On waveform-relaxation methods for wave-type equations

Gabriele Ciaramella

MOX, Politecnico di Milano
(joint work with Martin J. Gander and Ilario Mazzieri)

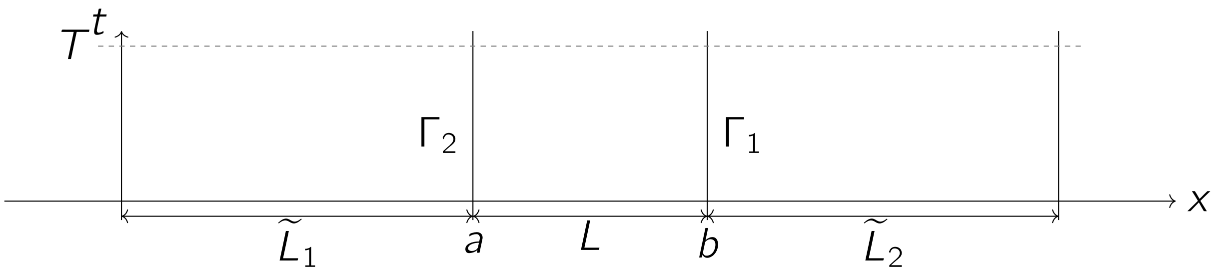
Wave equation

Consider a domain $\Omega = (0, 1)$, a time $T > 0$, and the wave equation

$$\begin{aligned} \mathcal{L}u := \partial_{tt}u - c^2\partial_{xx}u = 0 \qquad & \text{in } \Omega \times (0, T), \\ u(x, 0) = g_0(x) \text{ and } \partial_t u(x, 0) = g_1(x) \qquad & \text{for } x \in \Omega, \\ u(x, t) = 0 \qquad & \text{for } t \in [0, T] \text{ and } x \in \partial\Omega, \end{aligned}$$

where $c > 0$ is the wave speed.

Schwarz waveform relaxation method



Consider the Schwarz waveform relaxation (SWR) method: $\Omega = (0, b) \cup (a, 1)$, with $b > a$, and

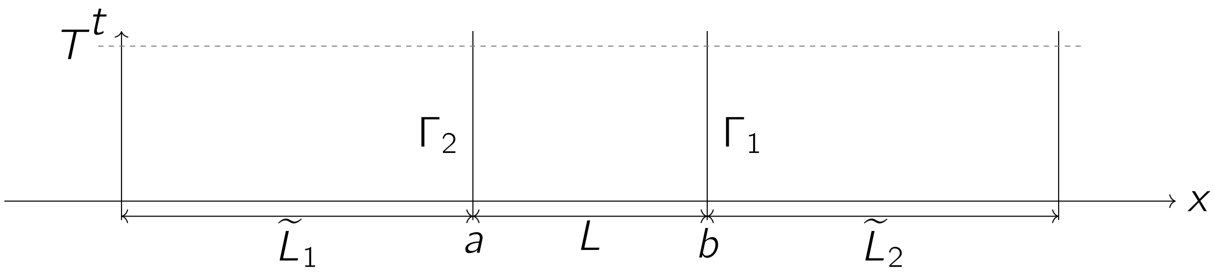
$$
\begin{aligned}
\mathcal{L}u_1^n &= 0 && \text{in } (0,b)\times(0,T),\\
u_1^n(x,0) = g_0(x) \text{ and } \partial_t u_1^n(x,0) &= g_1(x) && \text{for } x \in (0,b),\\
u_1^n(0,t) &= 0 && \text{for } t \in [0,T],\\
u_1^n(b,t) &= u_2^{n-1}(b,t) && \text{for } t \in [0,T],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}u_2^n &= 0 && \text{in } (a,1)\times(0,T),\\
u_2^n(x,0) = g_0(x) \text{ and } \partial_t u_2^n(x,0) &= g_1(x) && \text{for } x \in (a,1),\\
u_2^n(1,t) &= 0 && \text{for } t \in [0,T],\\
u_2^n(a,t) &= u_1^{n-1}(a,t) && \text{for } t \in [0,T],
\end{aligned}
$$

where n is the iteration index.

Schwarz waveform relaxation method



Consider the Schwarz waveform relaxation (SWR) method: $\Omega = (0, b) \cup (a, 1)$, with $b > a$, and

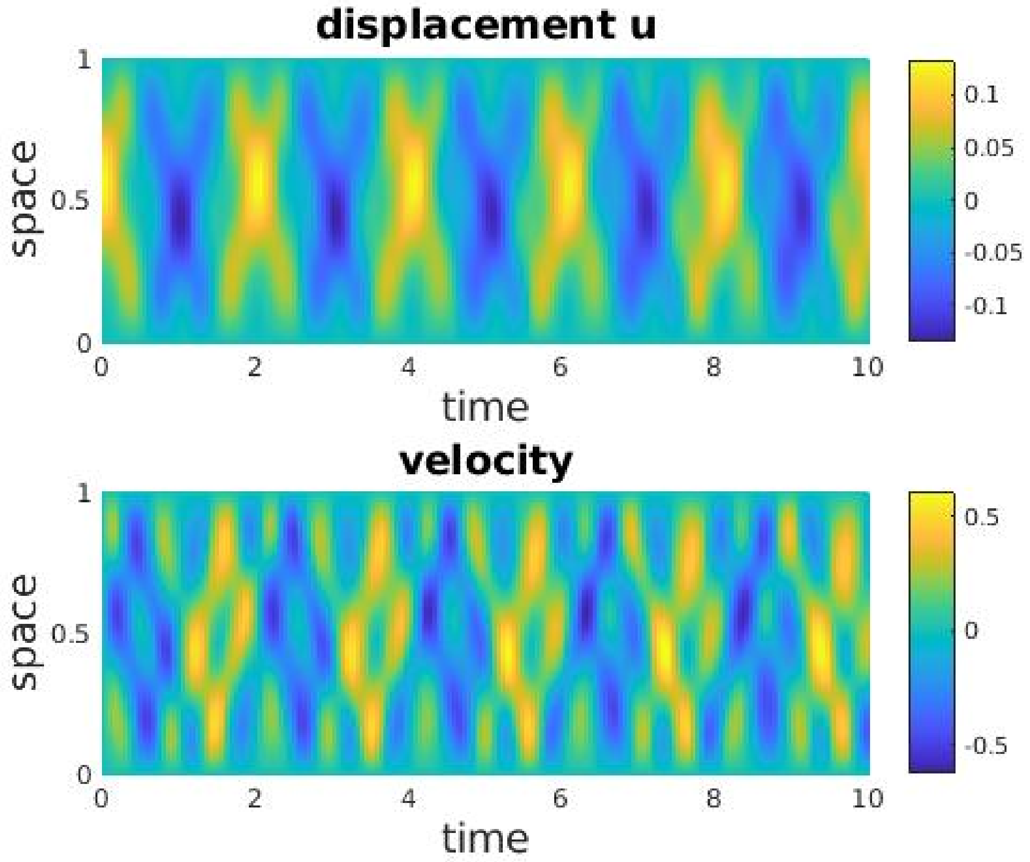
$$
\begin{aligned}
\mathcal{L}u_1^n &= 0 && \text{in } (0,b)\times(0,T),\\
u_1^n(x,0) = g_0(x) \text{ and } \partial_t u_1^n(x,0) &= g_1(x) && \text{for } x \in (0,b),\\
u_1^n(0,t) &= 0 && \text{for } t \in [0,T],\\
u_1^n(b,t) &= u_2^{n-1}(b,t) && \text{for } t \in [0,T],
\end{aligned}
$$

and

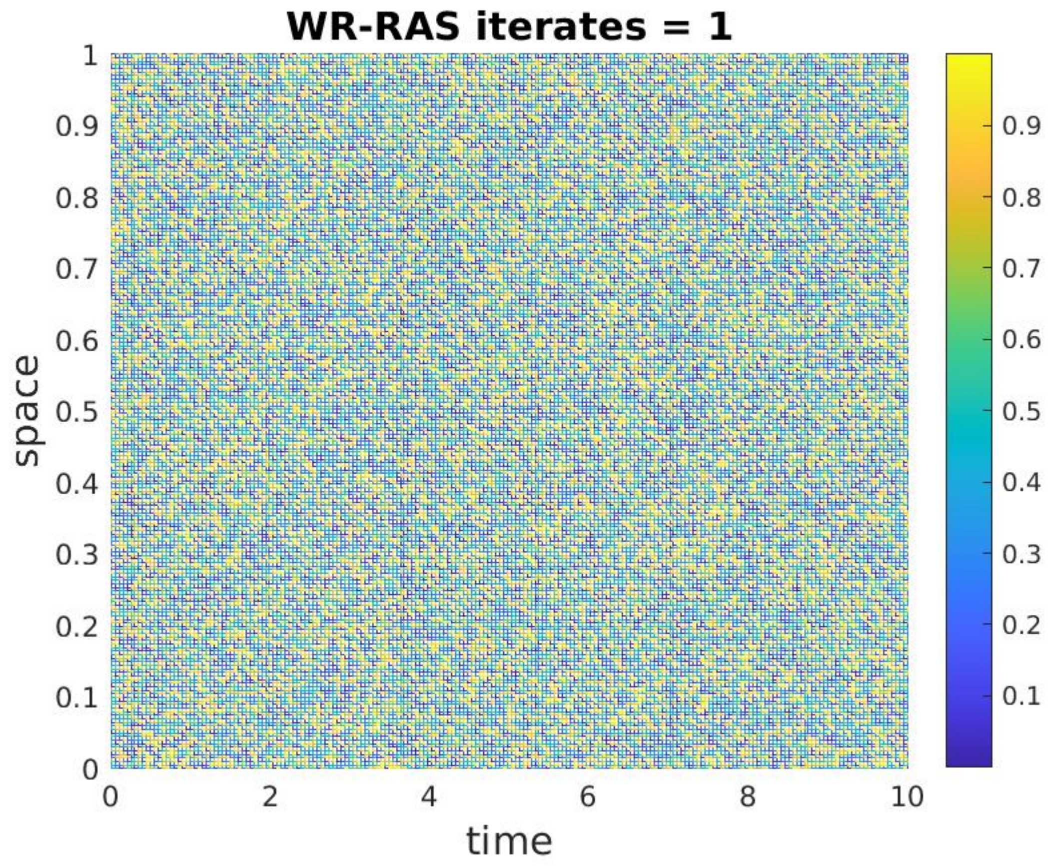
$$
\begin{aligned}
\mathcal{L}u_2^n &= 0 && \text{in } (a,1)\times(0,T),\\
u_2^n(x,0) = g_0(x) \text{ and } \partial_t u_2^n(x,0) &= g_1(x) && \text{for } x \in (a,1),\\
u_2^n(1,t) &= 0 && \text{for } t \in [0,T],\\
u_2^n(a,t) &= u_1^{n-1}(a,t) && \text{for } t \in [0,T],
\end{aligned}
$$

where n is the iteration index. (Exact solution for $n \geq \frac{Tc}{b-a}$.)

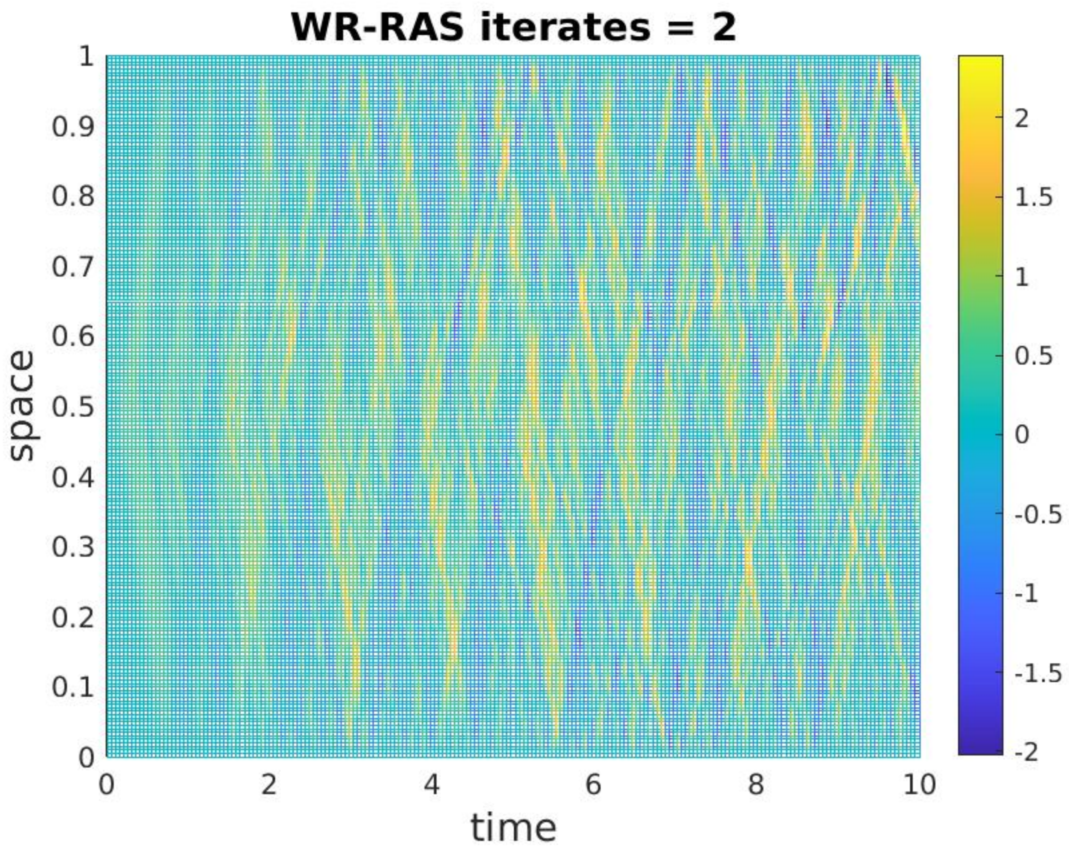
Numerical experiments ($T = 10$)



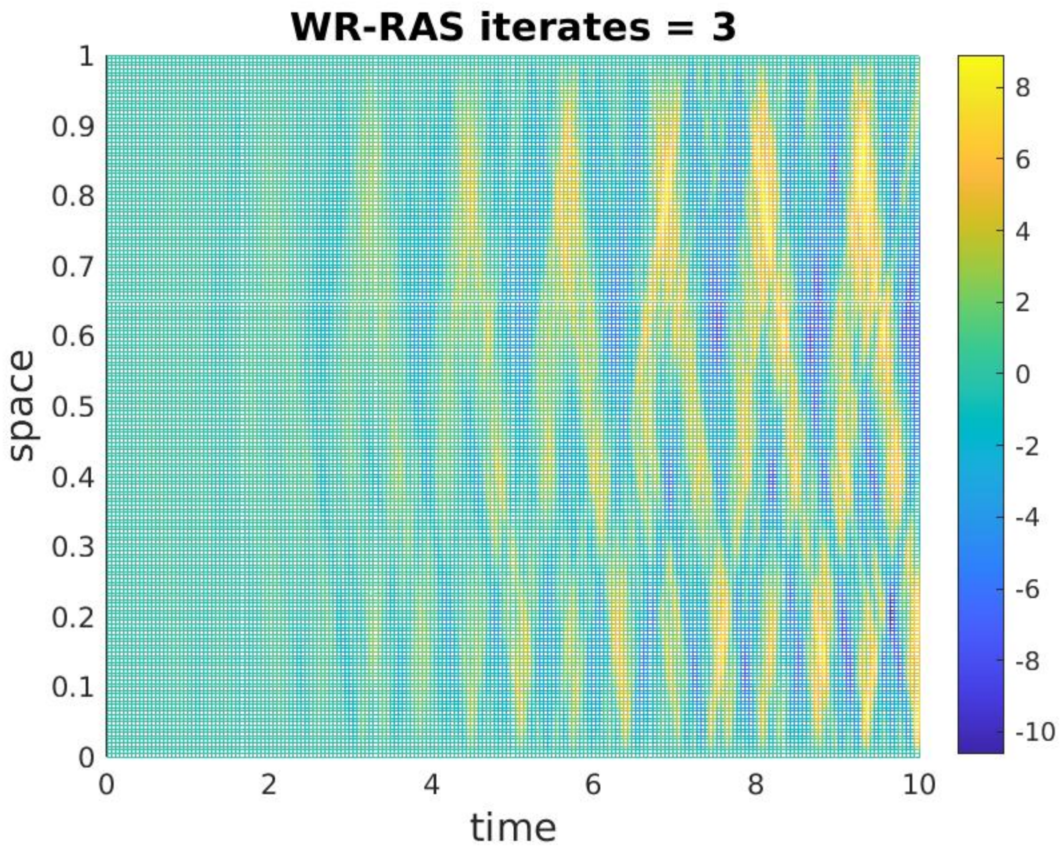
Numerical experiments ($T = 10$)



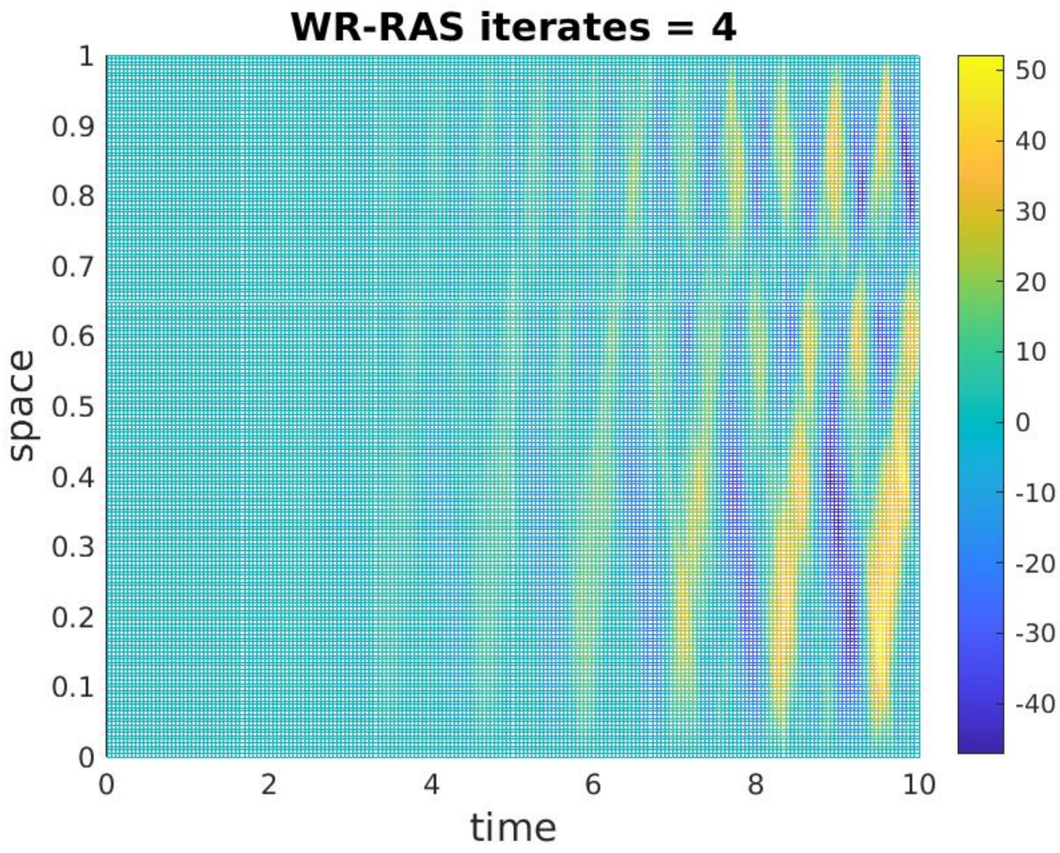
Numerical experiments ($T = 10$)



