omega(t)} \frac{d}{dt} u, z \rangle_Q}{\|z\|_Y}.$$

Variational formulation - Heat equation

- Find $u \in X$, such that

$$a(u,z) = \langle f, z\rangle_Q \quad \text{for all } z \in Y,$$

where

$$a(u,z) := \int_0^T \int_{\Omega(t)} \Big[\varrho(t,\boldsymbol{y}) \frac{d}{dt} u(t,\boldsymbol{y})\, z(t,\boldsymbol{y}) + [\Lambda(t,\boldsymbol{y}) \nabla_y u(t,\boldsymbol{y})] \cdot \nabla_y z(t,\boldsymbol{y})\Big]\, d\boldsymbol{y}\, dt$$

$$= \int_0^T \int_{\Omega(t)} \Big(\varrho(t,\boldsymbol{y}) \left[\frac{\partial}{\partial t} u(t,\boldsymbol{y}) + \boldsymbol{v}(t,\boldsymbol{y}) \cdot \nabla_y u(t,\boldsymbol{y})\right] z(t,\boldsymbol{y}) + [\Lambda(t,\boldsymbol{y}) \nabla_y u(t,\boldsymbol{y})] \cdot \nabla_y z(t,\boldsymbol{y})\Big)\, d\boldsymbol{y}\, dt$$

Variational formulation - Eddy current problem

- Find $u \in X$, such that

$$a(u,v) = \langle F_{\Omega(t)}, v\rangle \quad \text{for all } v \in Y,$$

where

$$a(u,v) = \int_Q \left(\sigma_{\Omega(t)} \frac{\mathrm{d}}{\mathrm{d}t} uv + \nu_{\Omega(t)}(|\nabla u|)\nabla_y u \cdot \nabla_y v\right) \mathrm{d}(\boldsymbol{y},t),$$

$$\langle F_{\Omega(t)}, v\rangle = \int_0^T \int_{\Omega_\mathrm{C}} j_z v \,\mathrm{d}(\boldsymbol{y},t) + \int_0^T \int_{\Omega_\mathrm{m}(t)} M^\perp \cdot \nabla_y v \,\mathrm{d}(\boldsymbol{y},t).$$

Variational formulation

- a is bounded, i.e.

$$|a(u,z)| \leq \sqrt{2}\,\|u\|_X \|z\|_Y .$$

Lemma 1 (Steinbach, 2015)

For $u \in X$ there holds the inf-sup stability condition

$$\frac{1}{\sqrt{2}}\,\|u\|_X \leq \sup_{0 \neq z \in Y} \frac{a(u,z)}{\|z\|_Y} .$$

Lemma 2

For all $z \in Y \backslash \{0\}$ there exists a $\widetilde{u} \in X$ such that

$$a(\widetilde{u}, z) \neq 0 .$$

- Unique solvability follows form Babuška-Nečas Theorem!

Galerkin variational formulation

- Consider admissible decomposition of Q into shape regular simplicial finite elements
- Introduce FE space $X_h = Y_h \subset X \subset Y$ of piecewise linear and continuous basis functions
- Space-time FE Galerkin variational form: Find $u_h \in X_h$, s.t.

$$a(u_h, z_h) = \langle f, z_h \rangle \quad \text{for all } z_h \in Y_h.$$

Lemma 3

For $u \in X_h$ there holds the discrete inf-sup stabilty condition

$$\frac{1}{\sqrt{2}} \|u_h\|_{X_h} \leq \sup_{0 \neq z_h \in Y_h} \frac{a(u_h, z_h)}{\|z_h\|_Y}.$$

- Standard stability and error estimates follow.

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Perturbed Galerkin variational formulation - Heat equation

- Find $u_h \in X_h$ such that

$$\int_0^T \int_{\Omega(t)} \left(\varrho(t,\boldsymbol{y}) \left[\frac{\partial}{\partial t} u_h(t,\boldsymbol{y}) + \boldsymbol{v}(t,\boldsymbol{y}) \cdot \nabla_y u_h(t,\boldsymbol{y}) \right] z_h(t,\boldsymbol{y}) \right.$$

$$\left. + \left[\Lambda(t,\boldsymbol{y}) \nabla_y u_h(t,\boldsymbol{y})\right] \cdot \nabla_y z_h(t,\boldsymbol{y}) \right) d\boldsymbol{y}\, dt$$

$$= \int_0^T \int_{\Omega(t)} f(t,\boldsymbol{y}) z_h(t,\boldsymbol{y})\, d\boldsymbol{y}\, dt.$$

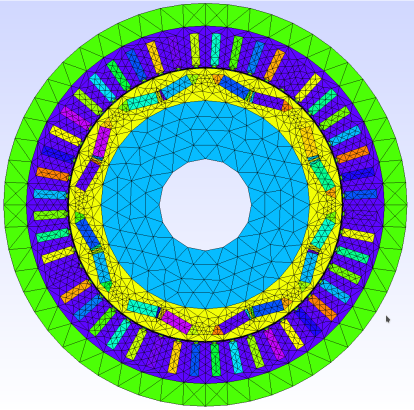
- Usually $\boldsymbol{v}(t,\boldsymbol{y})$ is not known, but approximation $\boldsymbol{v}_h(t,\boldsymbol{y})$.

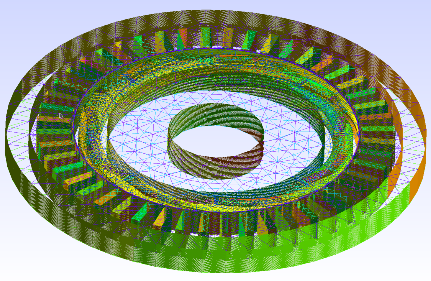
- Perturbed variational form: Find $\widetilde{u}_h \in X_h$ such that

$$\widetilde{a}(\widetilde{u}_h, z_h) = \langle f, z_h \rangle_Q \quad \text{for all } z_h \in Y_h.$$