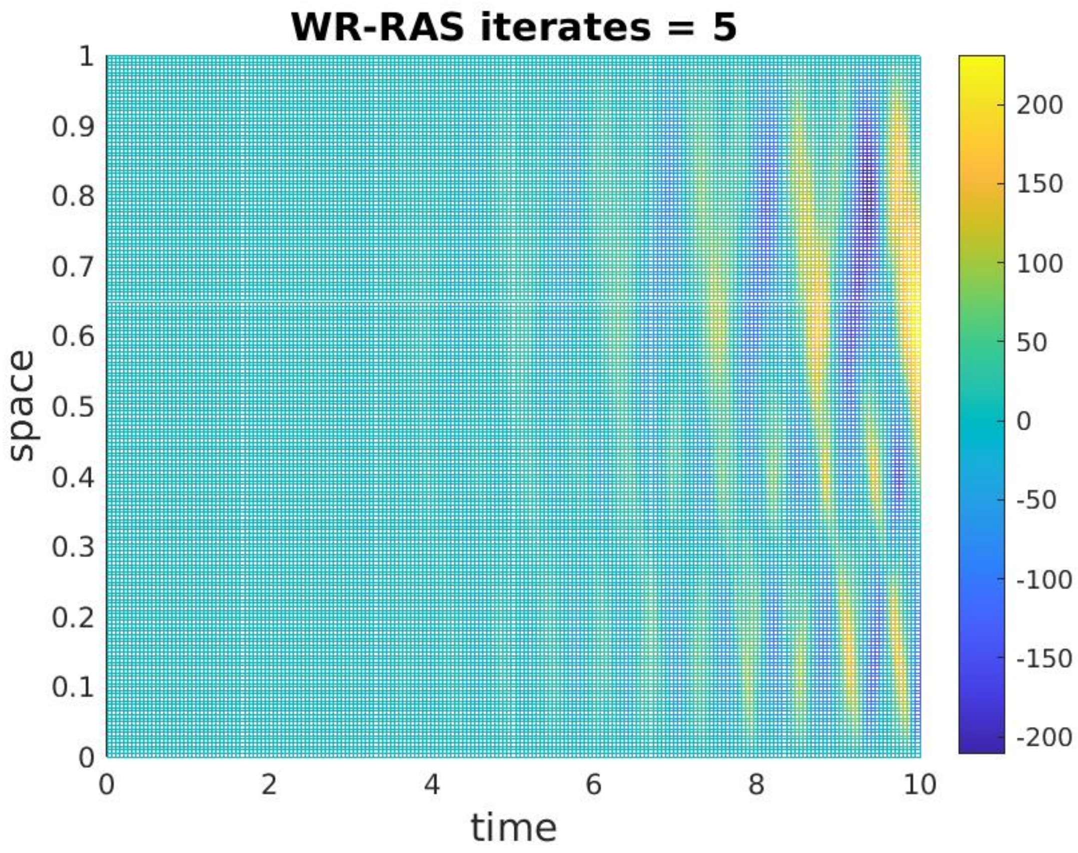
Numerical experiments ($T = 10$)



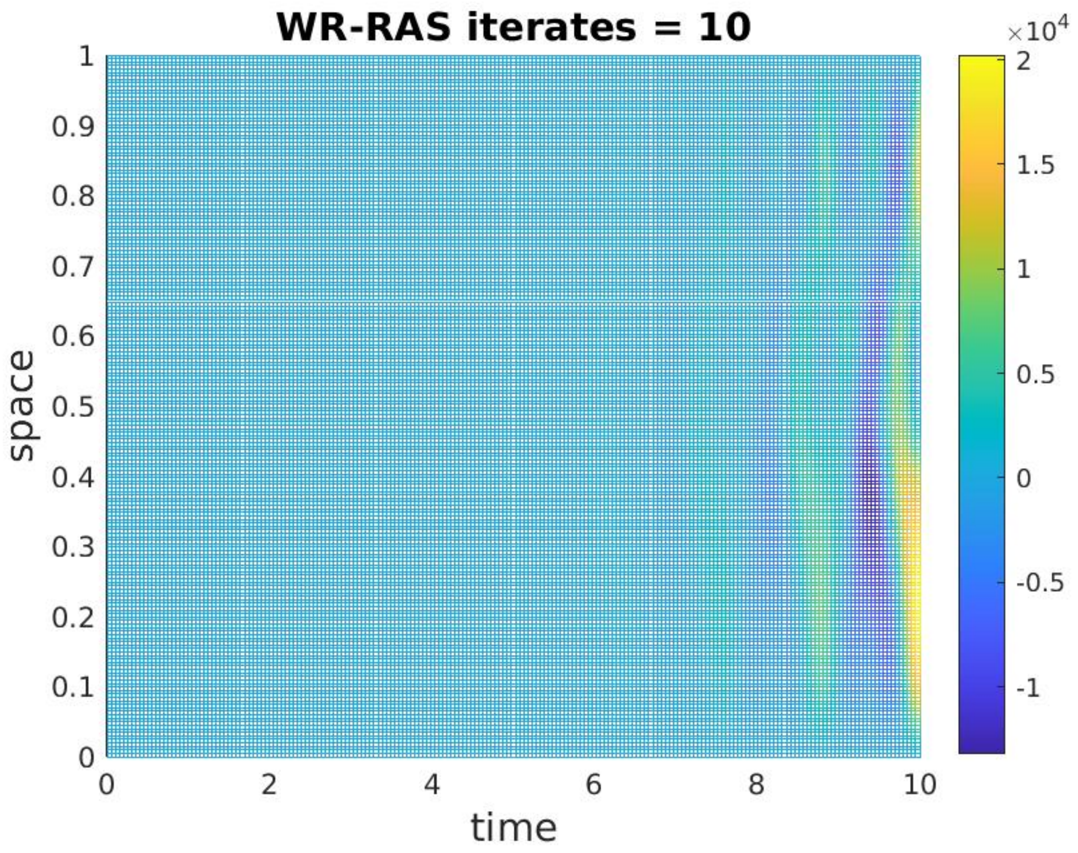
Numerical experiments ($T = 10$)



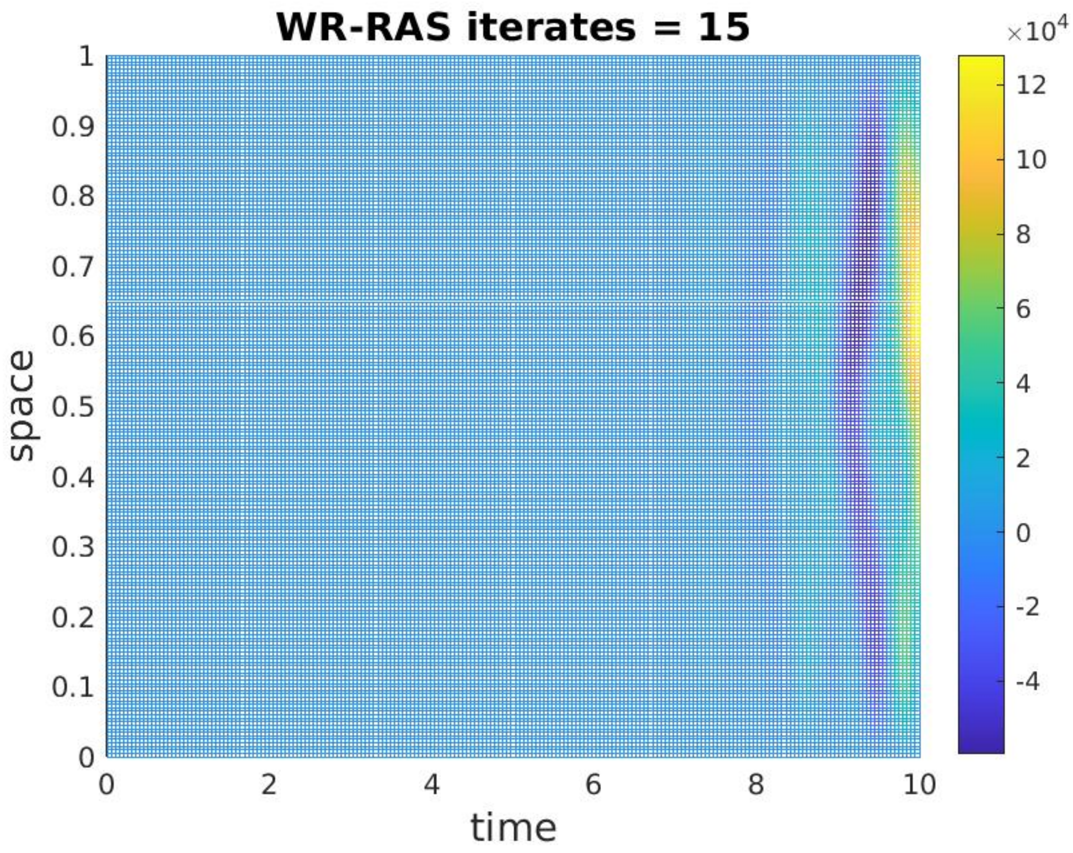
Numerical experiments ($T = 10$)



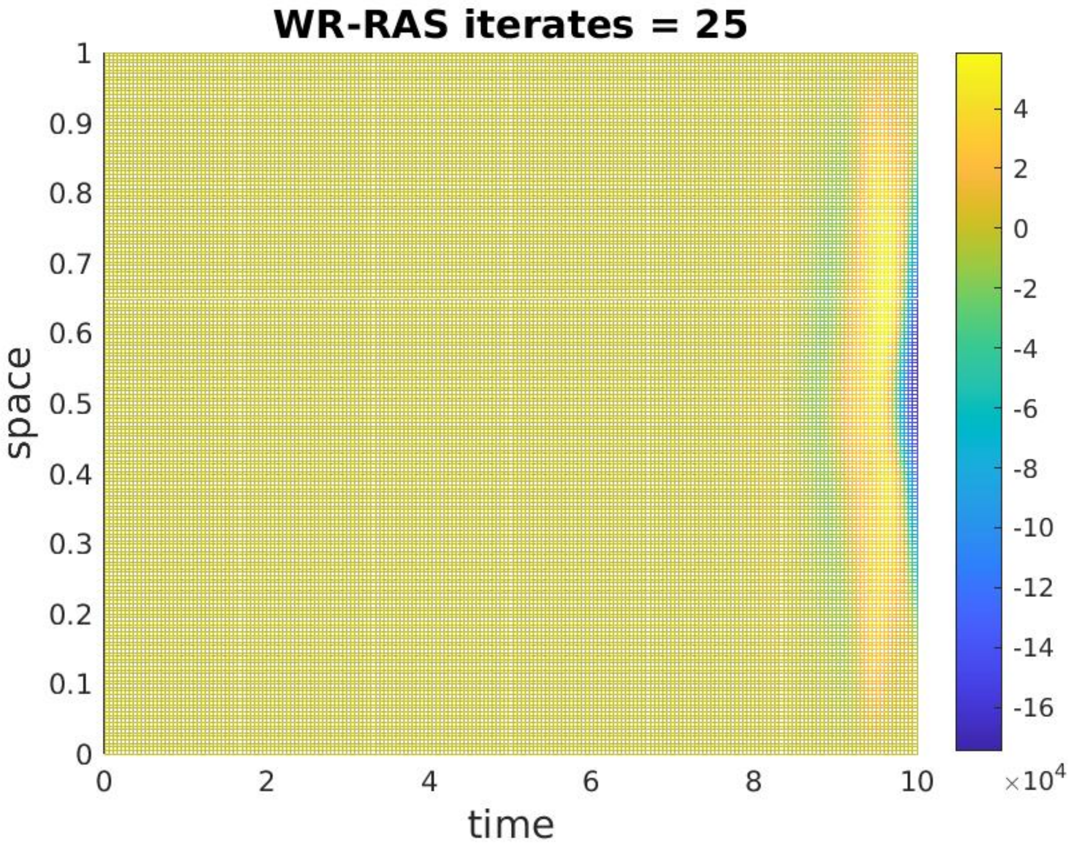
Numerical experiments ($T = 10$)



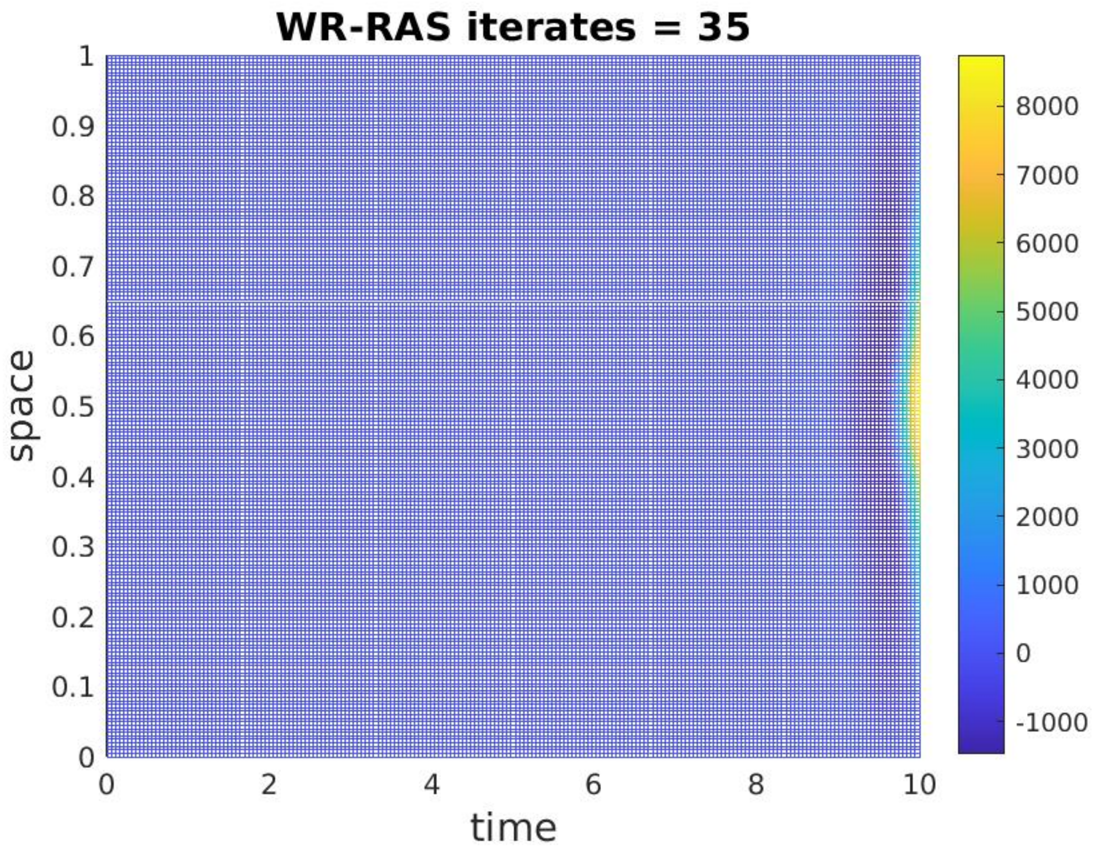
Numerical experiments ($T = 10$)



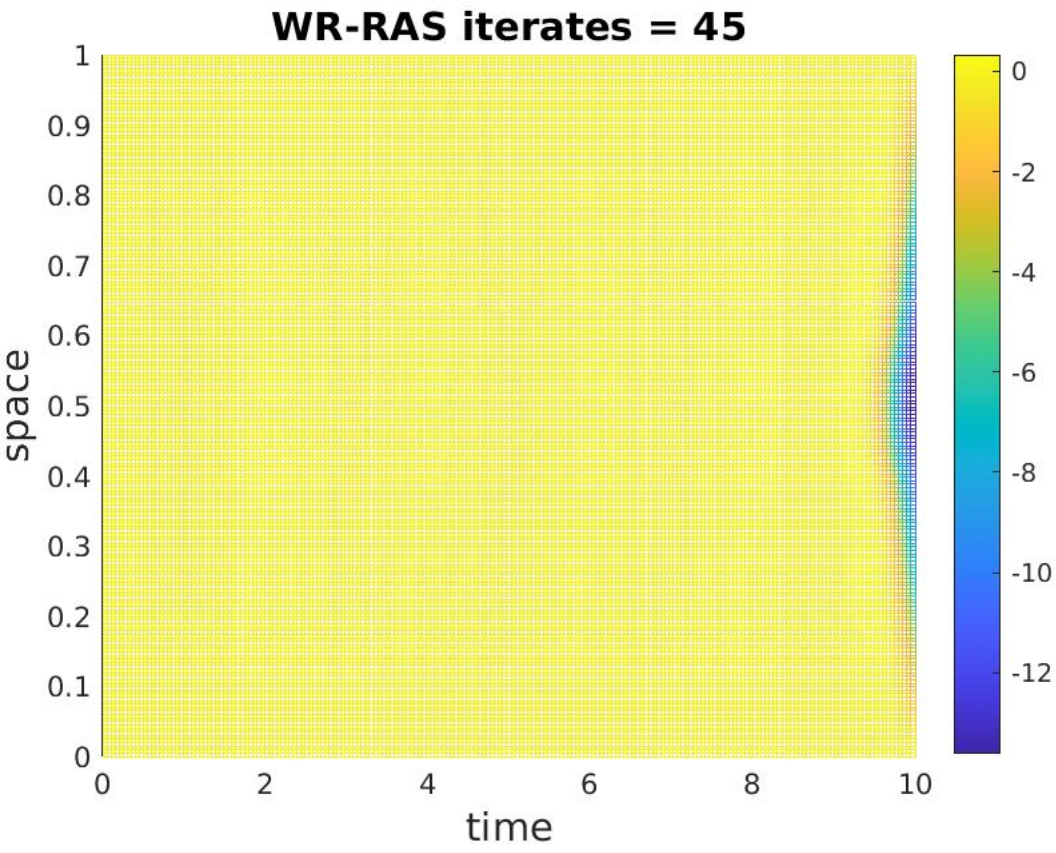
Numerical experiments ($T = 10$)



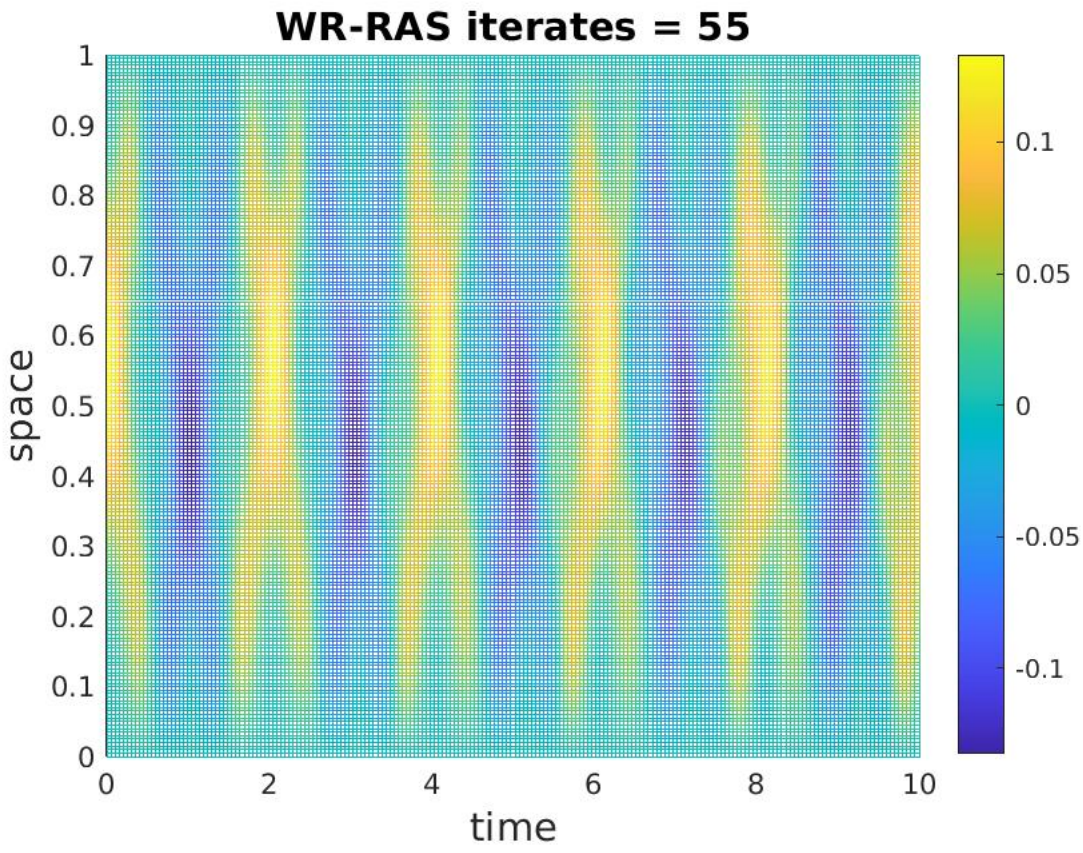
Numerical experiments ($T = 10$)



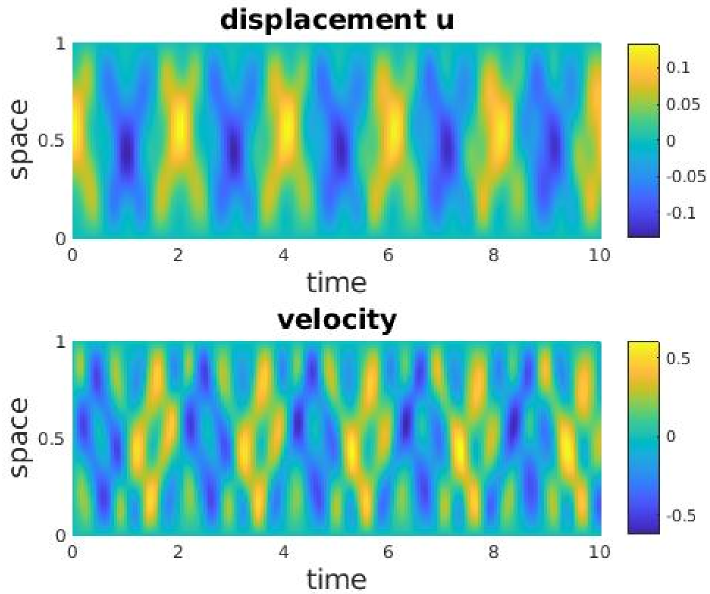
Numerical experiments ($T = 10$)