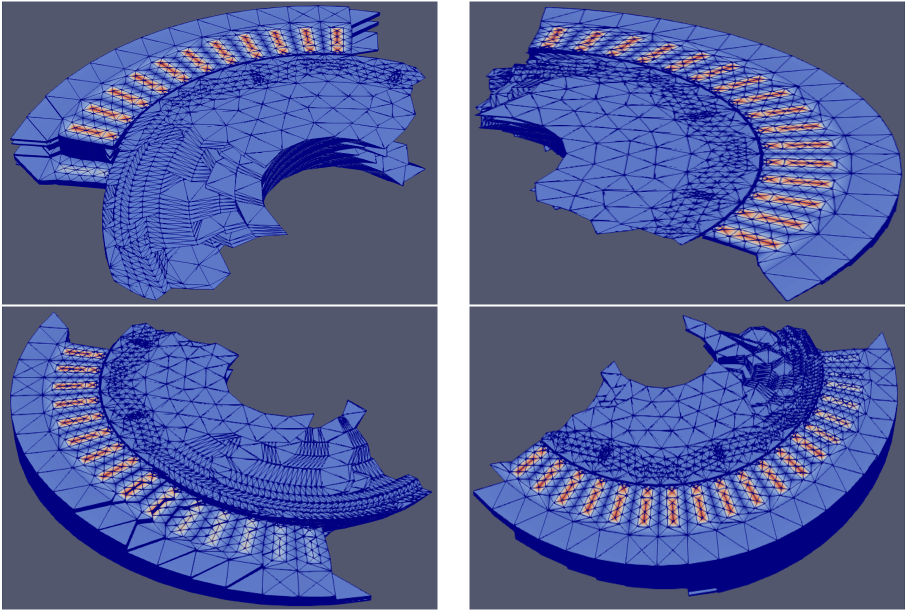
- Numerical Analysis via Strang-Lemma.

Mesh of the electric motor





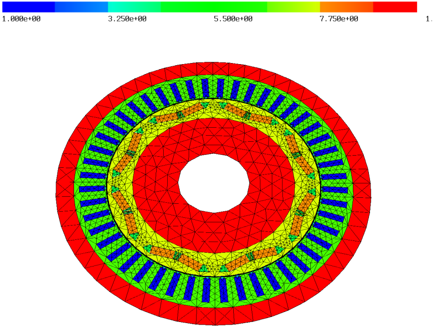
number of elements: 642716
number of points: 104144

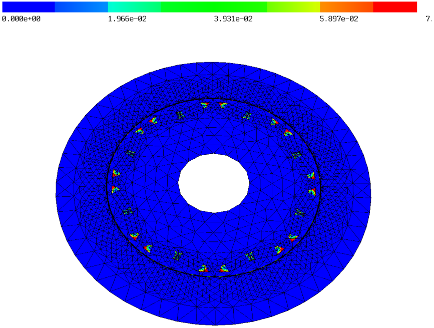


Computation of the velocity field

$$(u,p) \in X : \int\limits_{Q_{air}} \left[\frac{\partial u}{\partial t} \cdot v + \nu \nabla_y u : \nabla_y v - \mathrm{div}_y(u) q - \mathrm{div}_y(v) p \right] \, d(y,t) = 0,$$

for $(v,q) \in X$. Viscosity $\nu = 0.001$.





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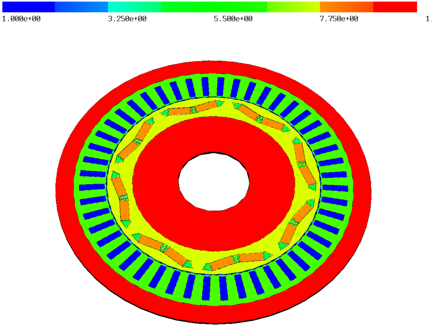

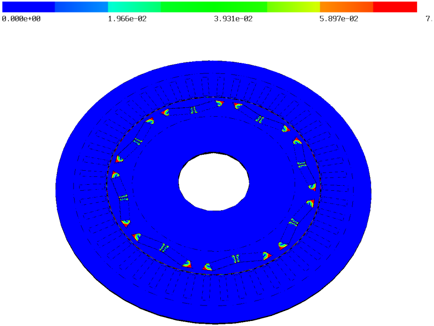

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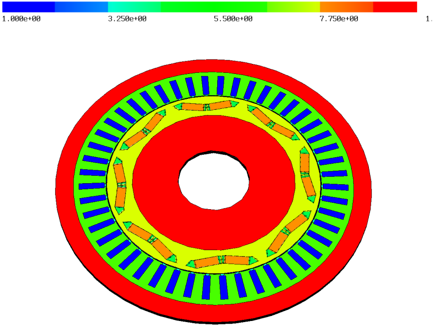

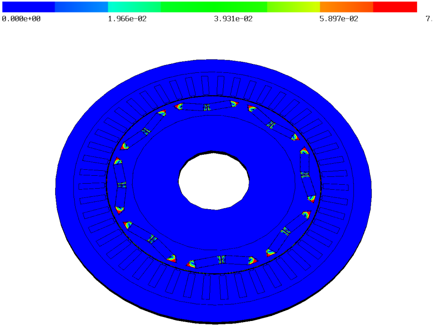

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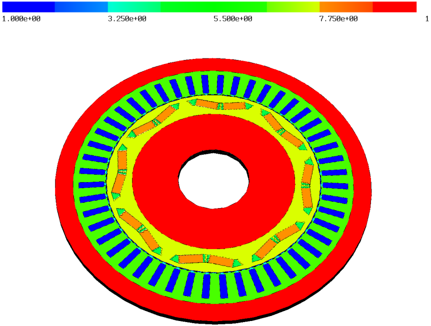

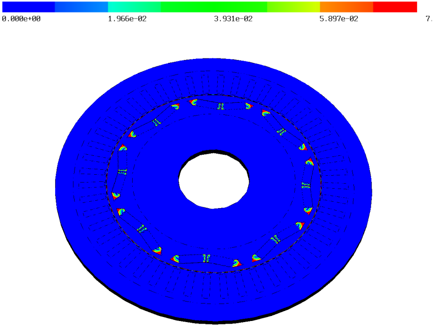

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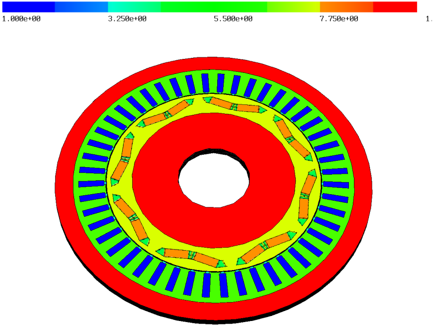