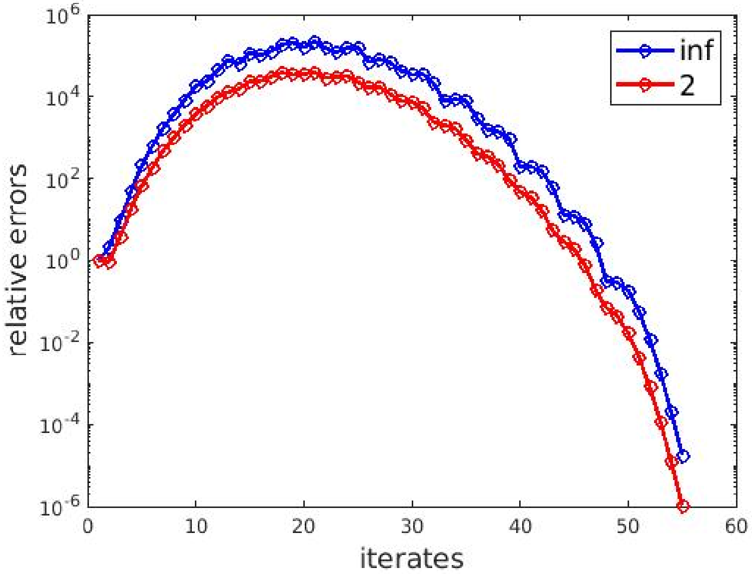


Numerical experiments ($T = 10$)

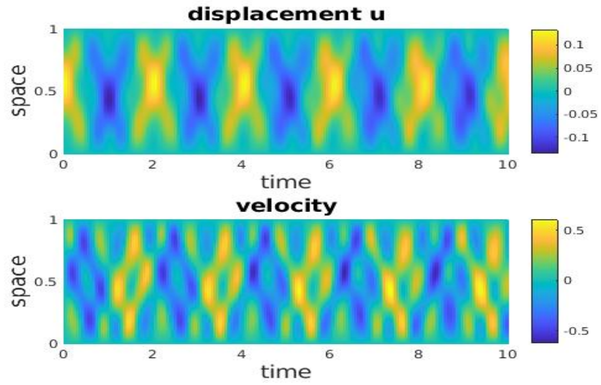


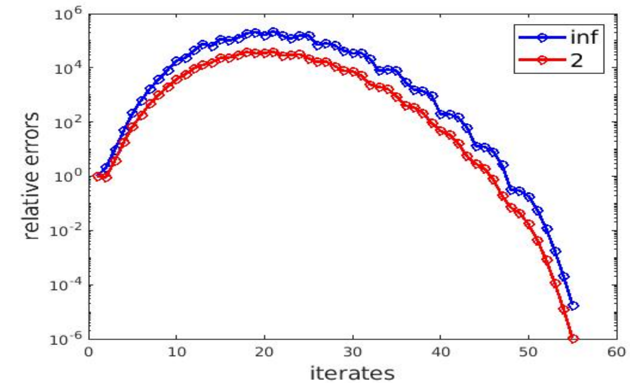
Numerical experiments ($T = 10$)



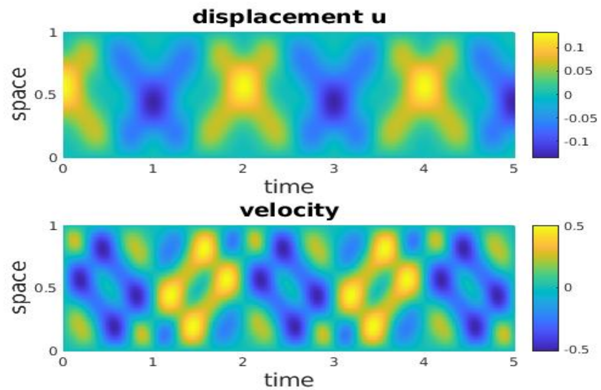


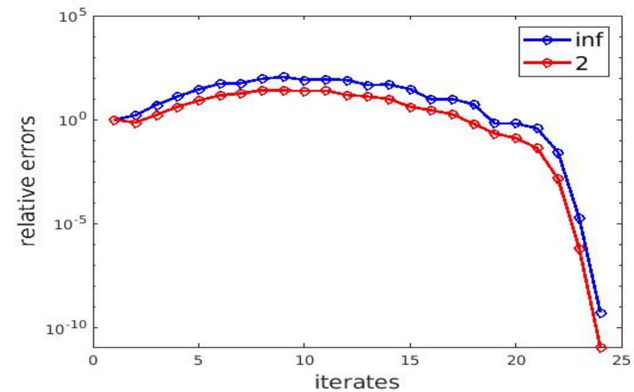
- $T = 10$


displacement u
space
time
velocity
space
time

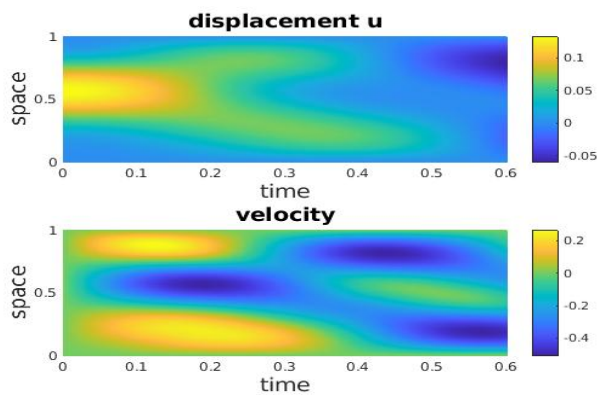

relative errors
iterates
inf
2

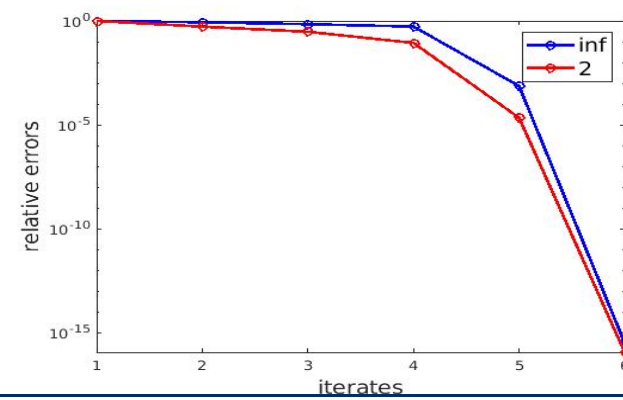
- $T = 5$


displacement u
space
time
velocity
space
time


relative errors
iterates
inf
2

- $T = 0.6$


displacement u
space
time
velocity
space
time


relative errors
iterates
inf
2

Numerical experiments

We observed that

- If T is sufficiently small then the error contracts.
- The error can propagate and grow in time.

Goal: understand/prove these behaviors to modify/improve the solution process.

Numerical experiments

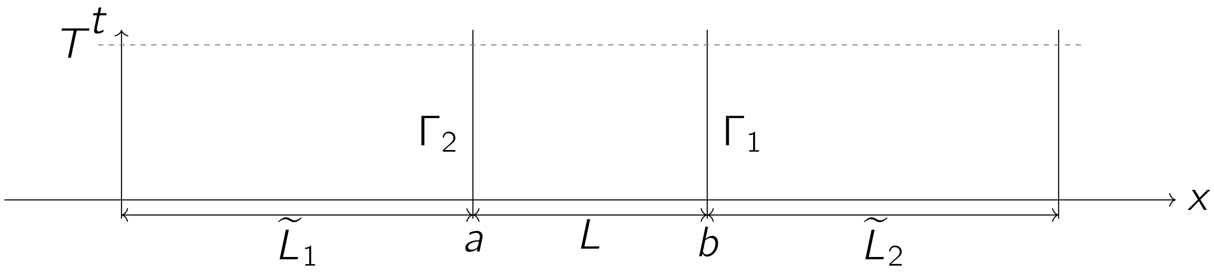
We observed that

- If T is sufficiently small then the error contracts.
- The error can propagate and grow in time.

Goal: understand/prove these behaviors to modify/improve the solution process.

(A Laplace/Fourier analysis does not reveal these behaviors.)

Convergence analysis



Let us define the iteration operator G:

$$G : g_0 \in C(\overline{\Gamma}_1) \mapsto g_1 \in C(\overline{\Gamma}_2) \mapsto g_2 \in C(\overline{\Gamma}_1),$$
$$G(g_0)(t) := g_2(t),$$

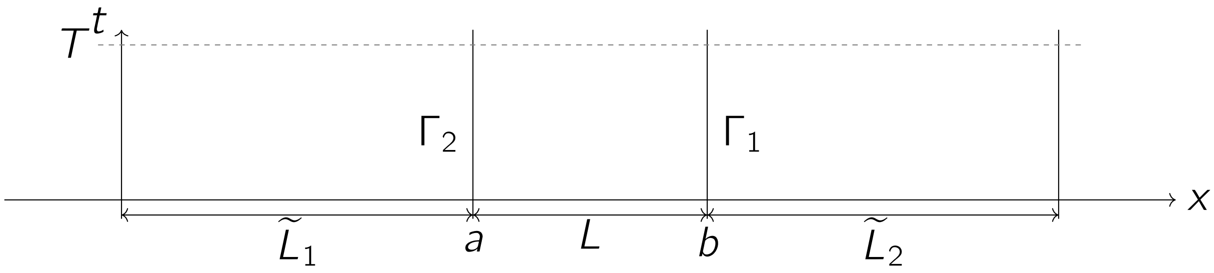
for any $g_0 \in C(\overline{\Gamma}_1)$, where $g_1 := e_1|_{\Gamma_2}$ and $g_2 := e_2|_{\Gamma_1}$ with e_1 and e_2 solutions to

$$\begin{aligned} \mathcal{L}e_1 &= 0 && \text{in } (0,b)\times(0,T),\\ e_1(x,0) = 0 \text{ and } \partial_t e_1(x,0) &= 0 && \text{for } x \in (0,b),\\ e_1(0,t) = 0 \text{ and } e_1(b,t) &= g_0(t) && \text{for } t \in [0,T], \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}e_2 &= 0 && \text{in } (a,1)\times(0,T),\\ e_2(x,0) = 0 \text{ and } \partial_t e_2(x,0) &= 0 && \text{for } x \in (a,1),\\ e_2(1,t) = 0 \text{ and } e_2(a,t) &= g_1(t) && \text{for } t \in [0,T]. \end{aligned}$$

Convergence analysis



Let us define the iteration operator G:

$$G : g_0 \in C(\overline{\Gamma}_1) \mapsto g_1 \in C(\overline{\Gamma}_2) \mapsto g_2 \in C(\overline{\Gamma}_1),$$
$$G(g_0)(t) := g_2(t),$$

for any $g_0 \in C(\overline{\Gamma}_1)$, where $g_1 := e_1|_{\Gamma_2}$ and $g_2 := e_2|_{\Gamma_1}$ with e_1 and e_2 solutions to

$$\begin{aligned} \mathcal{L}e_1 &= 0 && \text{in } (0,b)\times(0,T),\\ e_1(x,0) = 0 \text{ and } \partial_t e_1(x,0) &= 0 && \text{for } x \in (0,b),\\ e_1(0,t) = 0 \text{ and } e_1(b,t) &= g_0(t) && \text{for } t \in [0,T], \end{aligned}$$

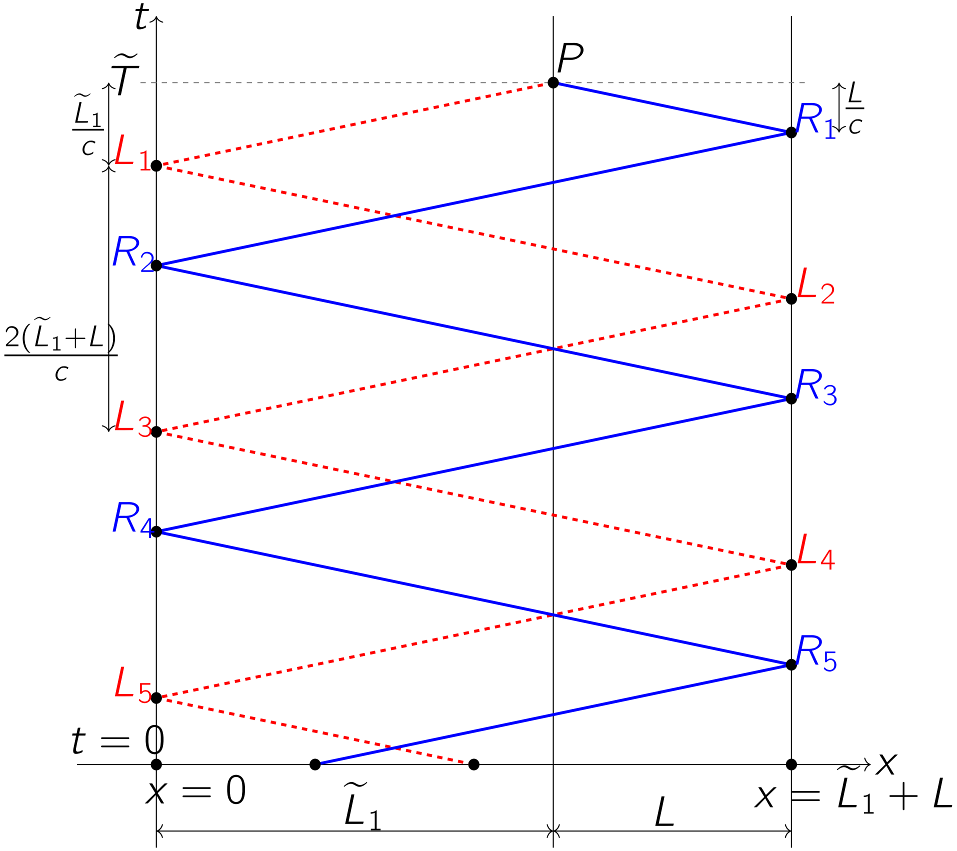
and

$$\begin{aligned} \mathcal{L}e_2 &= 0 && \text{in } (a,1)\times(0,T),\\ e_2(x,0) = 0 \text{ and } \partial_t e_2(x,0) &= 0 && \text{for } x \in (a,1),\\ e_2(1,t) = 0 \text{ and } e_2(a,t) &= g_1(t) && \text{for } t \in [0,T]. \end{aligned}$$

We study G using the operator norm $\|G\| := \sup_{\|g_0\|_\infty=1} \|G(g_0)\|_\infty$.

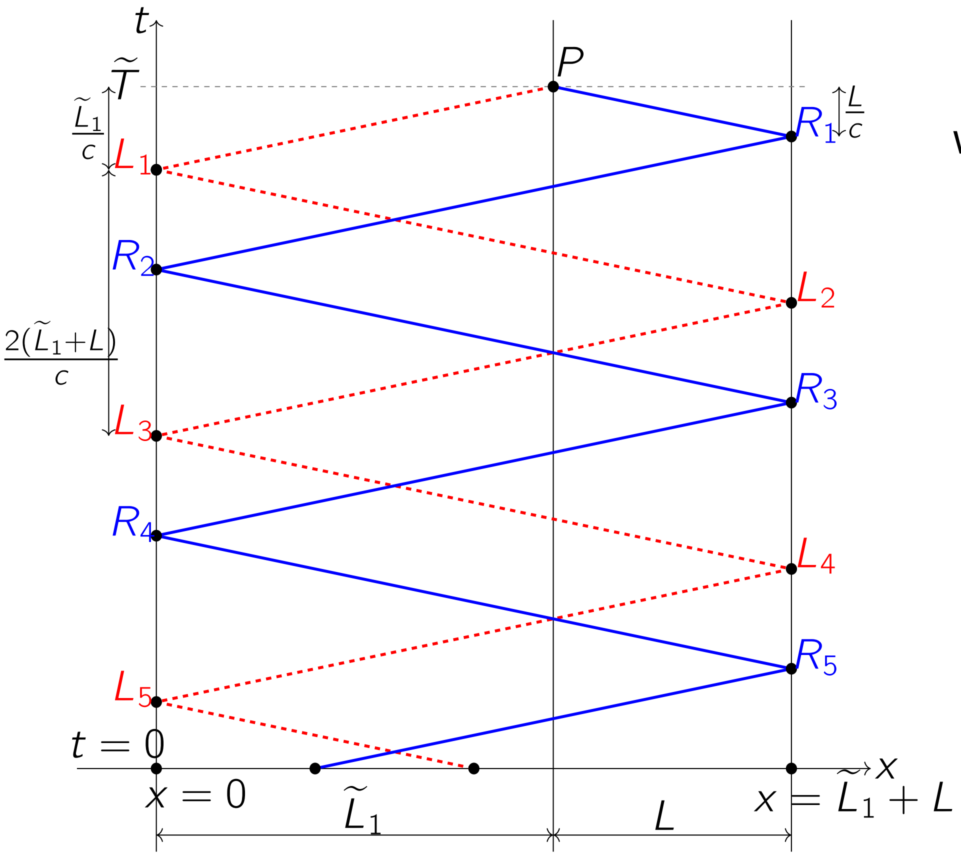
Convergence analysis

$$\mathcal{L}e_1 = 0$$
$$e_1(x,0) = 0 \text{ and } \partial_t e_1(x,0) = 0$$
$$e_1(0,t) = 0 \text{ and } e_1(b,t) = g_0(t)$$



Convergence analysis

$$\mathcal{L}e_1 = 0$$
$$e_1(x,0) = 0 \text{ and } \partial_t e_1(x,0) = 0$$
$$e_1(0,t) = 0 \text{ and } e_1(b,t) = g_0(t)$$



$$e_1(P) := \widehat{g}_R(\widetilde{T}) - \widehat{g}_L(\widetilde{T}),$$

where

$$\widehat{g}_R(\widetilde{T}) := \sum_{i=0}^{\infty} g_0\left(\widetilde{T} - \frac{L}{c} - \frac{2i|\Omega_1|}{c}\right) H_i^R(\widetilde{T}; L, |\Omega_1|),$$

$$\widehat{g}_L(\widetilde{T}) := \sum_{i=0}^{\infty} g_0\left(\widetilde{T} - \frac{\widetilde{L}_1}{c} - \frac{(2i+1)|\Omega_1|}{c}\right) H_i^L(\widetilde{T}; \widetilde{L}_1, |\Omega_1|),$$

and

$$H_i^R(t; X, Y) := \begin{cases} 1 & \text{if } t - \frac{X}{c} - \frac{2iY}{c} \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_i^L(t; X, Y) := \begin{cases} 1 & \text{if } t - \frac{X}{c} - \frac{(2i+1)Y}{c} \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Convergence analysis

$$
\begin{aligned}
G(g_0)(t) = \sum_{i=0}^{\infty} \Bigg\{ H_i^R(t;L,|\Omega_2|) \sum_{j=0}^{\infty} \Bigg[& g_0\Big(t - \frac{L+2i|\Omega_2|}{c} - \frac{L+2j|\Omega_1|}{c}\Big) H_j^R\Big(t - \frac{L+2i|\Omega_2|}{c}; L, |\Omega_1|\Big) \\
& - g_0\Big(t - \frac{L+2i|\Omega_2|}{c} - \frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c}\Big) H_j^L\Big(t - \frac{L+2i|\Omega_2|}{c}; \widetilde{L}_1, |\Omega_1|\Big) \Bigg] \\
& - H_i^L(t;\widetilde{L}_2,|\Omega_2|) \sum_{j=0}^{\infty} \Bigg[g_0\Big(t - \frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c} - \frac{L+2j|\Omega_1|}{c}\Big) H_j^R\Big(t - \frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c}; L, |\Omega_1|\Big) \\
& - g_0\Big(t - \frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c} - \frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c}\Big) H_j^L\Big(t - \frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c}; \widetilde{L}_1, |\Omega_1|\Big) \Bigg] \Bigg\}.
\end{aligned}
$$

Convergence analysis

$$
\begin{aligned}
G(g_0)(t)=\sum_{i=0}^{\infty}\Bigg\{ & H_i^R(t;L,|\Omega_2|)\sum_{j=0}^{\infty}\Bigg[g_0\Big(t-\frac{L+2i|\Omega_2|}{c}-\frac{L+2j|\Omega_1|}{c}\Big)H_j^R\Big(t-\frac{L+2i|\Omega_2|}{c};L,|\Omega_1|\Big)\\
&-g_0\Big(t-\frac{L+2i|\Omega_2|}{c}-\frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c}\Big)H_j^L\Big(t-\frac{L+2i|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\Big)\Bigg]\\
&-H_i^L(t;\widetilde{L}_2,|\Omega_2|)\sum_{j=0}^{\infty}\Bigg[g_0\Big(t-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c}-\frac{L+2j|\Omega_1|}{c}\Big)H_j^R\Big(t-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};L,|\Omega_1|\Big)\\
&-g_0\Big(t-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c}-\frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c}\Big)H_j^L\Big(t-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\Big)\Bigg]\Bigg\}.
\end{aligned}
$$

In order to maximize $G(g_0)(t)$, we look at $t=T$ and pick any $\widetilde{g}_0$ such that

$$
\widetilde{g}_0(t)=\begin{cases} 1 & \text{for } t=T-\frac{L+2i|\Omega_2|}{c}-\frac{L+2j|\Omega_1|}{c} \text{ or } t=T-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c}-\frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c},\\ -1 & \text{for } t=T-\frac{L+2i|\Omega_2|}{c}-\frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c} \text{ or } t=T-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c}-\frac{L+2j|\Omega_1|}{c}.\end{cases}
$$

Notice that such a function exists since red points and blue points are distinct.