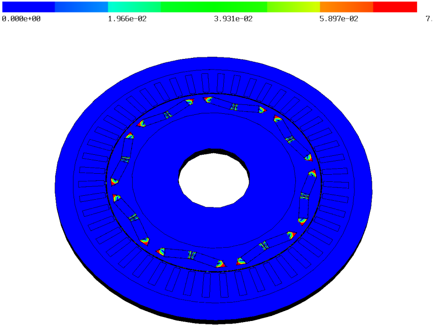

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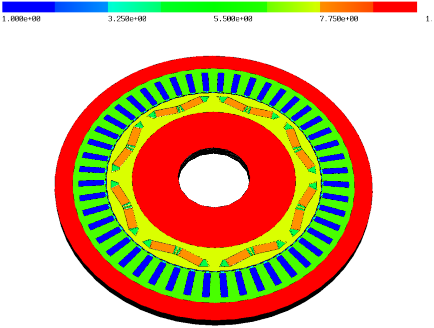

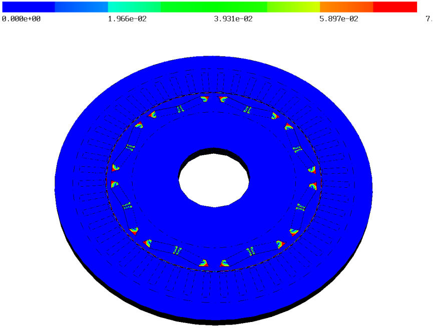

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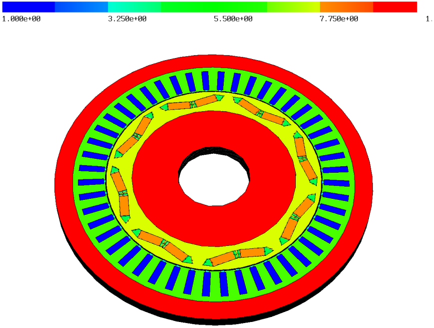

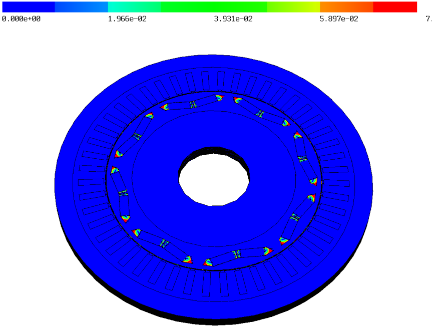

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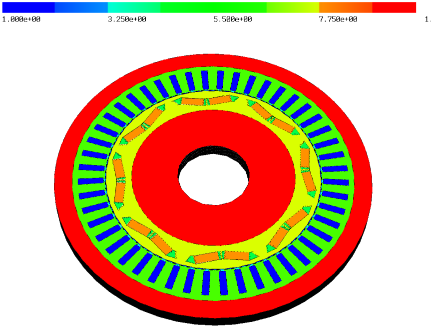

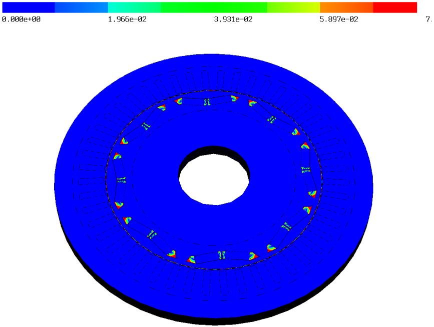

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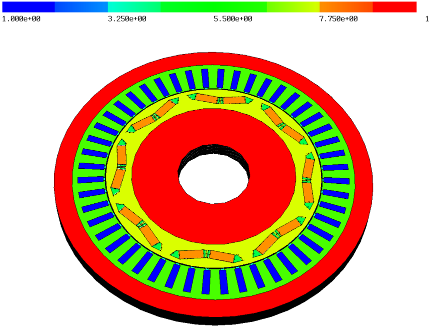

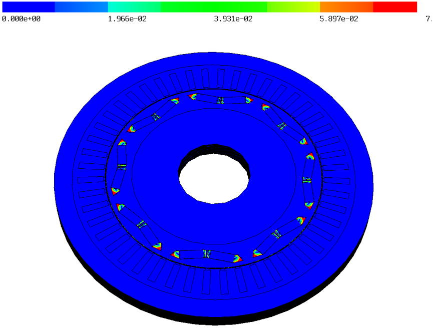

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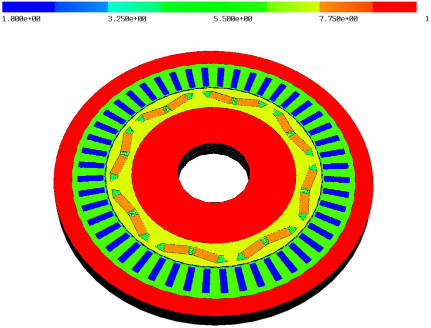

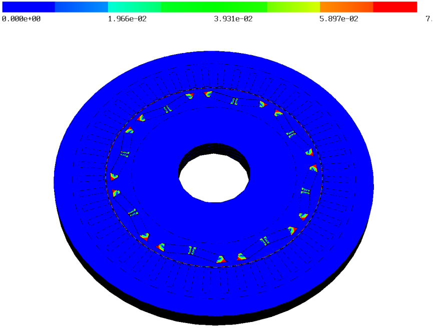

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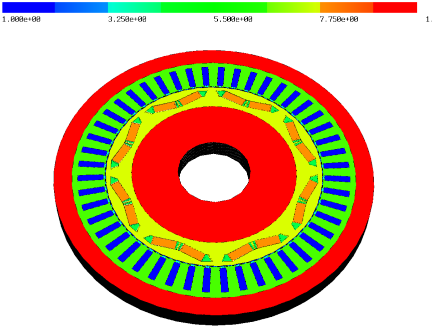
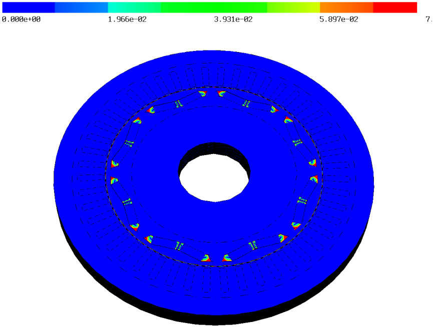




Heat equation

$$c_{air}\rho_{air}\frac{\partial u}{\partial t} - \mathrm{div}_y(\Lambda_{air}\nabla_y u) + b_{air}\mathrm{div}_y(uV) = f_{air} \quad \text{in } Q_{air}$$

$$c_{iron}\rho_{iron}\frac{\partial u}{\partial t} - \mathrm{div}_y(\Lambda_{iron}\nabla_y u) + b_{iron}\mathrm{div}_y(uV) = f_{iron} \quad \text{in } Q \setminus \overline{Q}_{air}$$