Convergence analysis

$$
\begin{aligned}
G(g_0)(t)=\sum_{i=0}^{\infty}\Bigg\{ & H_i^R(t;L,|\Omega_2|)\sum_{j=0}^{\infty}\Bigg[g_0\left(t-\frac{L+2i|\Omega_2|}{c}-\frac{L+2j|\Omega_1|}{c}\right)H_j^R\left(t-\frac{L+2i|\Omega_2|}{c};L,|\Omega_1|\right)\\
& -g_0\left(t-\frac{L+2i|\Omega_2|}{c}-\frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c}\right)H_j^L\left(t-\frac{L+2i|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\right)\Bigg]\\
& -H_i^L(t;\widetilde{L}_2,|\Omega_2|)\sum_{j=0}^{\infty}\Bigg[g_0\left(t-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c}-\frac{L+2j|\Omega_1|}{c}\right)H_j^R\left(t-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};L,|\Omega_1|\right)\\
& -g_0\left(t-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c}-\frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c}\right)H_j^L\left(t-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\right)\Bigg]\Bigg\}.
\end{aligned}
$$

In order to maximize $G(g_0)(t)$, we look at $t = T$ and pick any $\widetilde{g}_0$ such that

$$
\widetilde{g}_0(t) = \begin{cases} 1 & \text{for } t = T - \frac{L+2i|\Omega_2|}{c} - \frac{L+2j|\Omega_1|}{c} \text{ or } t = T - \frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c} - \frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c},\\ -1 & \text{for } t = T - \frac{L+2i|\Omega_2|}{c} - \frac{\widetilde{L}_1+(2j+1)|\Omega_1|}{c} \text{ or } t = T - \frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c} - \frac{L+2j|\Omega_1|}{c}. \end{cases}
$$

Notice that such a function exists since red points and blue points are distinct.

This choice implies that

$$
\|G\| = \max_{\|g_0\|_\infty=1}\max_{t\in[0,T]}|G(g_0)(t)| = |G(\widetilde{g}_0)(T)|.
$$

Convergence analysis

Hence, we get

$$\begin{aligned}\|G\|=\sum_{i=0}^{\infty}\Bigg\{ & H_i^R(T;L,|\Omega_2|)\sum_{j=0}^{\infty}\left[H_j^R\left(T-\frac{L+2i|\Omega_2|}{c};L,|\Omega_1|\right)+H_j^L\left(T-\frac{L+2i|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\right)\right] \\ & +H_i^L(T;\widetilde{L}_2,|\Omega_2|)\sum_{j=0}^{\infty}\left[H_j^R\left(T-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};L,|\Omega_1|\right)+H_j^L\left(T-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\right)\right]\Bigg\}.\end{aligned}$$

Convergence analysis

Hence, we get

$$\|G\|=\sum_{i=0}^{\infty}\Bigg\{H_i^R(T;L,|\Omega_2|)\sum_{j=0}^{\infty}\Bigg[H_j^R\Big(T-\frac{L+2i|\Omega_2|}{c};L,|\Omega_1|\Big)+H_j^L\Big(T-\frac{L+2i|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\Big)\Bigg]$$
$$+H_i^L(T;\widetilde{L}_2,|\Omega_2|)\sum_{j=0}^{\infty}\Bigg[H_j^R\Big(T-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};L,|\Omega_1|\Big)+H_j^L\Big(T-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\Big)\Bigg]\Bigg\}.$$

Using the definitions of H_j^R and H_j^L, we can estimate that

$$\|G\| \approx \frac{c^2T^2}{(|\Omega_1|+|\Omega_2|)^2}.$$

Moreover,

$$\|G\|<1 \ \text{ for } T<\frac{2\min(\widetilde{L}_1,\widetilde{L}_2)+2L}{c}.$$

It is not best to use the SWR for large T.

Convergence analysis

Hence, we get

$$\|G\|=\sum_{i=0}^{\infty}\Bigg\{H_i^R(T;L,|\Omega_2|)\sum_{j=0}^{\infty}\Bigg[H_j^R\Big(T-\frac{L+2i|\Omega_2|}{c};L,|\Omega_1|\Big)+H_j^L\Big(T-\frac{L+2i|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\Big)\Bigg]$$

$$+H_i^L(T;\widetilde{L}_2,|\Omega_2|)\sum_{j=0}^{\infty}\Bigg[H_j^R\Big(T-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};L,|\Omega_1|\Big)+H_j^L\Big(T-\frac{\widetilde{L}_2+(2i+1)|\Omega_2|}{c};\widetilde{L}_1,|\Omega_1|\Big)\Bigg]\Bigg\}.$$

Using the definitions of H_j^R and H_j^L, we can estimate that

$$\|G\| \approx \frac{c^2T^2}{(|\Omega_1|+|\Omega_2|)^2}.$$

Moreover,

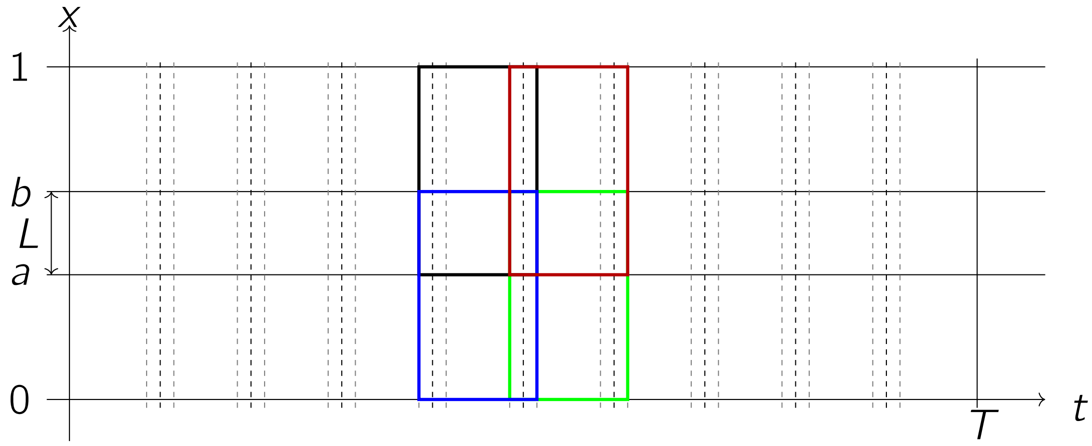
$$\|G\| < 1 \text{ for } T < \frac{2\min(\widetilde{L}_1,\widetilde{L}_2)+2L}{c}.$$

It is not best to use the SWR for large T.

What happens if we decompose also $[0,T]$ into “small enough” subdomains?

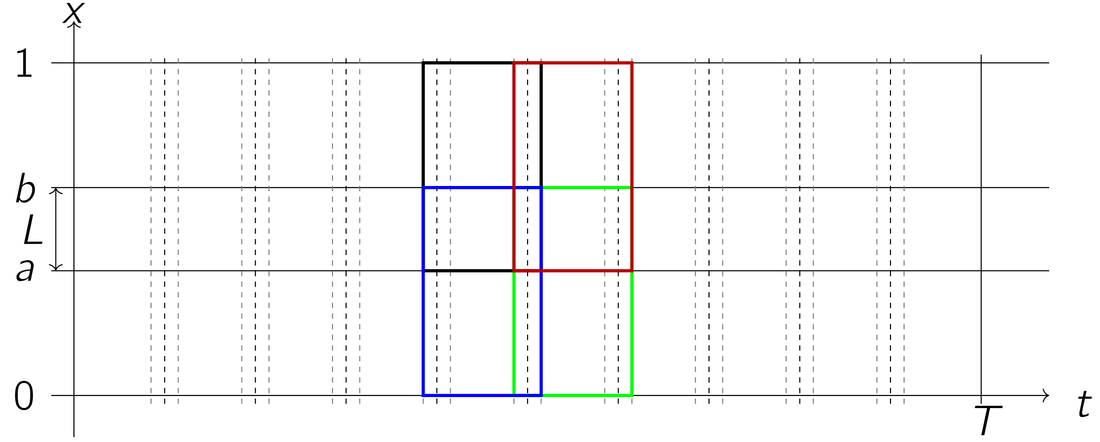
Space-time decomposition and XT-RAS method

We keep the 2-subdomain space decomposition and split $[0, T]$ in N_T overlapping subdomains:



Space-time decomposition and XT-RAS method

We keep the 2-subdomain space decomposition and split $[0, T]$ in N_T overlapping subdomains:

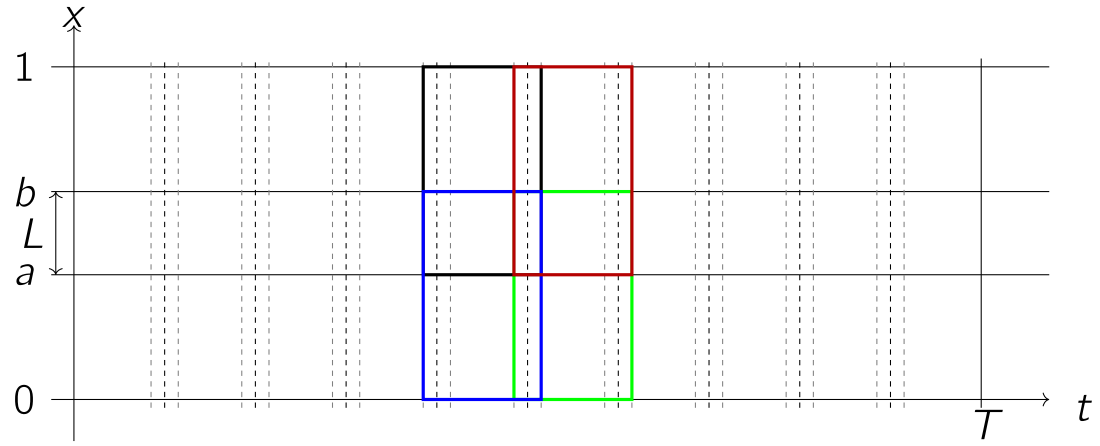


Let us discretize our problem (using, e.g., FD or FE in space and Newmark or Leapfrog in time) and write the discrete problem in the compact form

$$A\mathbf{u} = \mathbf{f}, \quad A \in \mathbb{R}^{N_h \times N_h}, \mathbf{u}, \mathbf{f} \in \mathbb{R}^{N_h}.$$

Space-time decomposition and XT-RAS method

We keep the 2-subdomain space decomposition and split $[0, T]$ in N_T overlapping subdomains:



Let us discretize our problem (using, e.g., FD or FE in space and Newmark or Leapfrog in time) and write the discrete problem in the compact form

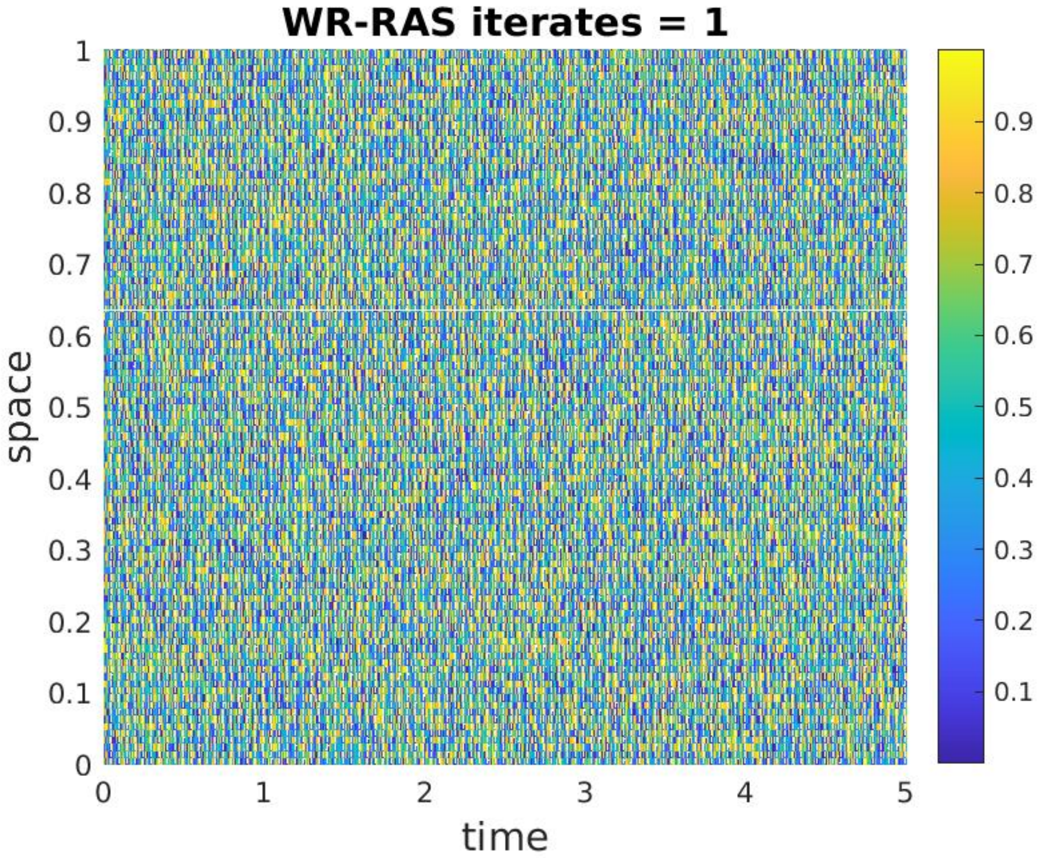
$$A\mathbf{u} = \mathbf{f}, \quad A \in \mathbb{R}^{N_h \times N_h}, \mathbf{u}, \mathbf{f} \in \mathbb{R}^{N_h}.$$

The SWR method can be easily generalized in a classical RAS form (now XT)

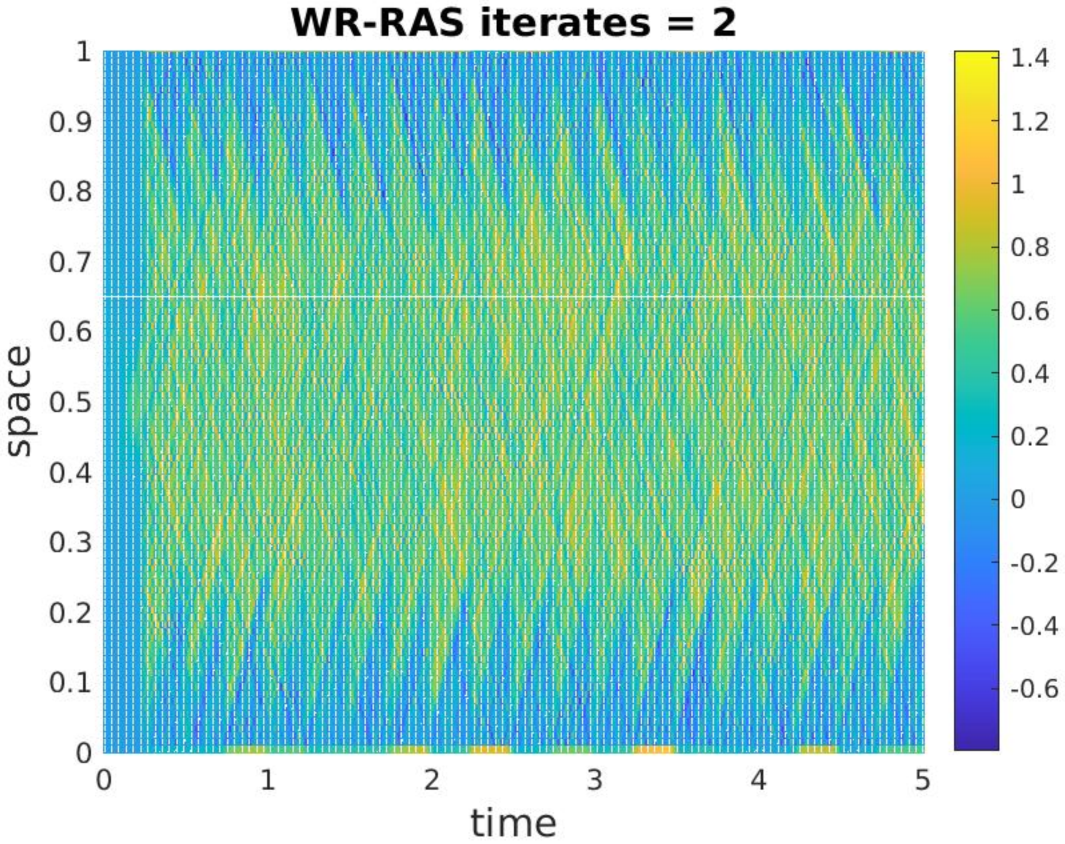
$$\mathbf{u}^{n+1} = \mathbf{u}^n + \sum_{k=1}^{N_X N_T} \widetilde{R}_k^\top A_k^{-1} R_k(\mathbf{f} - A\mathbf{u}^n),$$

where R_k and $\widetilde{R}_k$ are XT restriction matrices (the usual RAS matrices including a partition of unity) and A_k are XT matrices corresponding to the local subproblems.

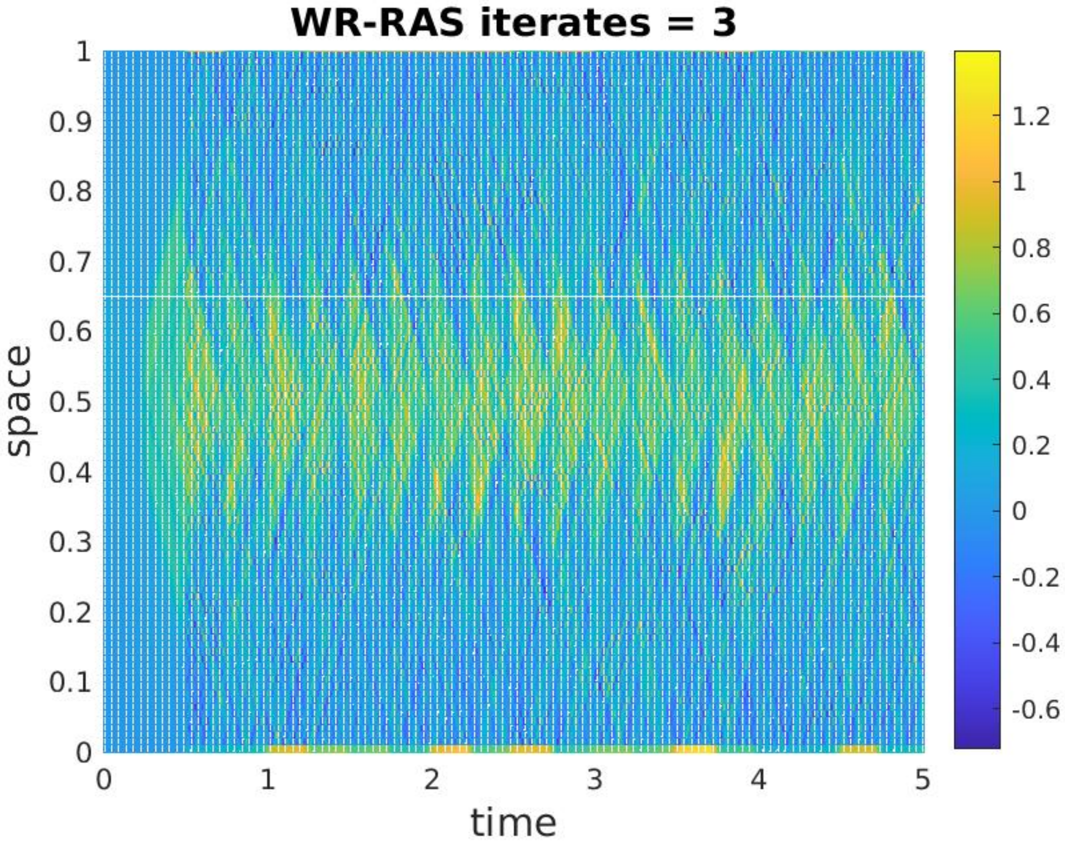
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



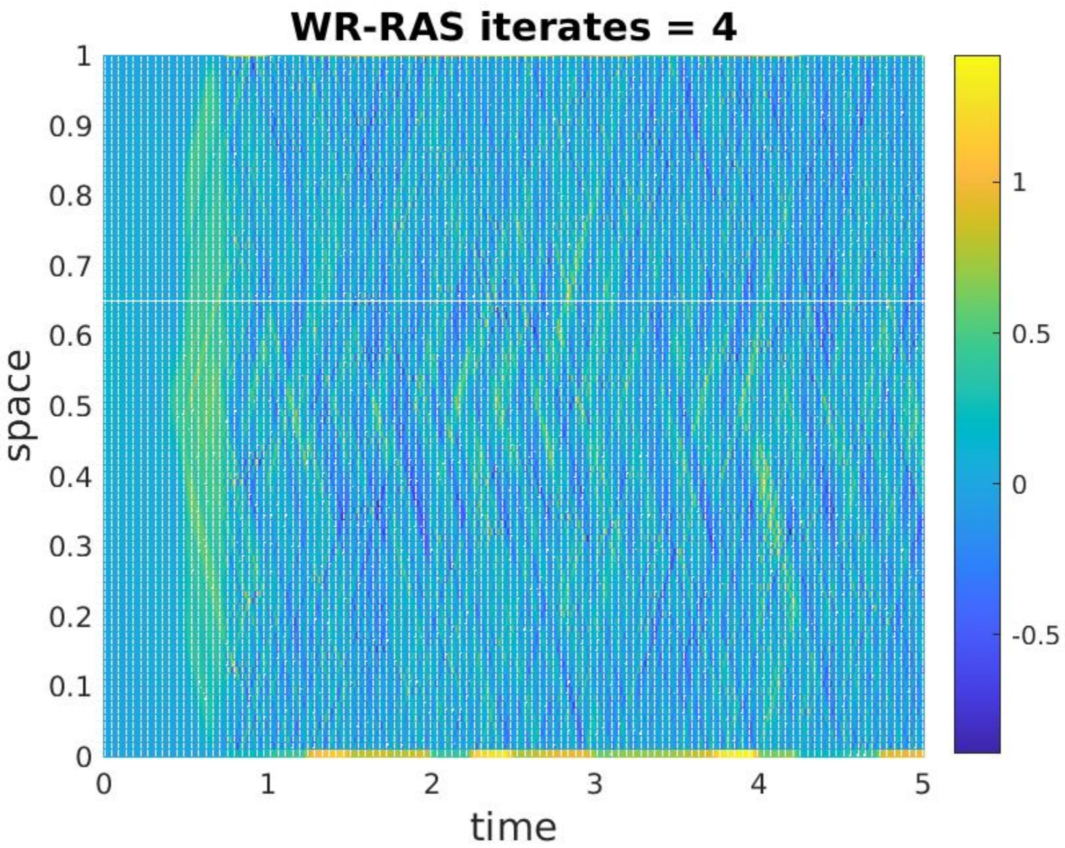
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



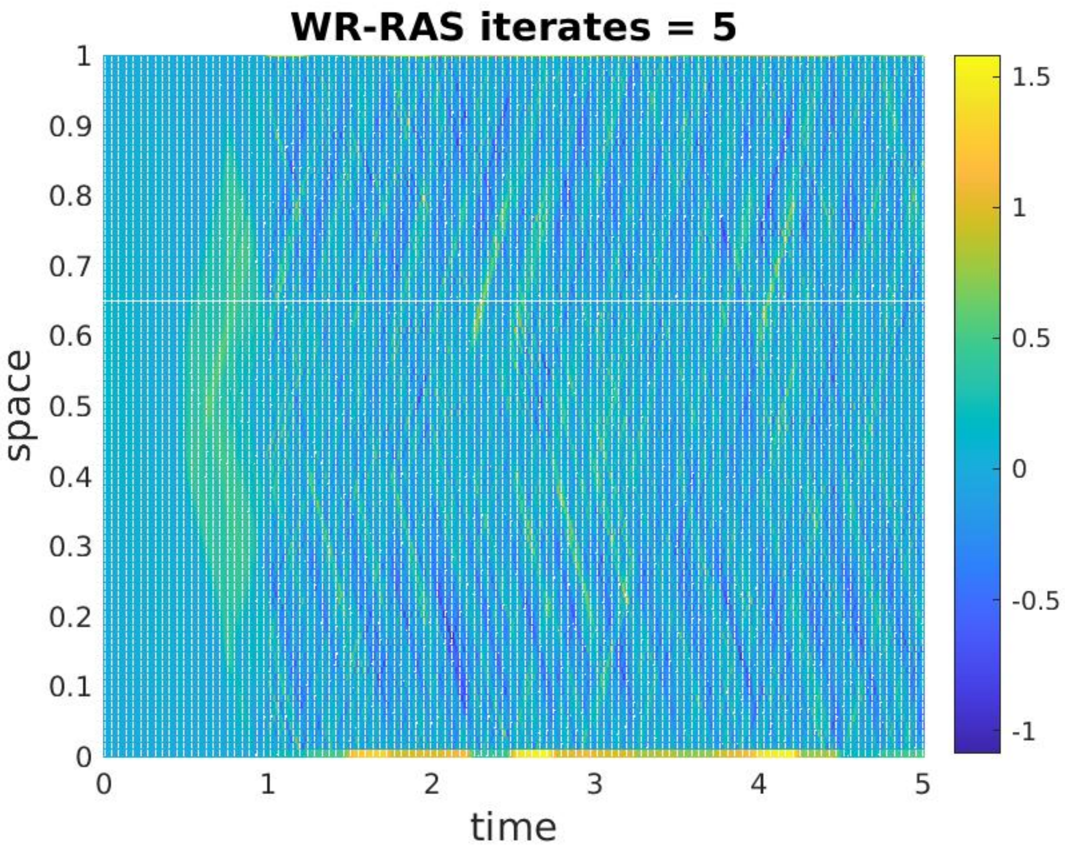
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



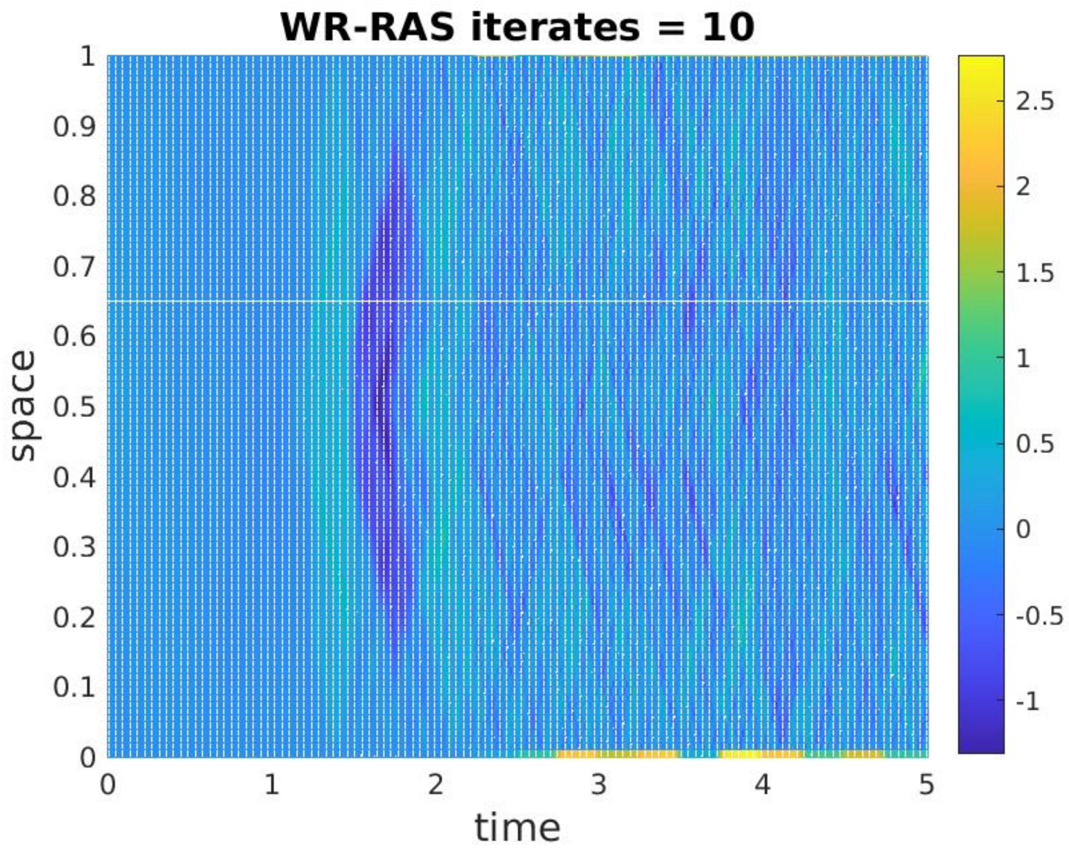
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



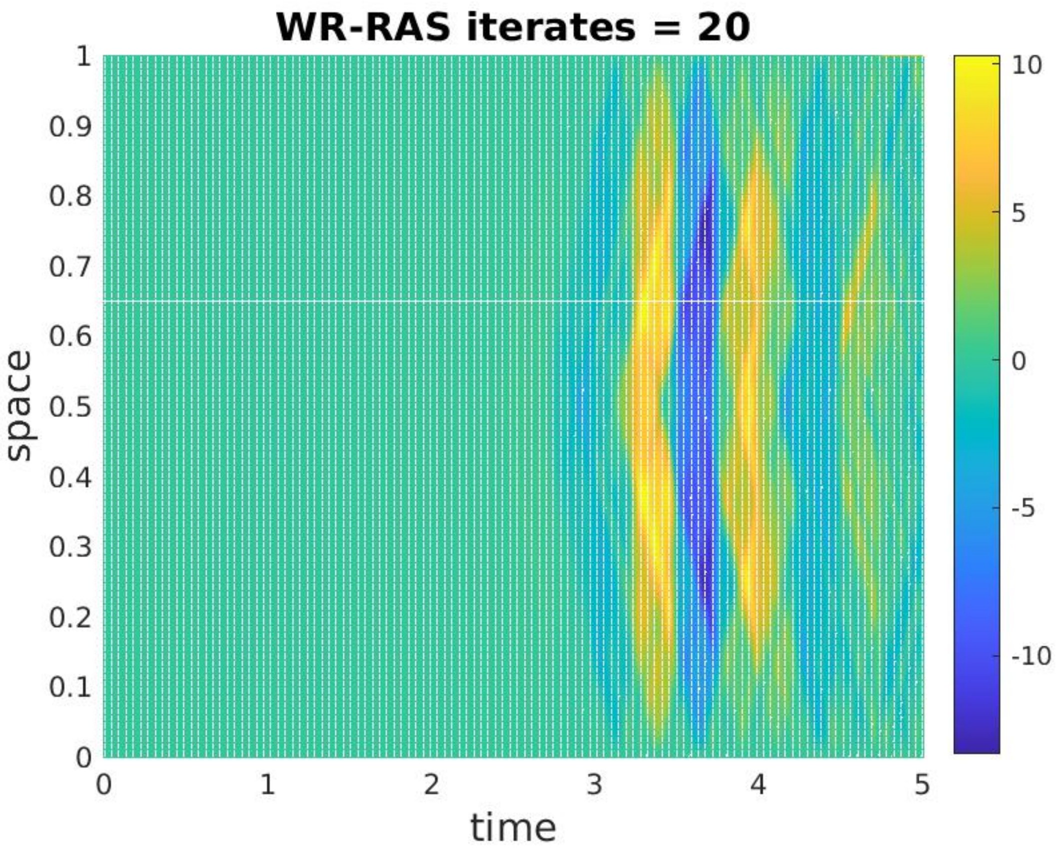
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



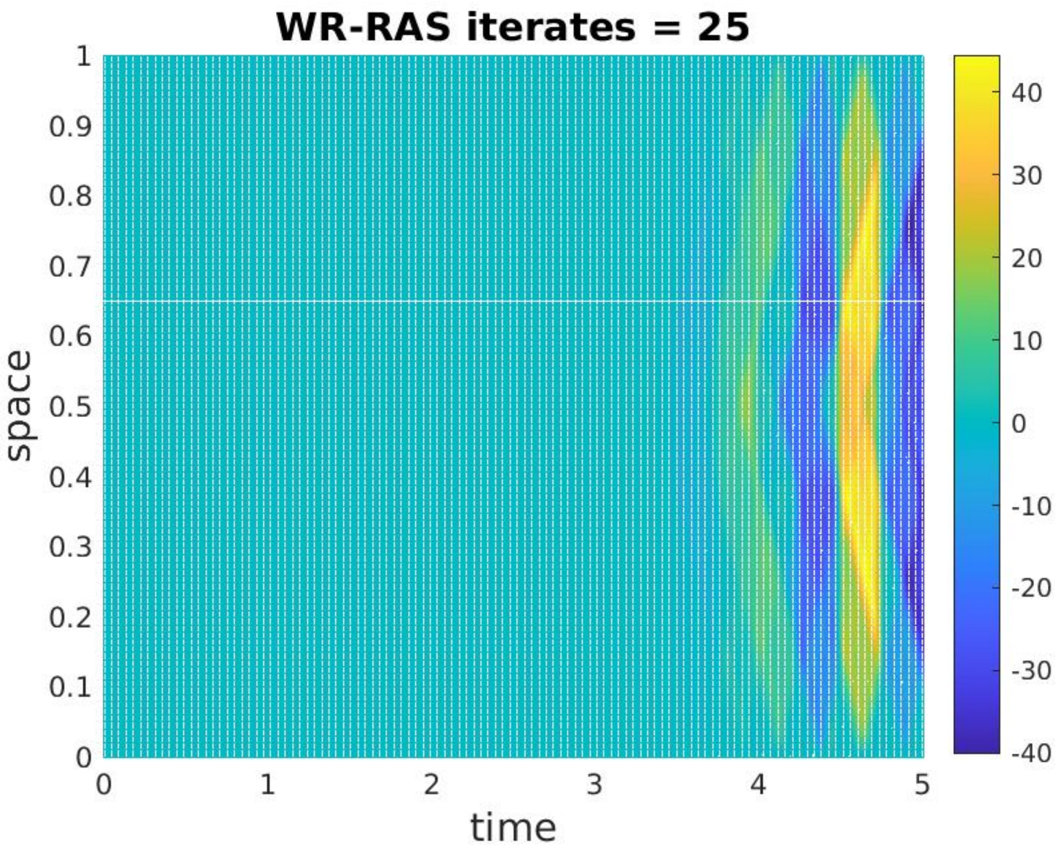
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



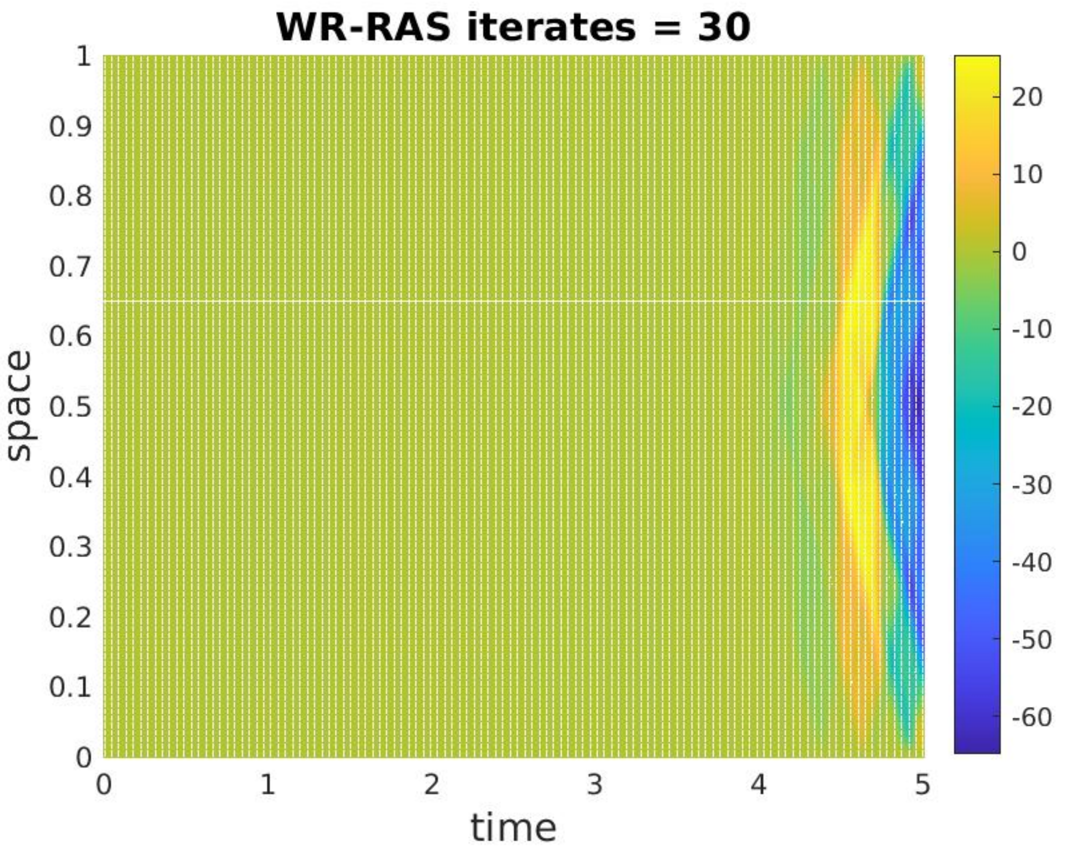
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



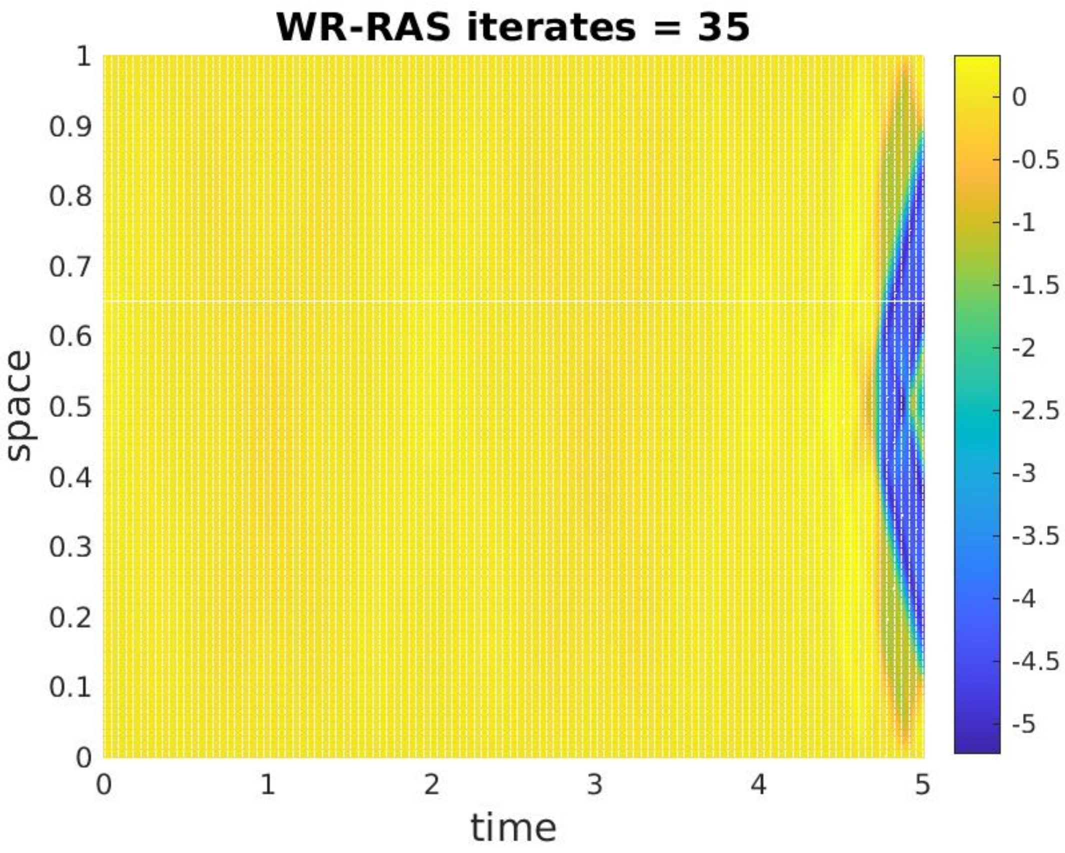
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