$c_{air}\rho_{air} = 1005 * 1.293,$
$c_{iron}\rho_{iron} = 465 * 7860,$

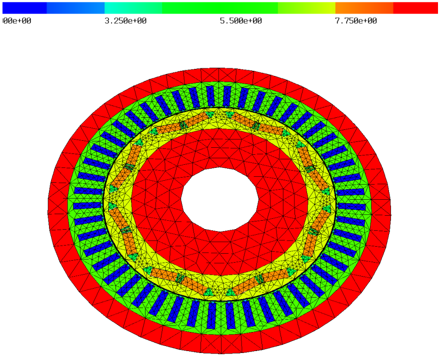
$\Lambda_{air} = 0.026, \ \Lambda_{iron} = 40,$

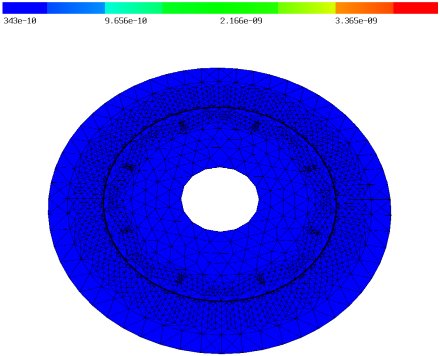
$b_{air} = c_{air}\rho_{air}, \ b_{iron} = c_{iron}\rho_{iron}$

$$f_{air} = 0, \ f_{iron} = \begin{cases} 1, & Q_{coils} \\ 0, & \text{otherwise} \end{cases}$$

Heat equation

$$
\begin{aligned}
c_{air}\rho_{air}\frac{\partial u}{\partial t} - \mathrm{div}_y(\Lambda_{air}\nabla_y u) + b_{air}\mathrm{div}_y(uV) &= f_{air} \quad \text{in } Q_{air} \\
c_{iron}\rho_{iron}\frac{\partial u}{\partial t} - \mathrm{div}_y(\Lambda_{iron}\nabla_y u) + b_{iron}\mathrm{div}_y(uV) &= f_{iron} \quad \text{in } Q \setminus \overline{Q}_{air}
\end{aligned}
$$





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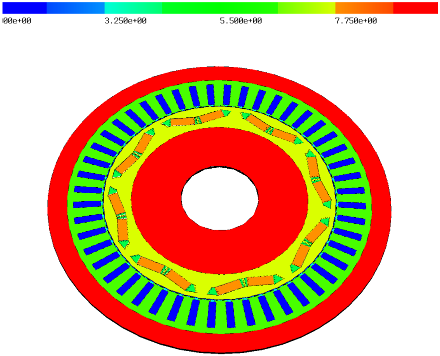
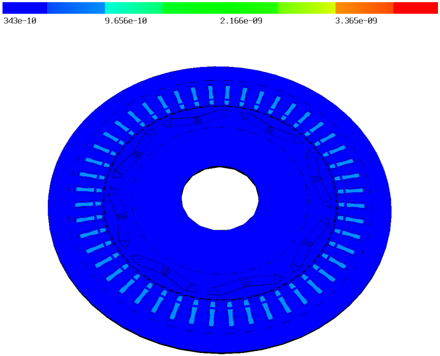
$$





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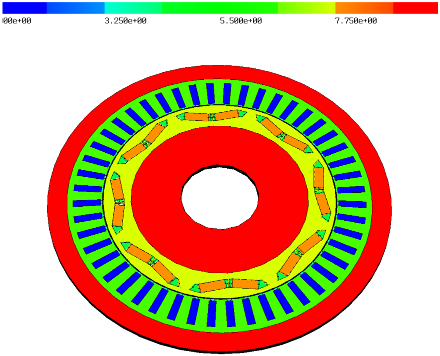

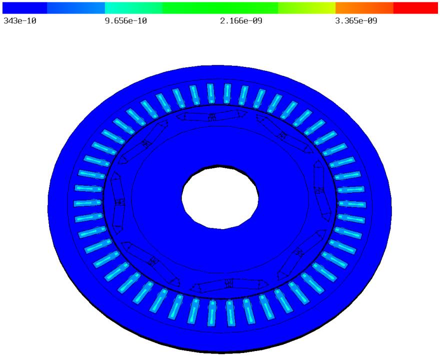

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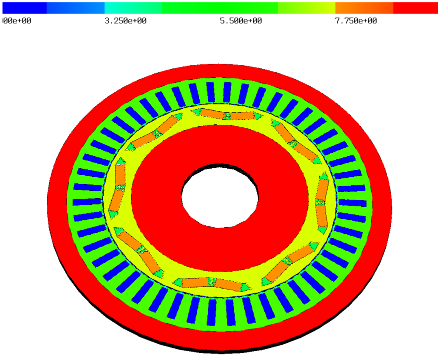
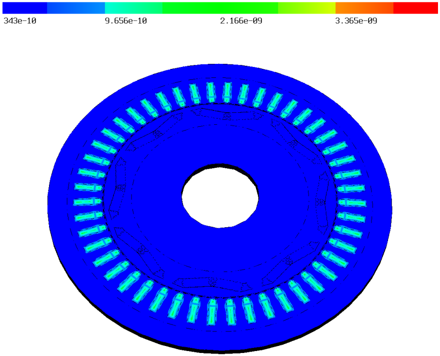
PinT 2022, 2022-07-14

Heat equation

$$c_{air}\rho_{air}\frac{\partial u}{\partial t} - \text{div}_y(\Lambda_{air}\nabla_y u) + b_{air}\text{div}_y(uV) = f_{air} \quad \text{in } Q_{air}$$

$$c_{iron}\rho_{iron}\frac{\partial u}{\partial t} - \text{div}_y(\Lambda_{iron}\nabla_y u) + b_{iron}\text{div}_y(uV) = f_{iron} \quad \text{in } Q \setminus \overline{Q}_{air}$$





Heat equation

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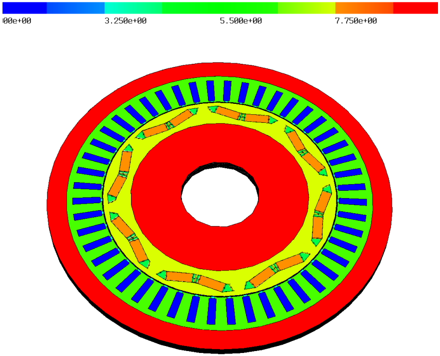

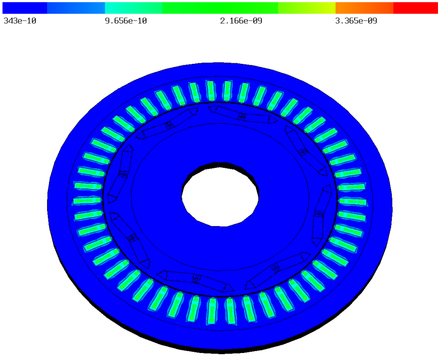

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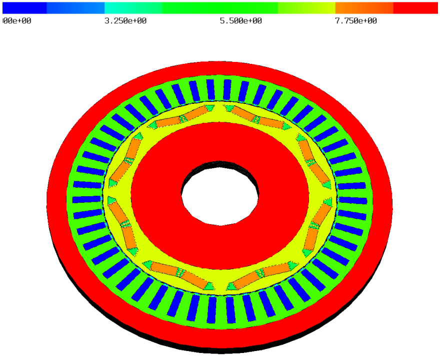

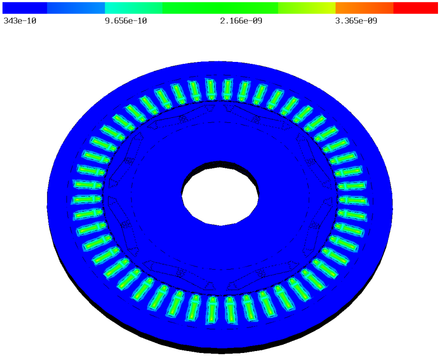

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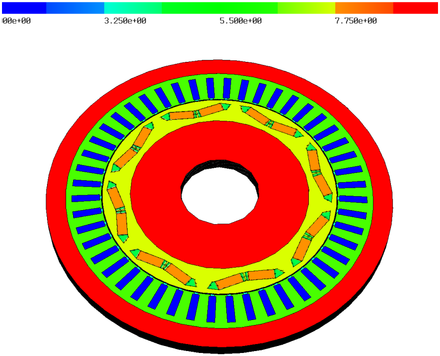

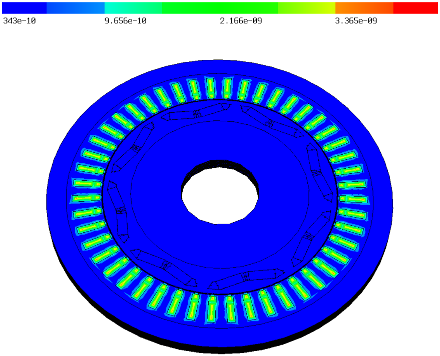

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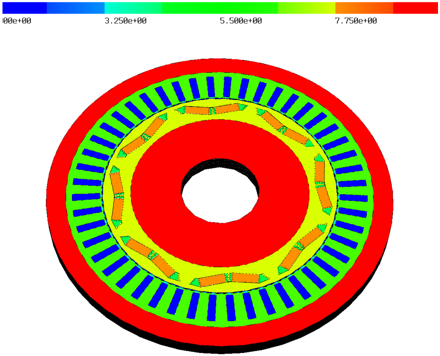

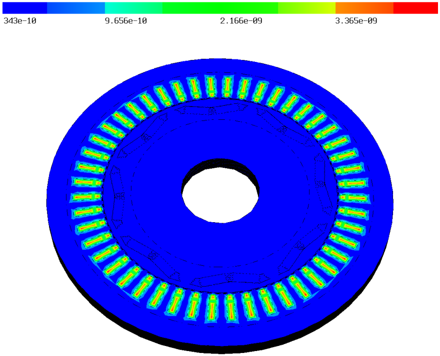

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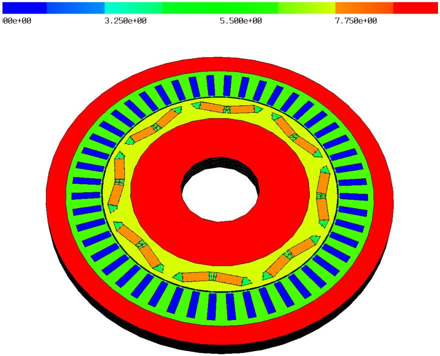

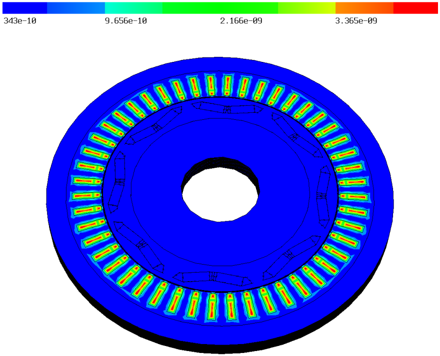

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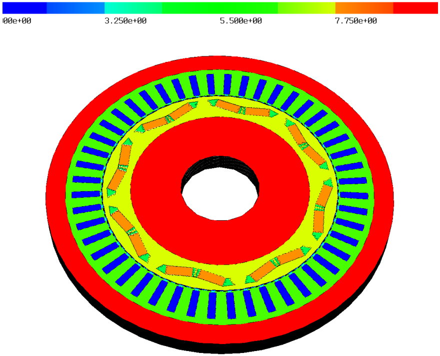