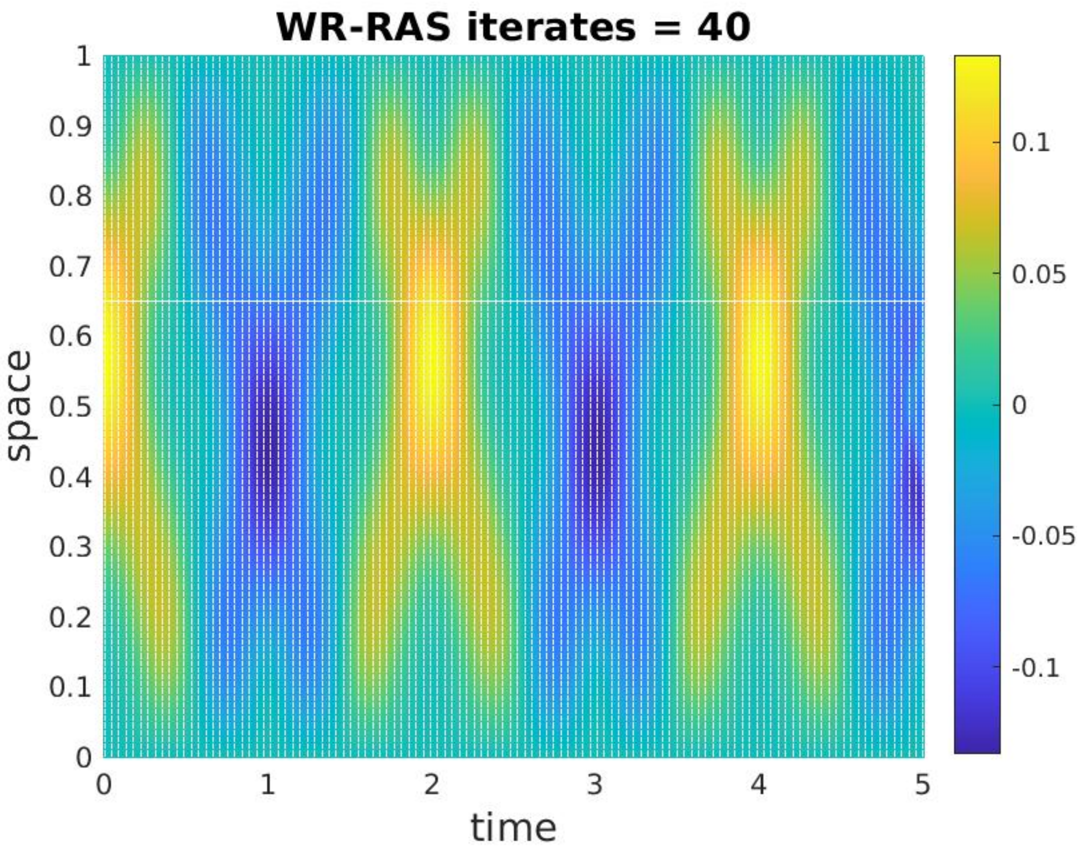
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



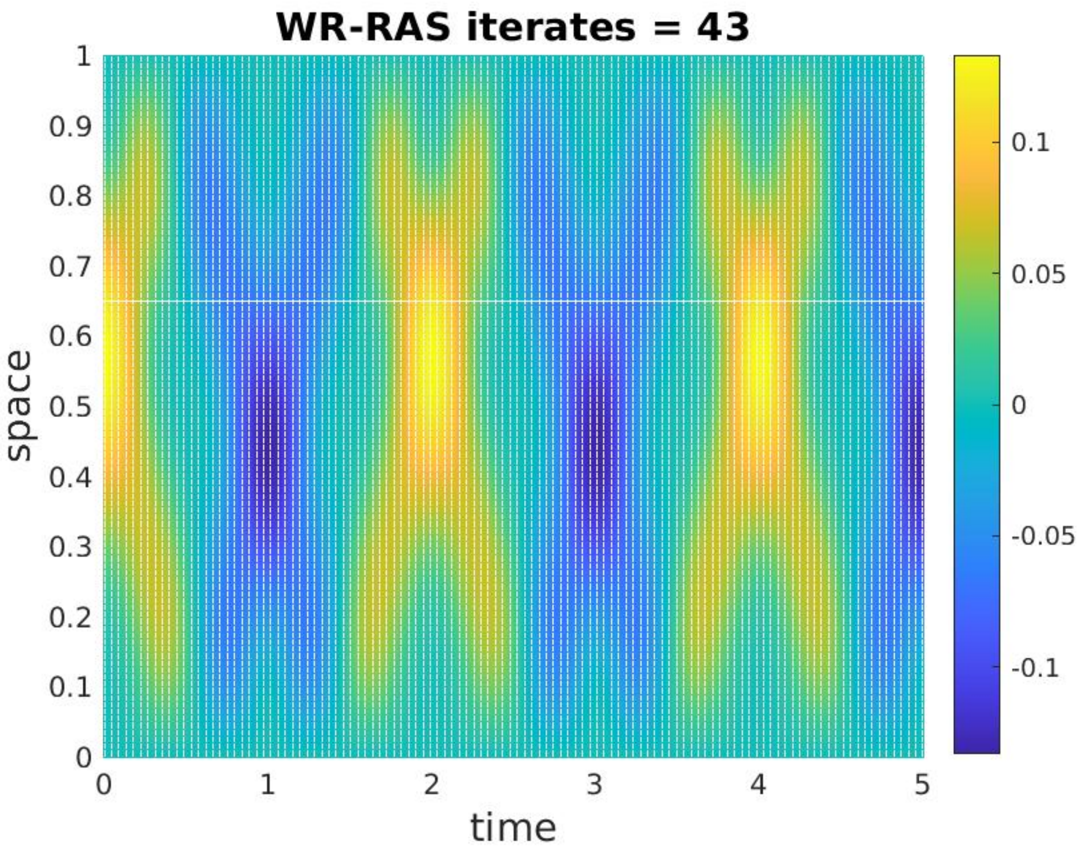
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)

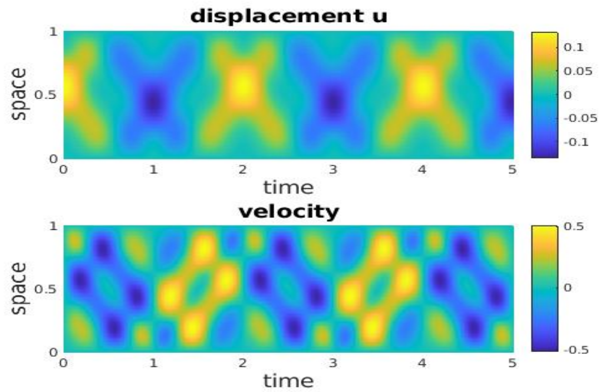


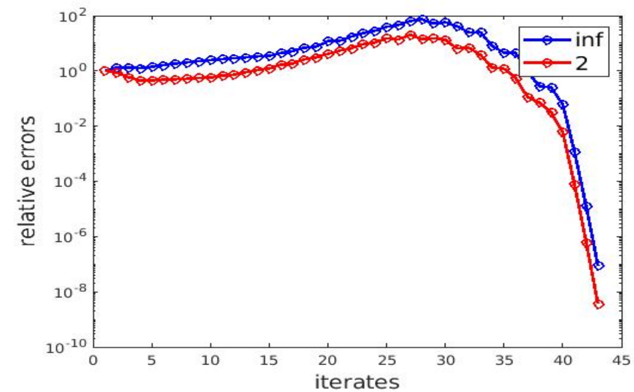
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)



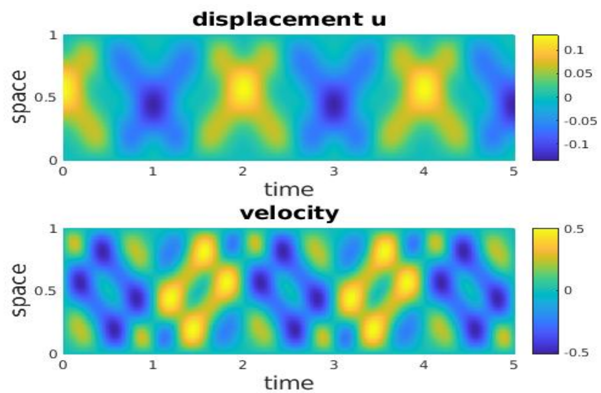
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)

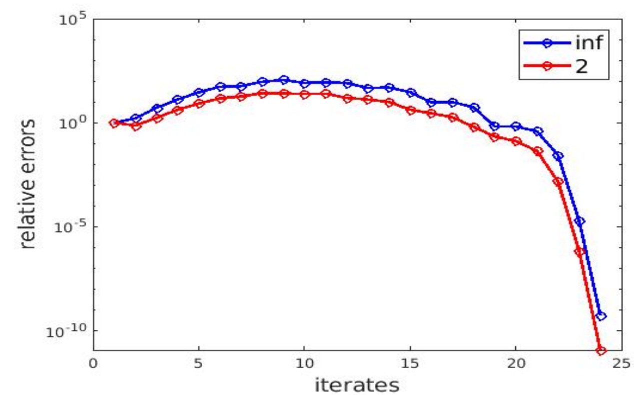
$N_T = 20$





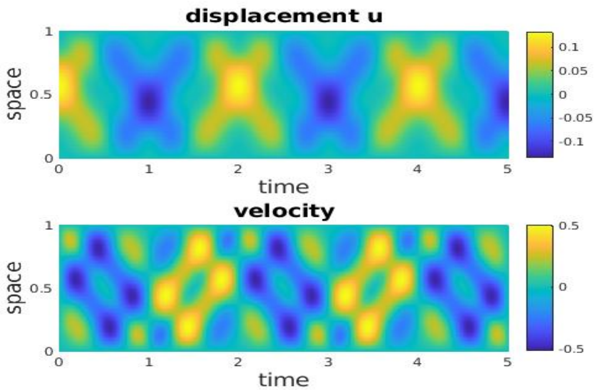
$N_T = 1$

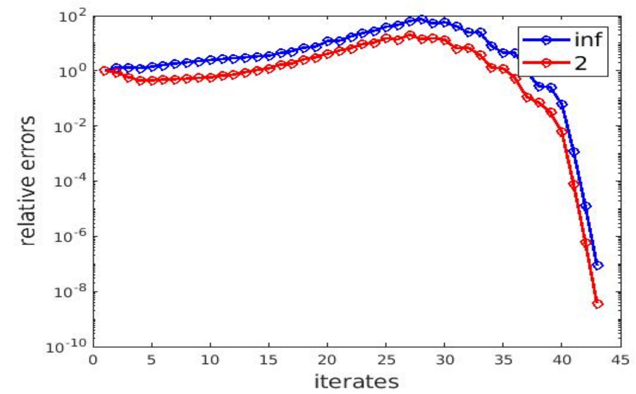




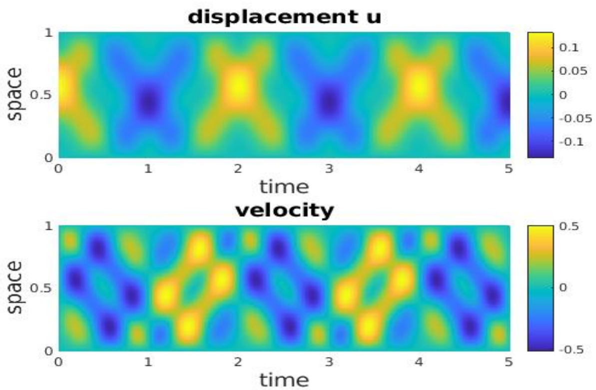
XT-RAS: Numerical experiments ($T = 5$, $N_X = 2$, $N_T = 20$)

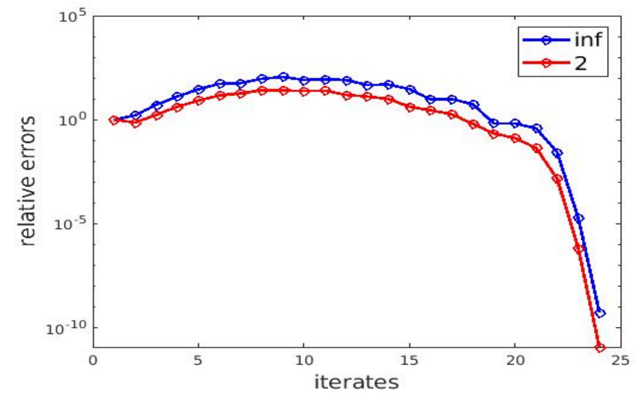
$N_T = 20$





$N_T = 1$



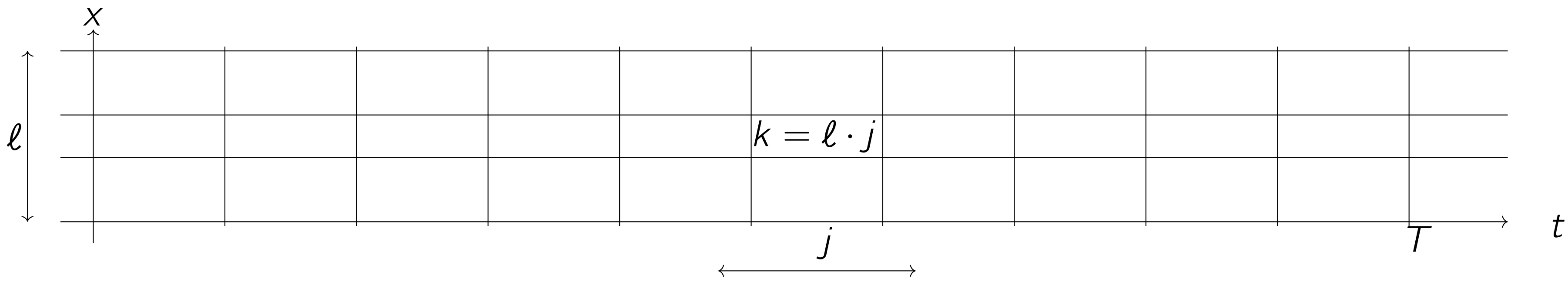


Remarks:

- There is no much gain in using a time decomposition.
- The "good information" need to propagate through the time subdomains to reach T.
- The error propagates and grows through the subdomains. $\rightarrow$ Useless subdomain solves.

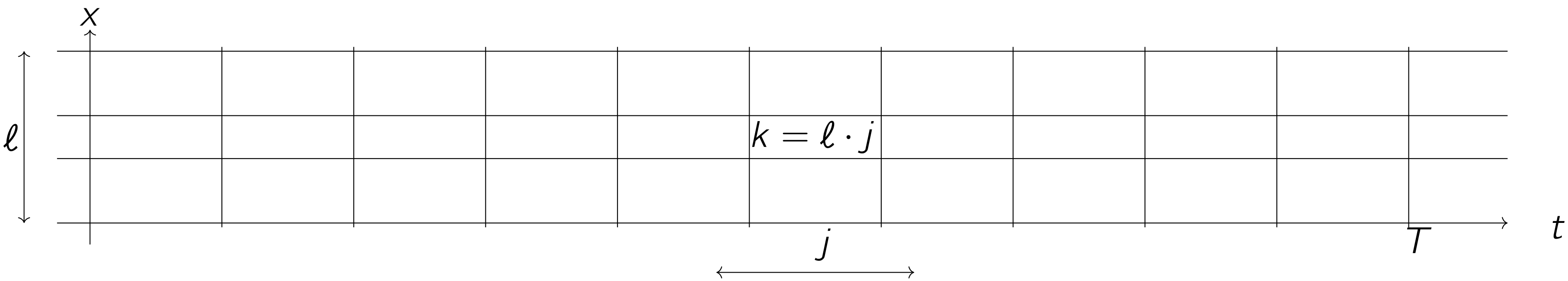
Pipeline XT-RAS method

Let us enumerate the subdomains as follows



Pipeline XT-RAS method

Let us enumerate the subdomains as follows



We modify the XT-RAS formula:

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \sum_{k \in \mathcal{K}_n} \widetilde{R}_k^\top A_k^{-1} R_k (\mathbf{f} - A\mathbf{u}^n),$$

with the set $\mathcal{K}_n$ defined as

$$\mathcal{K}_n := \left\{ k = \ell \cdot j \,:\, \ell \in \mathcal{K}_X \text{ and } j \in \mathbb{N}_+ \text{ with } j \leq \min\left(N_T, \left\lfloor \frac{n}{N_{WR}} \right\rfloor + 1 \right) \text{ and } \|R_j(\mathbf{f} - A\mathbf{u}^n)\|_\infty > \epsilon \right\},$$

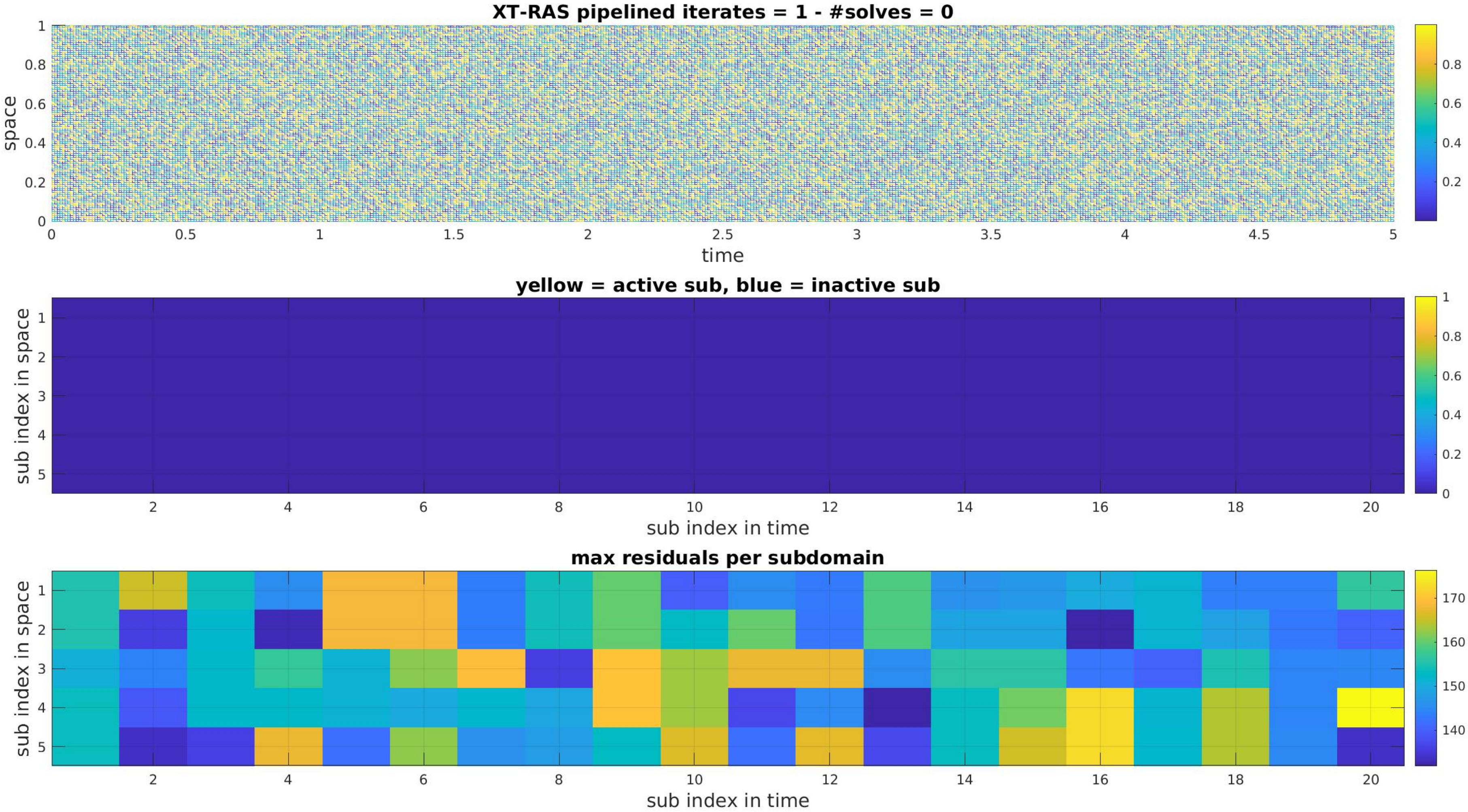
where $\mathcal{K}_X := \{1, 2, \ldots, N_X\}$ and $N_{WR} \in \mathbb{N}_+$ and $\epsilon > 0$. (In what follows $N_{WR} = 4$ and $\epsilon = 10^{-6}$.)

Notice that:

- This set defines a propagation in time of XT-RAS subsolves. The front of this propagation is governed by the red condition while the tail by the blue condition.
- This allows us to avoid useless XT-RAS subsolves by controlling the error propagation/growth and by avoiding the solution of “already solved” subproblems.

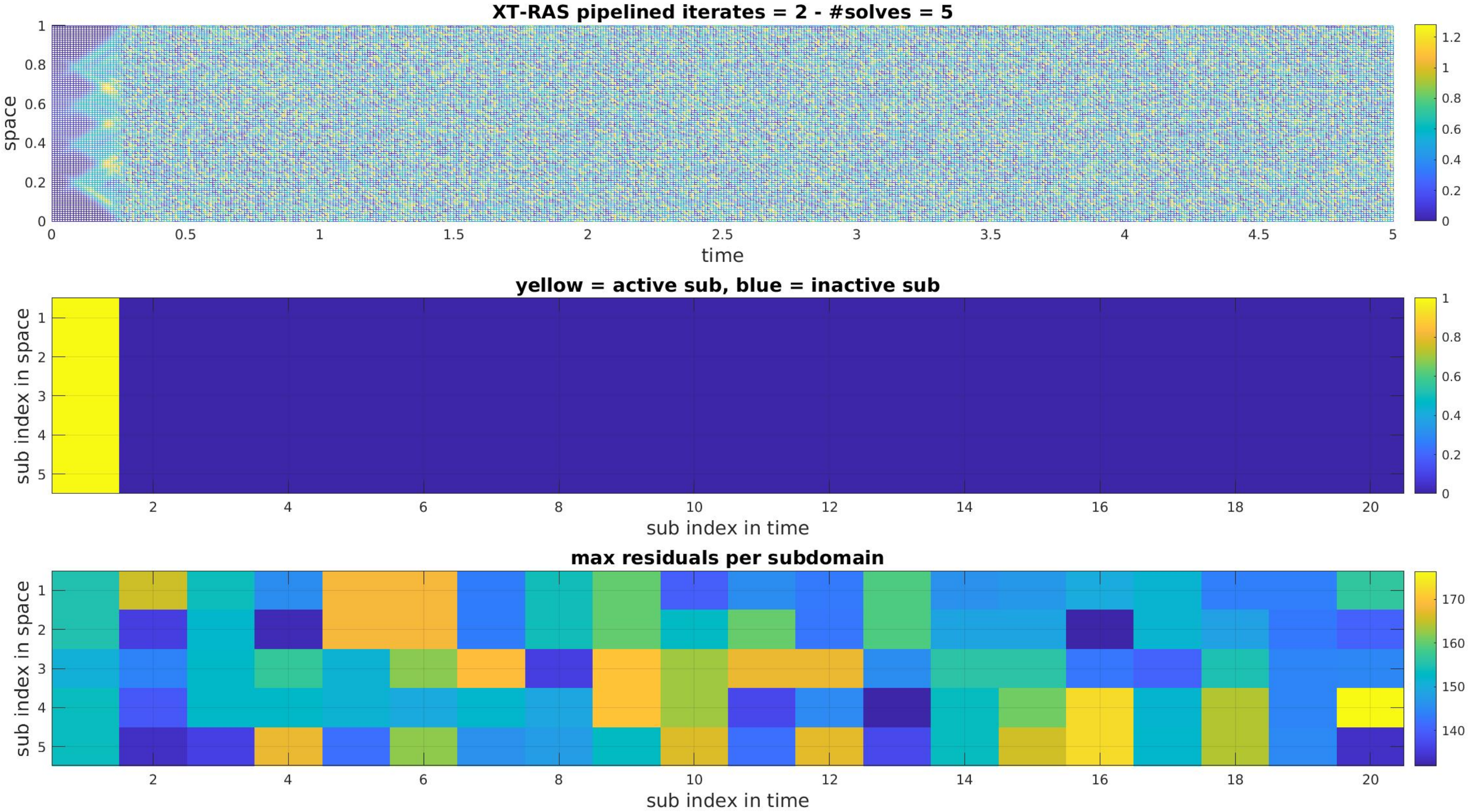
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



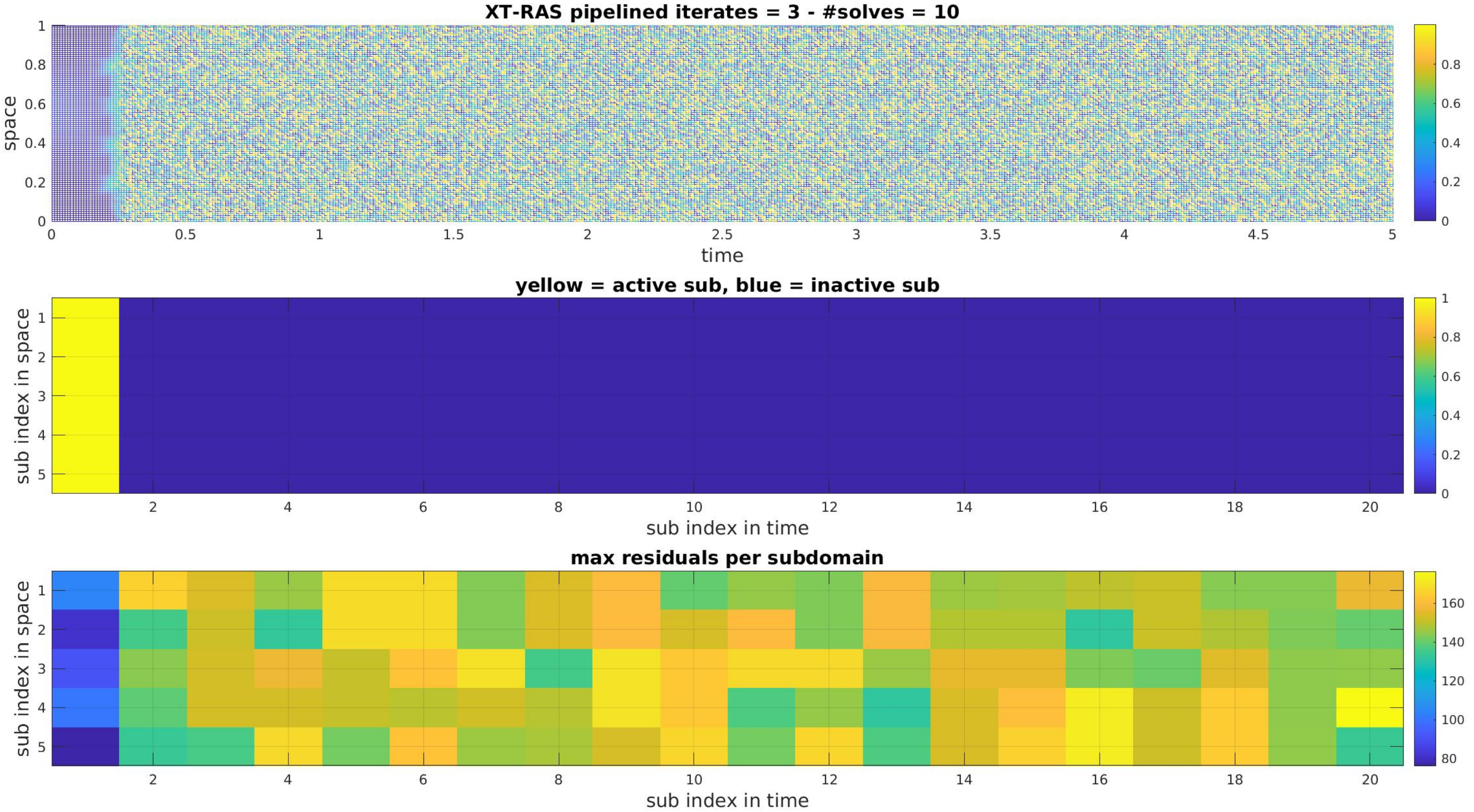
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



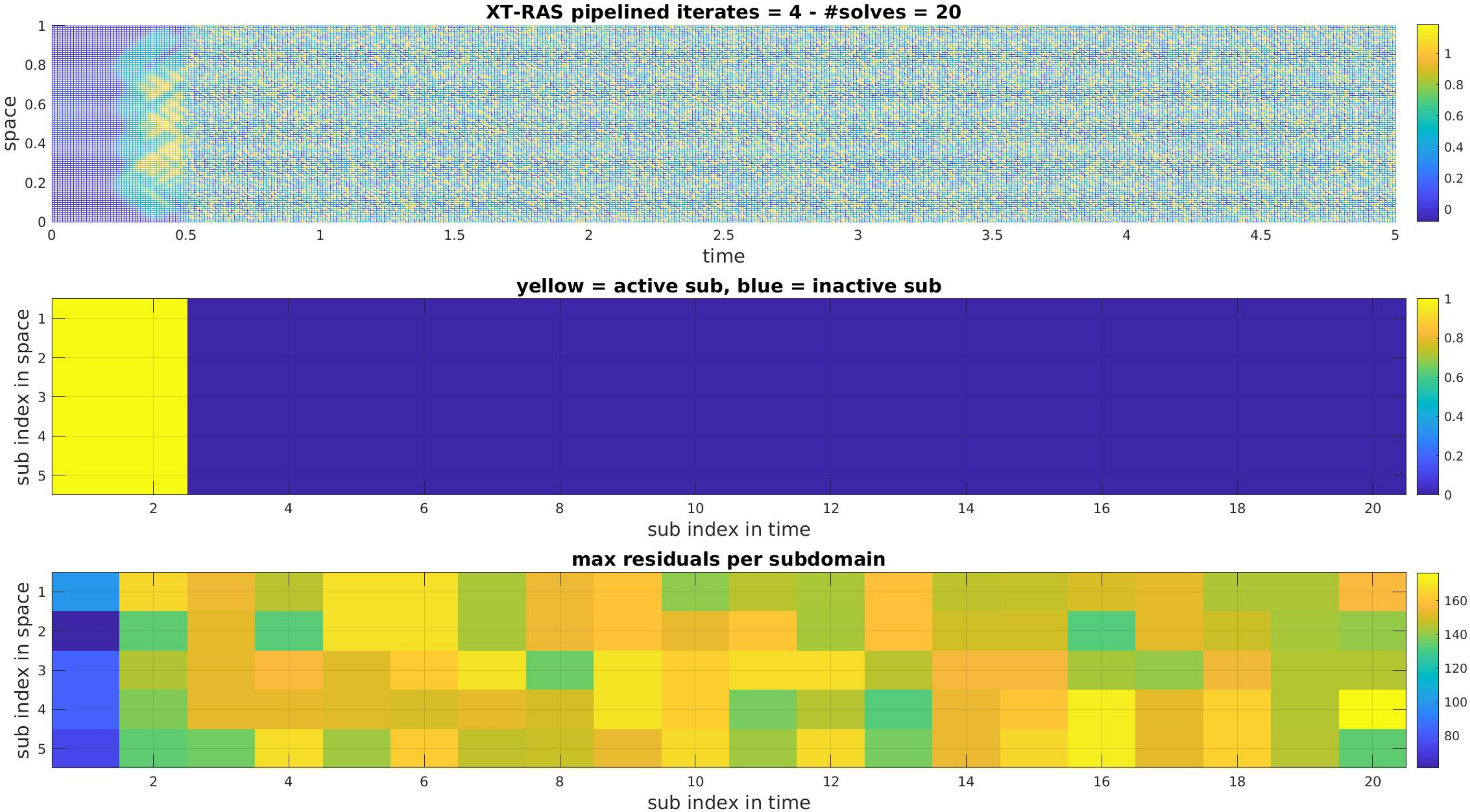
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



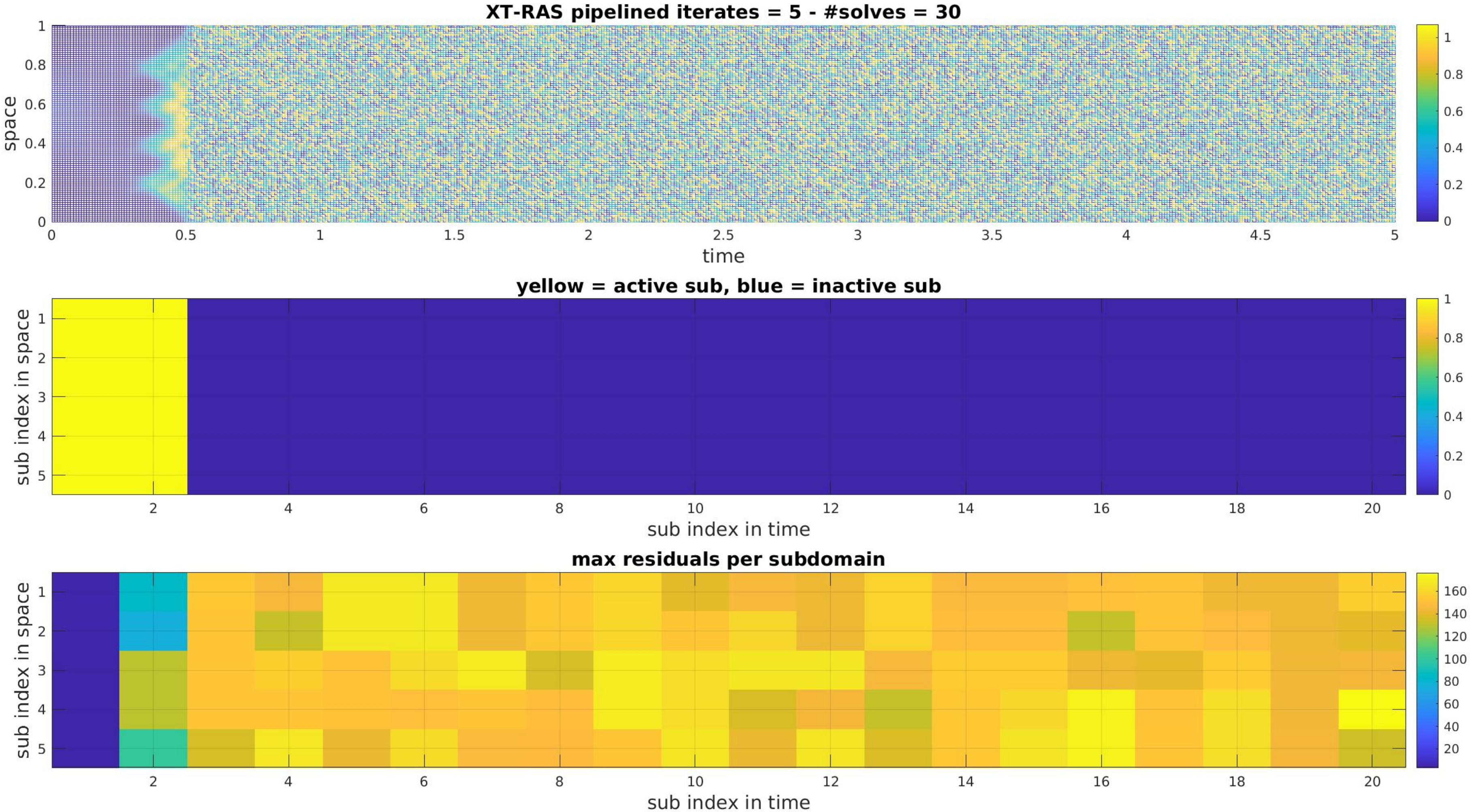
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



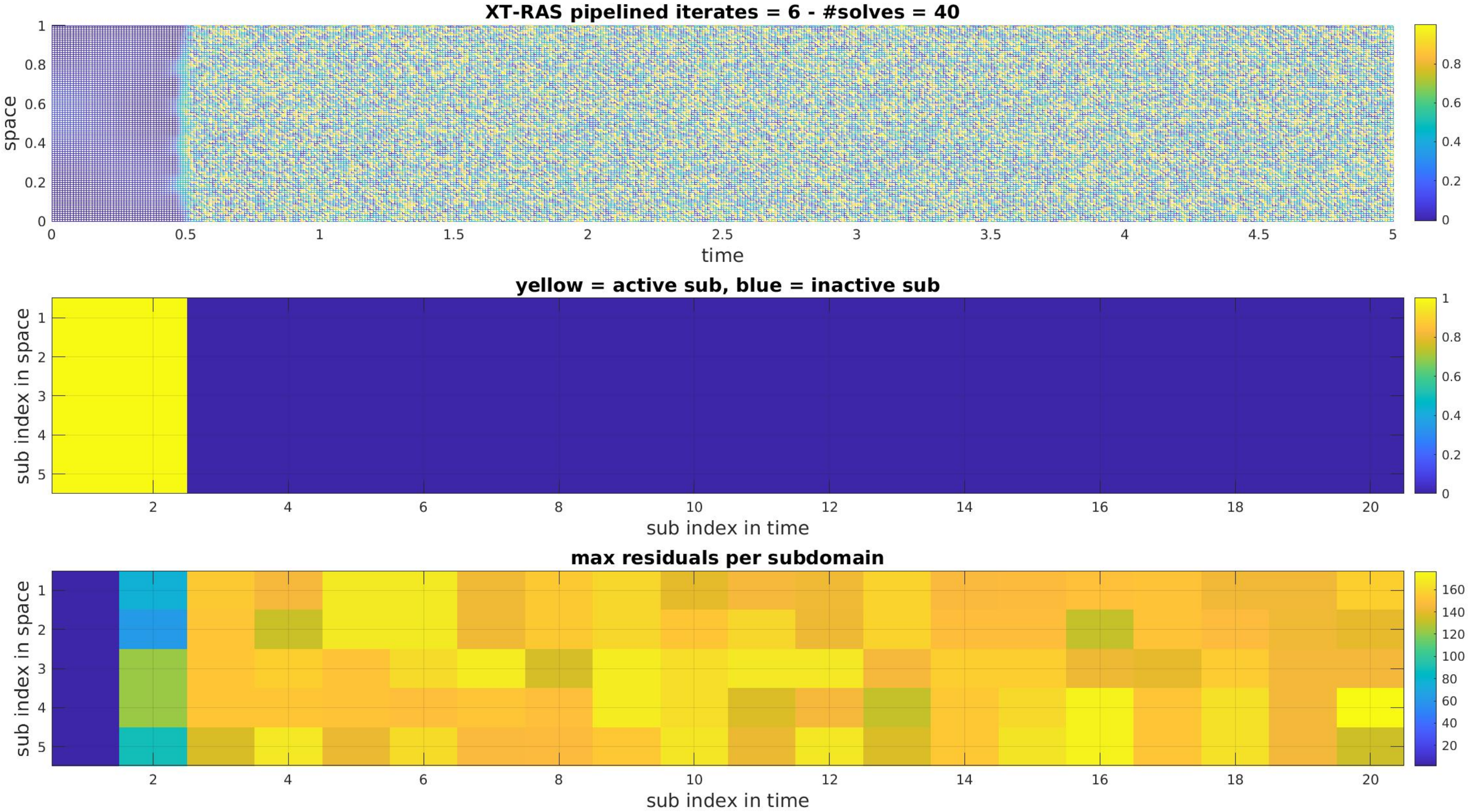
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