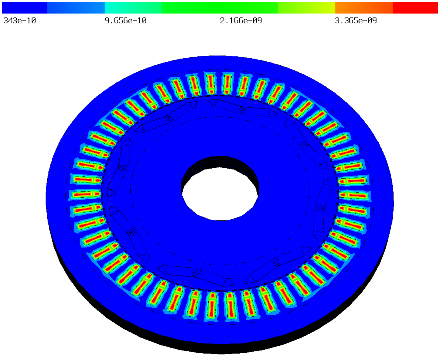

M. Gobrial, TU Graz

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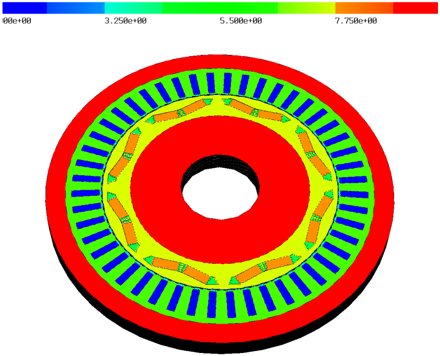
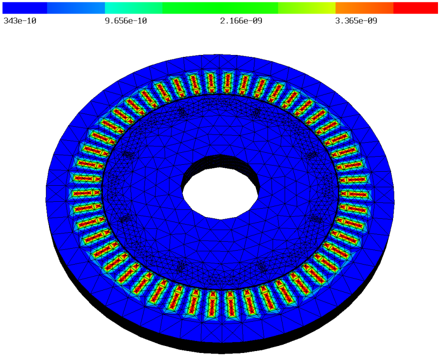
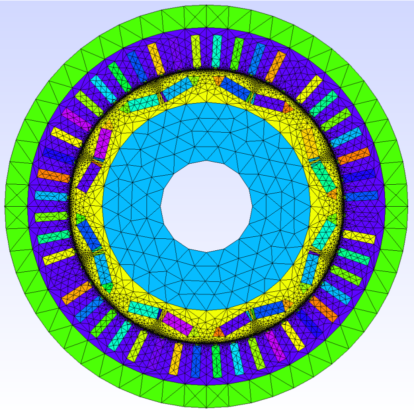
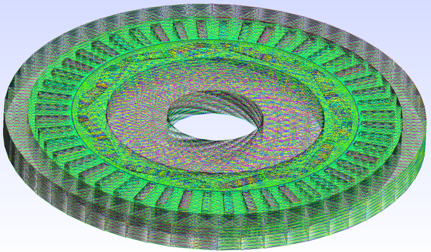


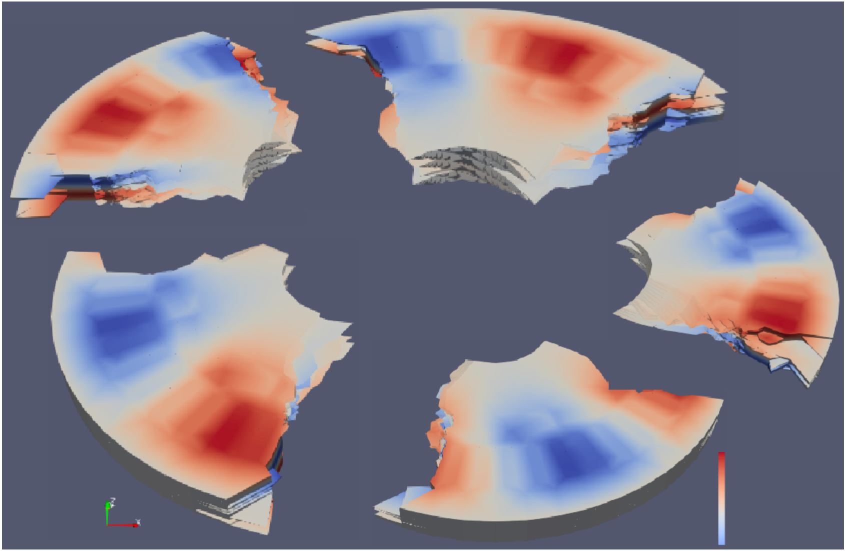


Mesh of the electric motor





number of elements: 1978689
number of points: 333288



Eddy current problem

$$\int_Q \sigma_{\Omega(t)} \frac{\mathrm{d}}{\mathrm{d}t} uv + \nu_{\Omega(t)}(|\nabla u|) \nabla_y u \cdot \nabla_y v \, \mathrm{d}(\boldsymbol{y}, t) = \langle F_{\Omega(t)}, v \rangle \quad v \in Y,$$

where

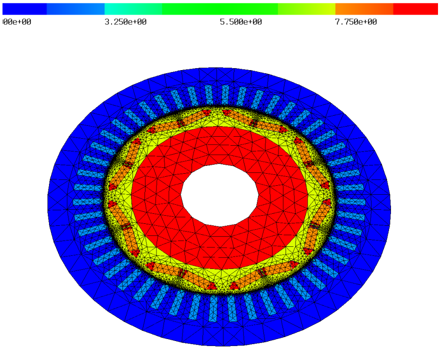
$$\langle F_{\Omega(t)}, v \rangle = \int_0^T \int_{\Omega_\mathrm{C}} j_z v \, \mathrm{d}(\boldsymbol{y}, t) + \int_0^T \int_{\Omega_\mathrm{m}(t)} M^\perp \cdot \nabla_y v \, \mathrm{d}(\boldsymbol{y}, t).$$

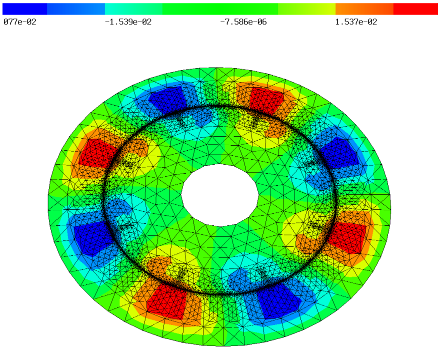
$$M^\perp := \nu_\mathrm{mag} B_R \begin{pmatrix} -M_2 \\ M1 \end{pmatrix}, \; \nu_\mathrm{mag} = \nu_0 / 1.05, \; B_R = 1.05 * 1.158$$

$$\nu_{\Omega(t)}(|\nabla u|) = \begin{cases} \nu_0 & \text{air, coils,} \\ \nu_\mathrm{mag} & \text{magnets,} \\ \nu(|\nabla u|) & \text{iron.} \end{cases} \quad \sigma_{\Omega(t)} = \begin{cases} 0 & \text{air,} \\ 0 & \text{coils,} \\ 0 & \text{magnets,} \\ 0 & \text{iron.} \end{cases}$$

Eddy current problem

$$u \in X : \int_Q \sigma_{\Omega(t)} \frac{\mathsf{d}}{\mathsf{d}t} uv + \nu_{\Omega(t)}(|\nabla u|)\nabla_y u \cdot \nabla_y v \, \mathsf{d}(\boldsymbol{y}, t) = \langle F_{\Omega(t)}, v \rangle \quad v \in Y$$





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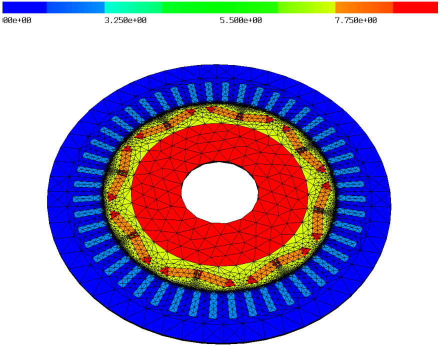

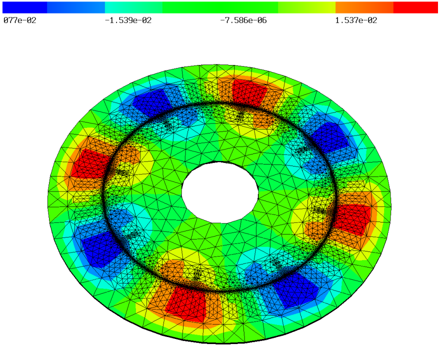

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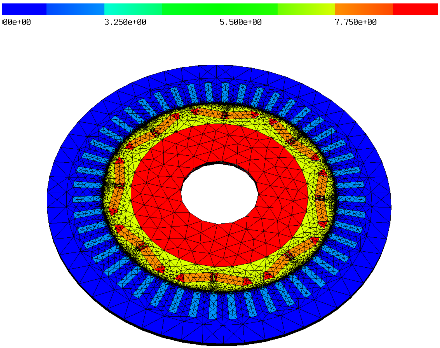

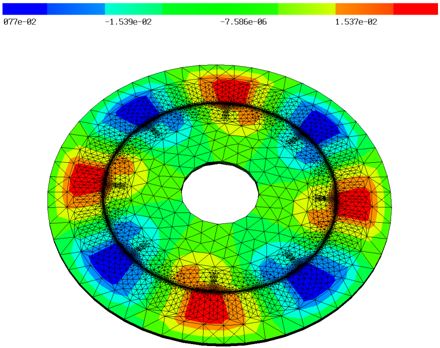

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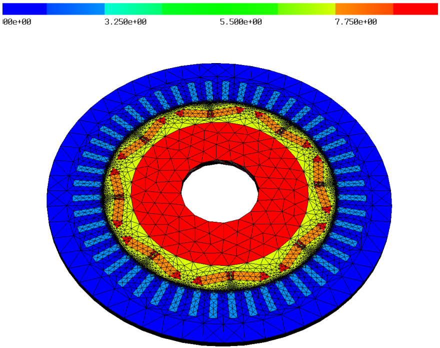

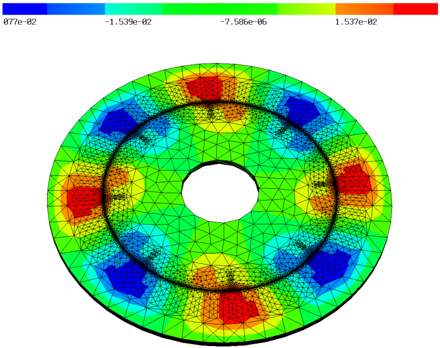

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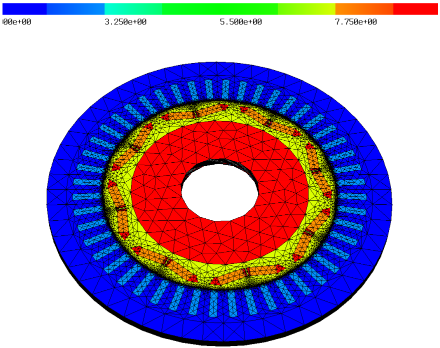

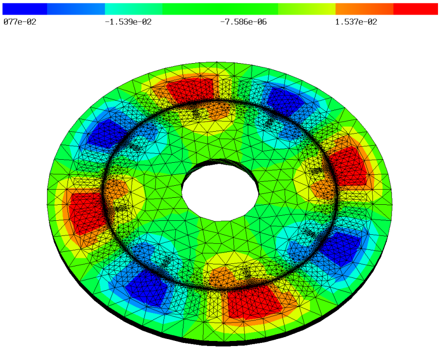

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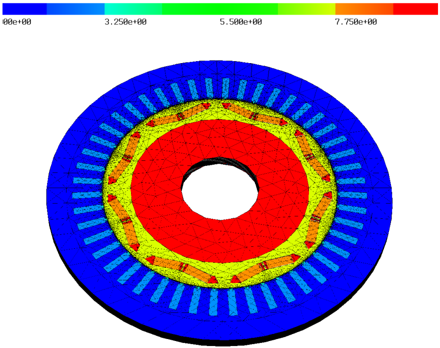

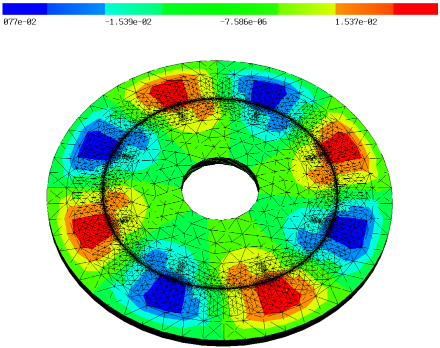

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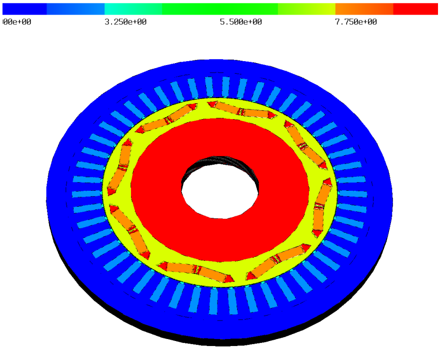

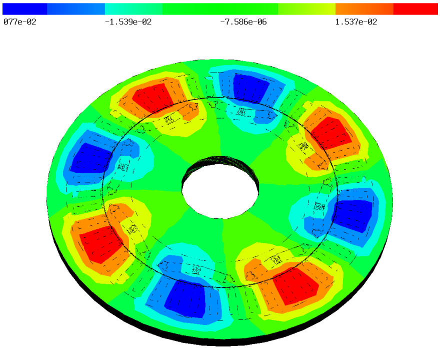

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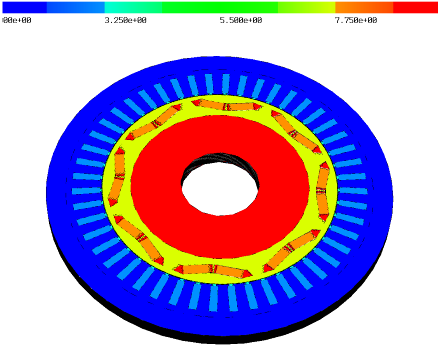

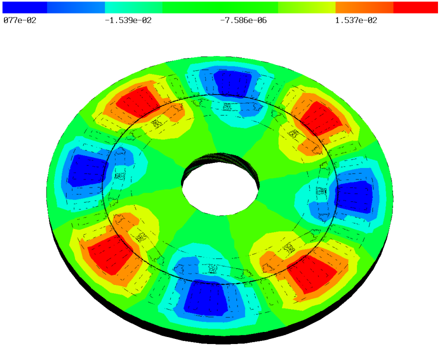

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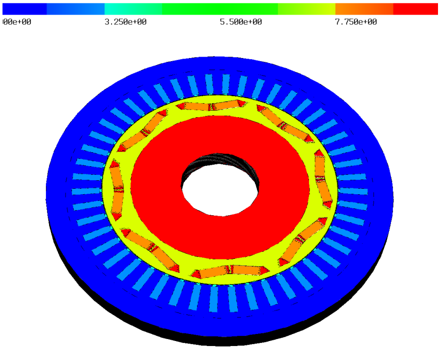

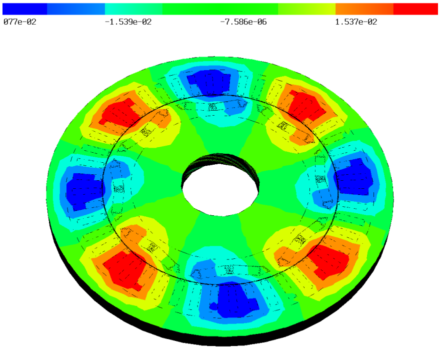

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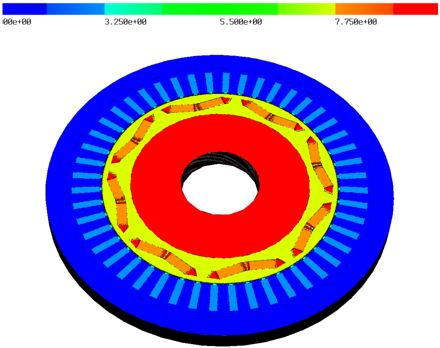

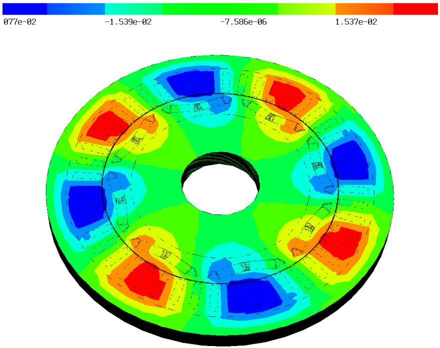

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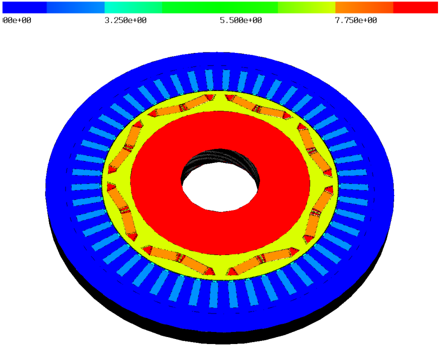

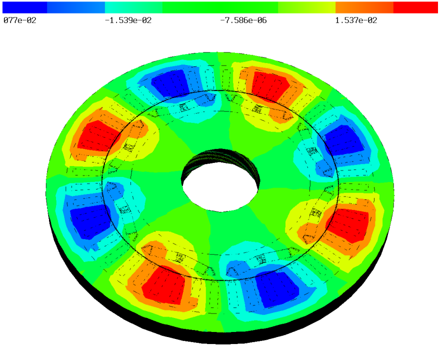

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Iron losses

- For every element τ in iron, calculate $B_r^\tau(0)$, $B_r^\tau(\Delta t)$, $B_r^\tau(2\Delta t)$, ... , $B_r^\tau(T)$ and $B_\phi^\tau(0)$, $B_\phi^\tau(\Delta t)$, $B_\phi^\tau(2\Delta t)$, ... , $B_\phi^\tau(T)$.
- Calculate Fourier-Coefficients

$$a_k^{r/\phi} = \frac{2}{T} \int_0^T B_{r/\phi}^\tau(t) \cos(k(\frac{2\pi}{T} t - \pi))\, dt,$$

$$b_k^{r/\phi} = \frac{2}{T} \int_0^T B_{r/\phi}^\tau(t) \sin(k(\frac{2\pi}{T} t - \pi))\, dt,$$

for $k = 1, ..., 20$. Then:

$$A_k^{r/\phi} = \sqrt{(a_k^{r/\phi})^2 + (b_k^{r/\phi})^2}.$$

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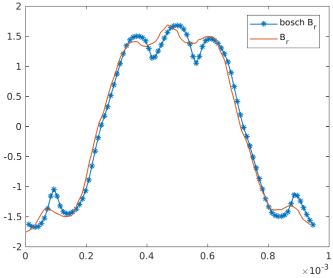
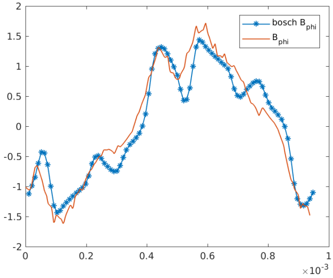
Iron losses

- $P = P_{eddy} + P_{hys}$ with $f_k = k\frac{1}{T}$

$$P_{eddy} = \sum_{k=1}^{20} 0.468 * (A_k^r)^2 * f_k^2 + 0.468 * (A_k^\phi)^2 * f_k^2$$

$$P_{hys} = \sum_{k=1}^{20} 195 * (A_k^r)^2 * f_k + 195 * (A_k^\phi)^2 * f_k^2$$





Summary and Outlook

- Heat equation for the motor
- Eddy current problem for 90 degree rotation
- Parallel computations for space time VF

Summary and Outlook

- Heat equation for the motor
- Eddy current problem for 90 degree rotation
- Parallel computations for space time VF

- Motor for one rotation or even more
- Shape/Topological optimization of electrical machines.
- Numerical scheme:
  - Parallelization
  - Domain decomposition methods
  - Preconditioners