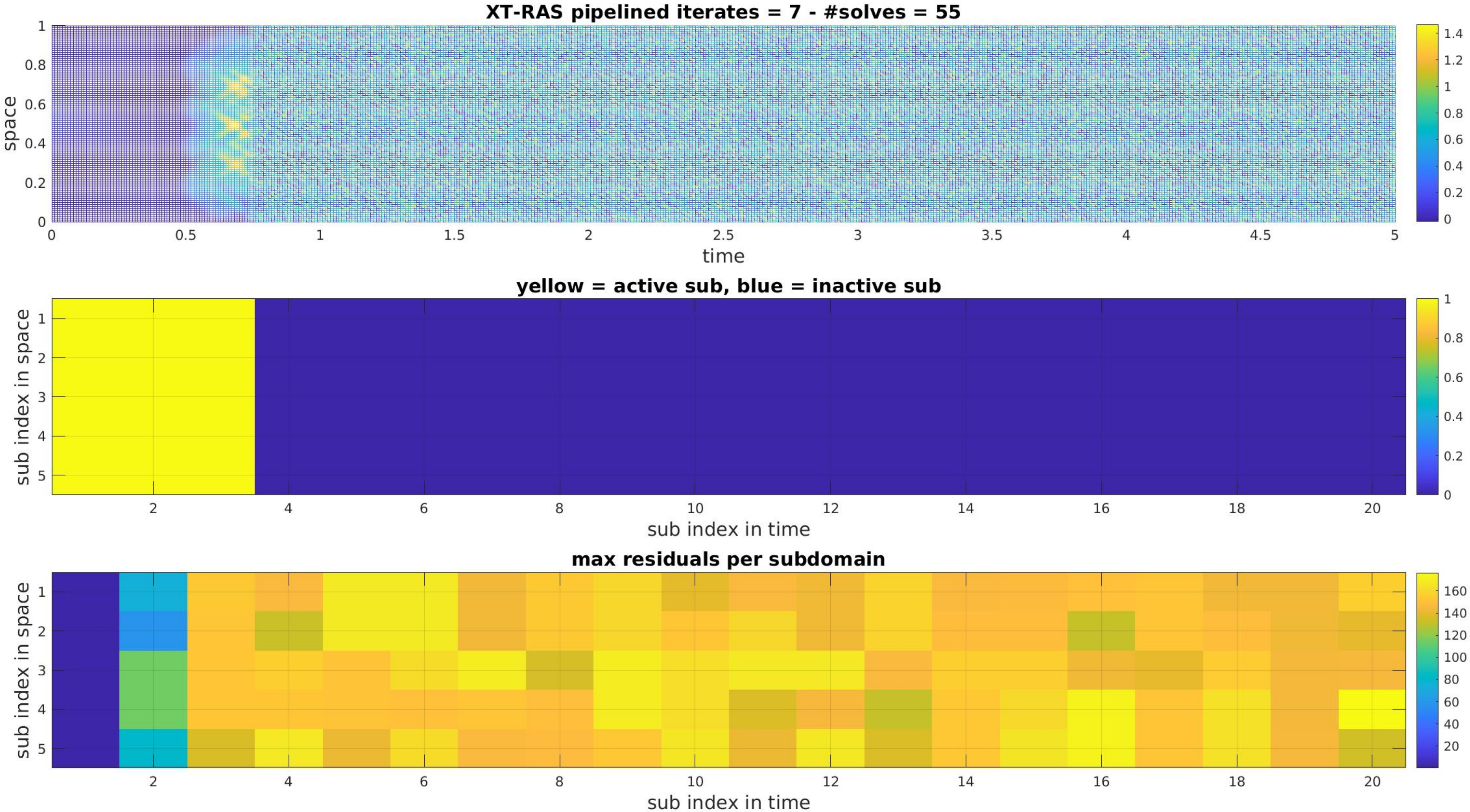
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



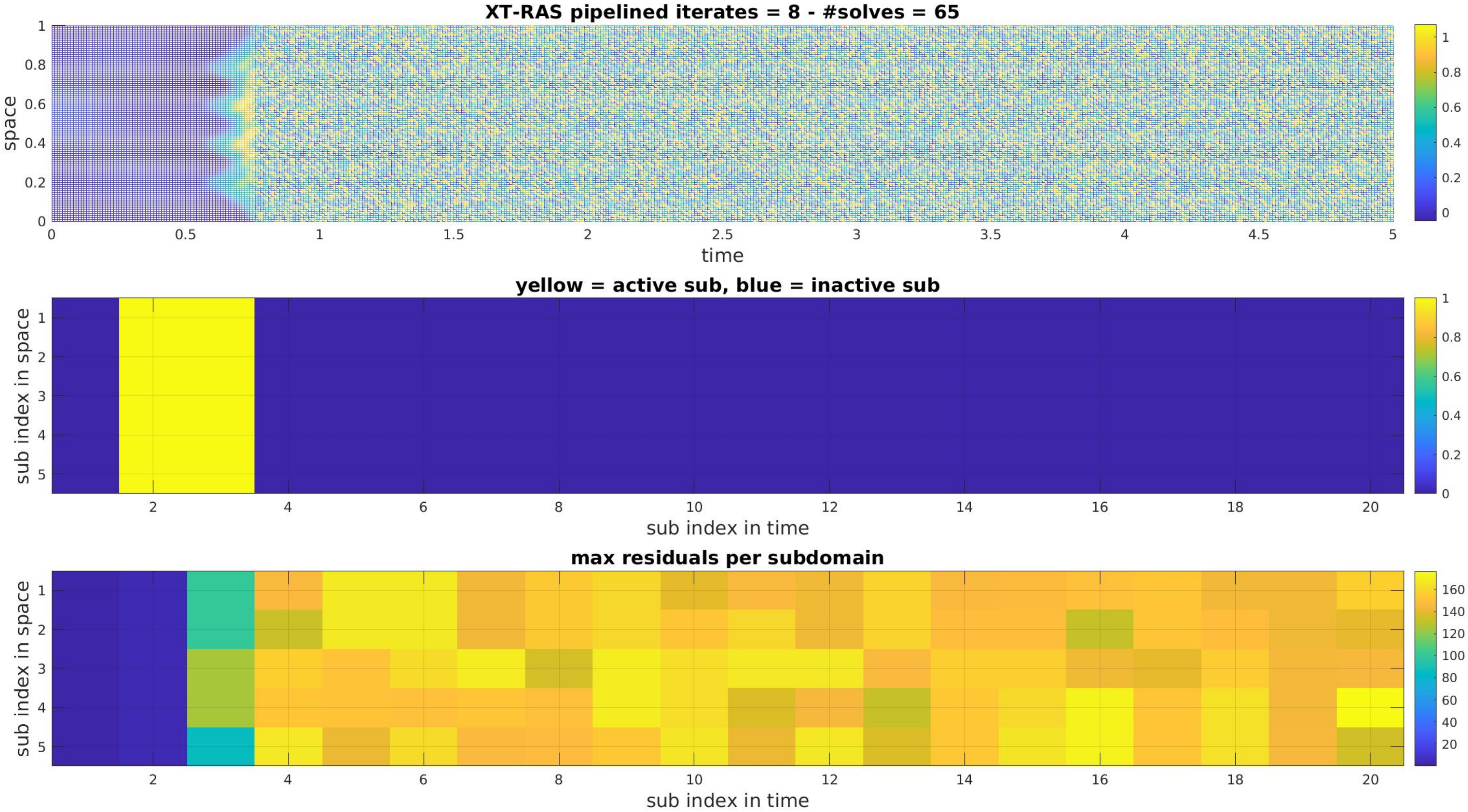
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



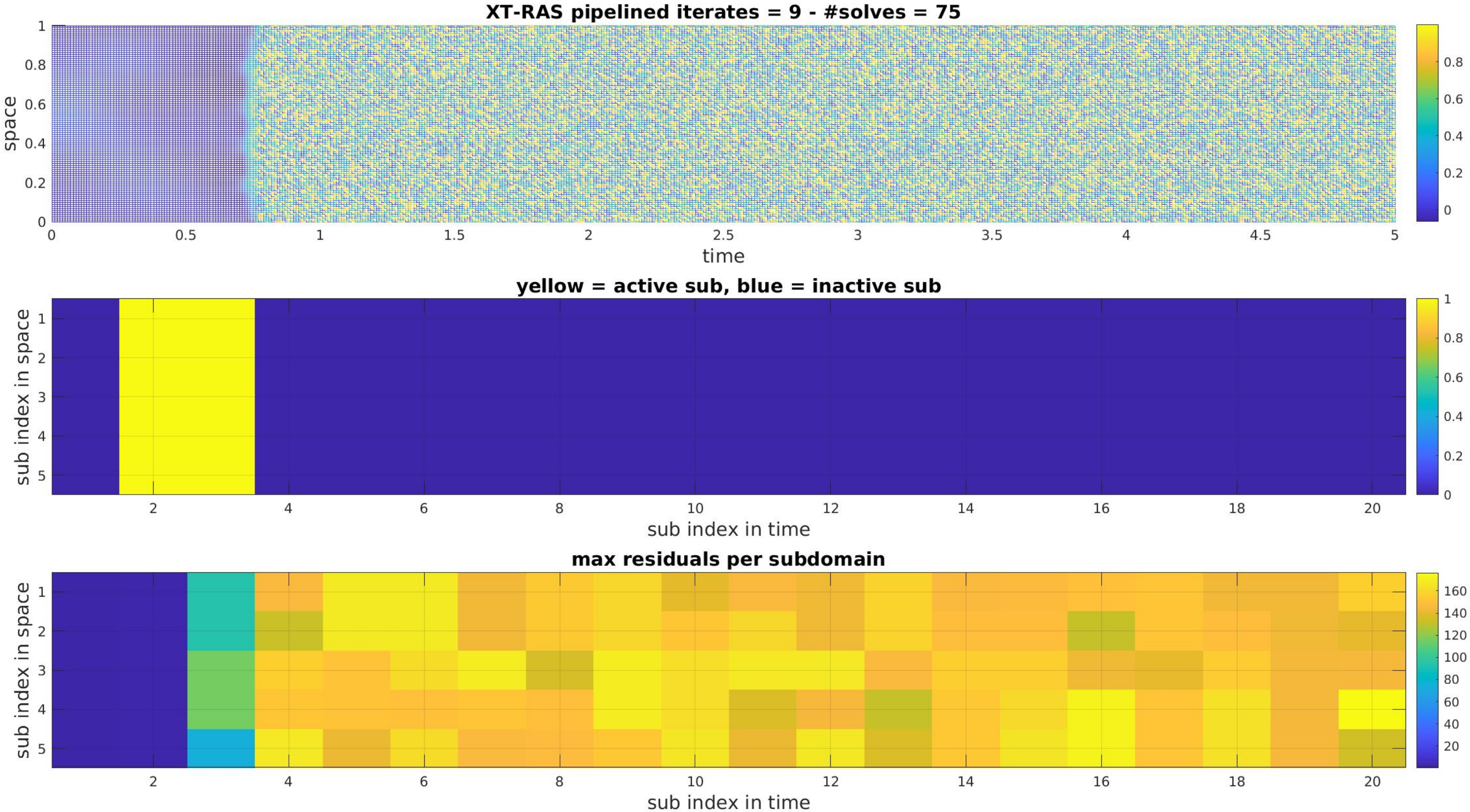
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



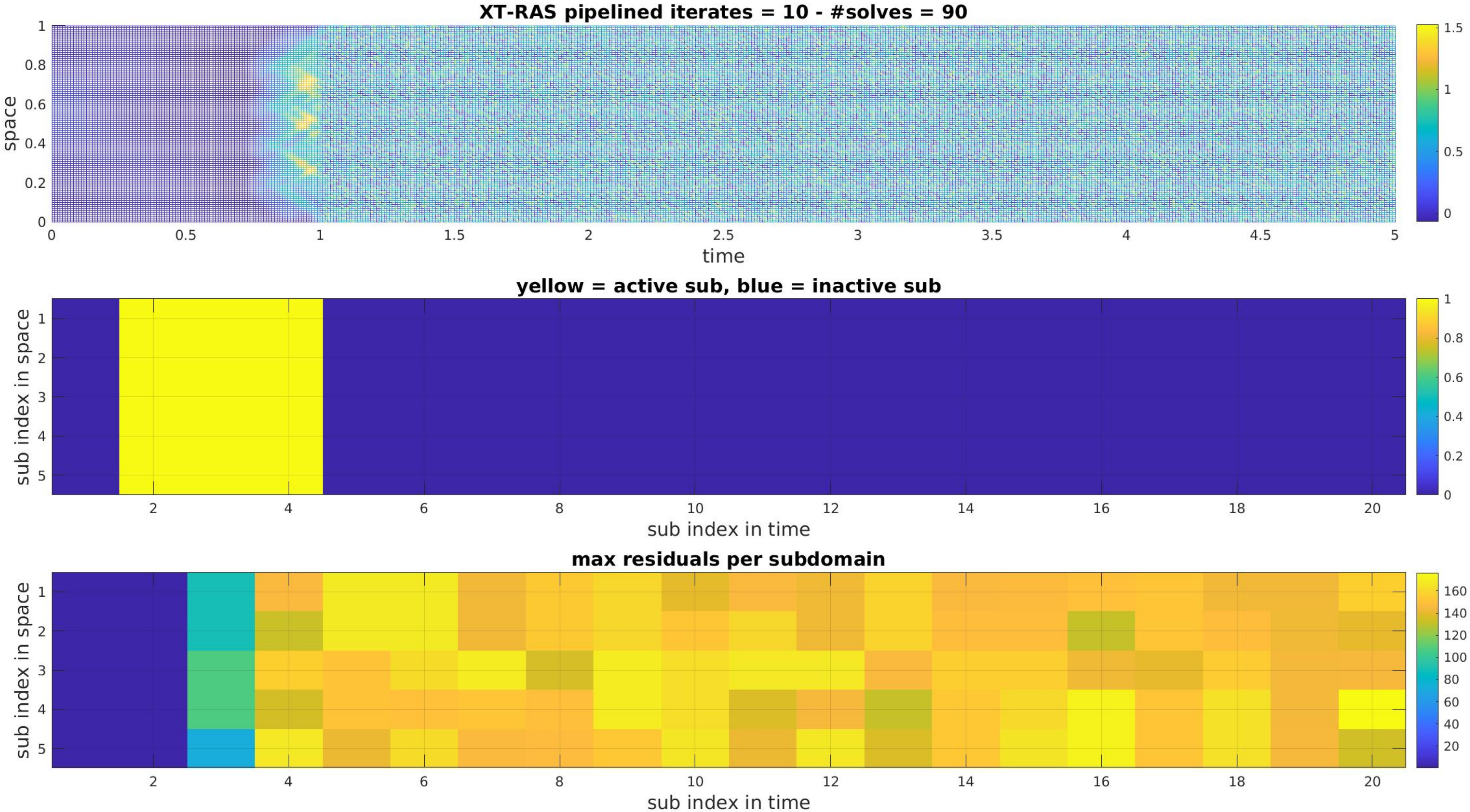
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



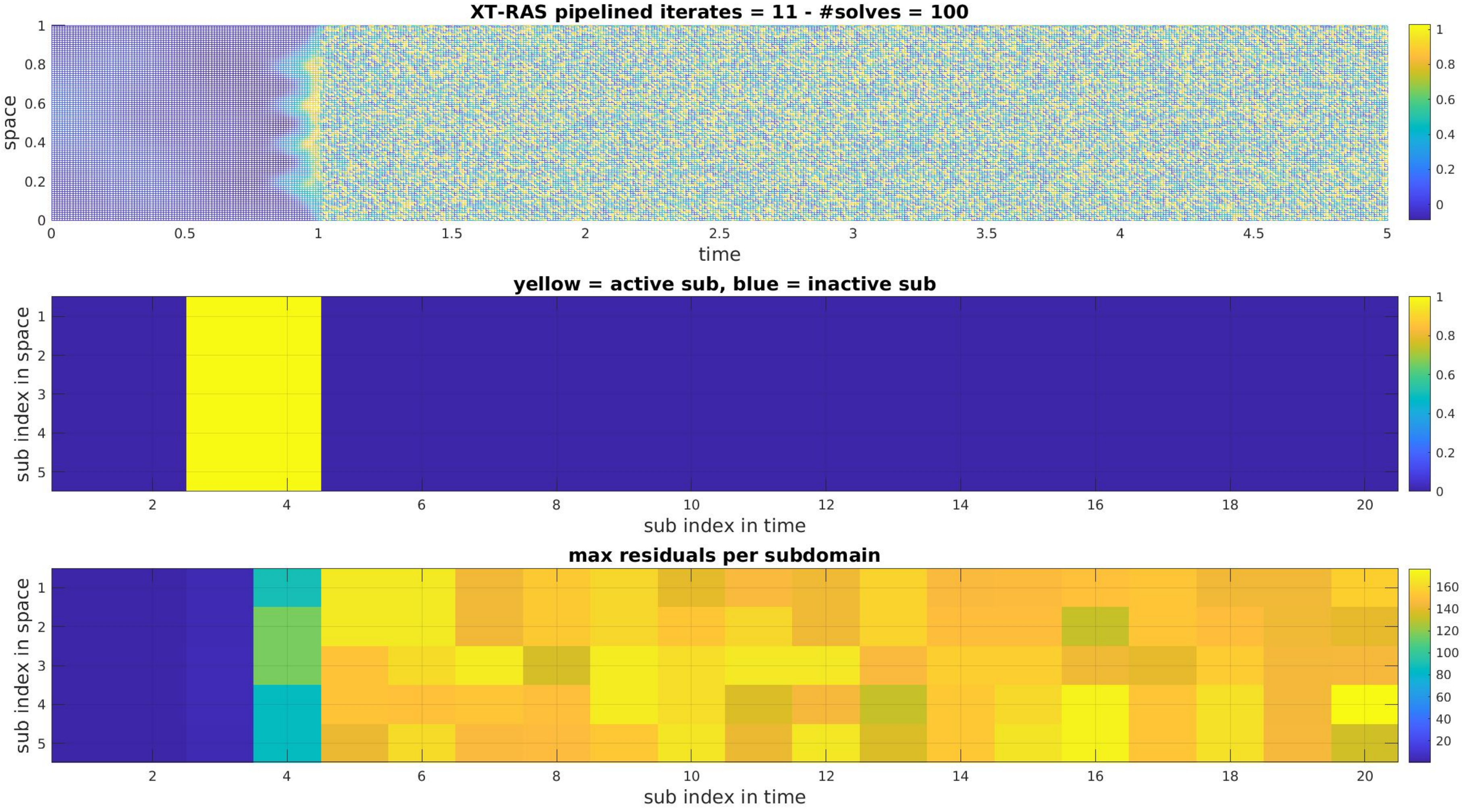
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



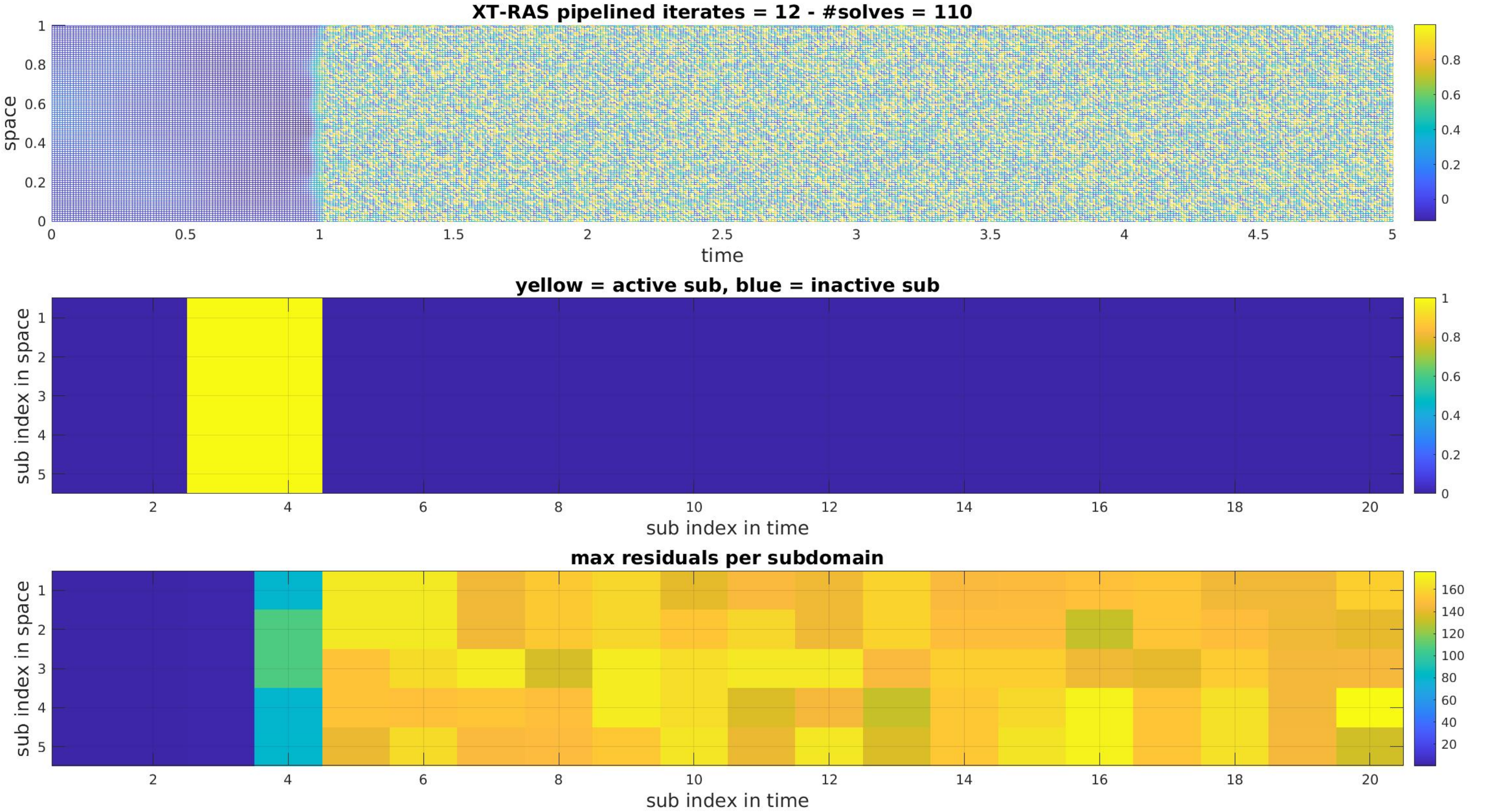
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



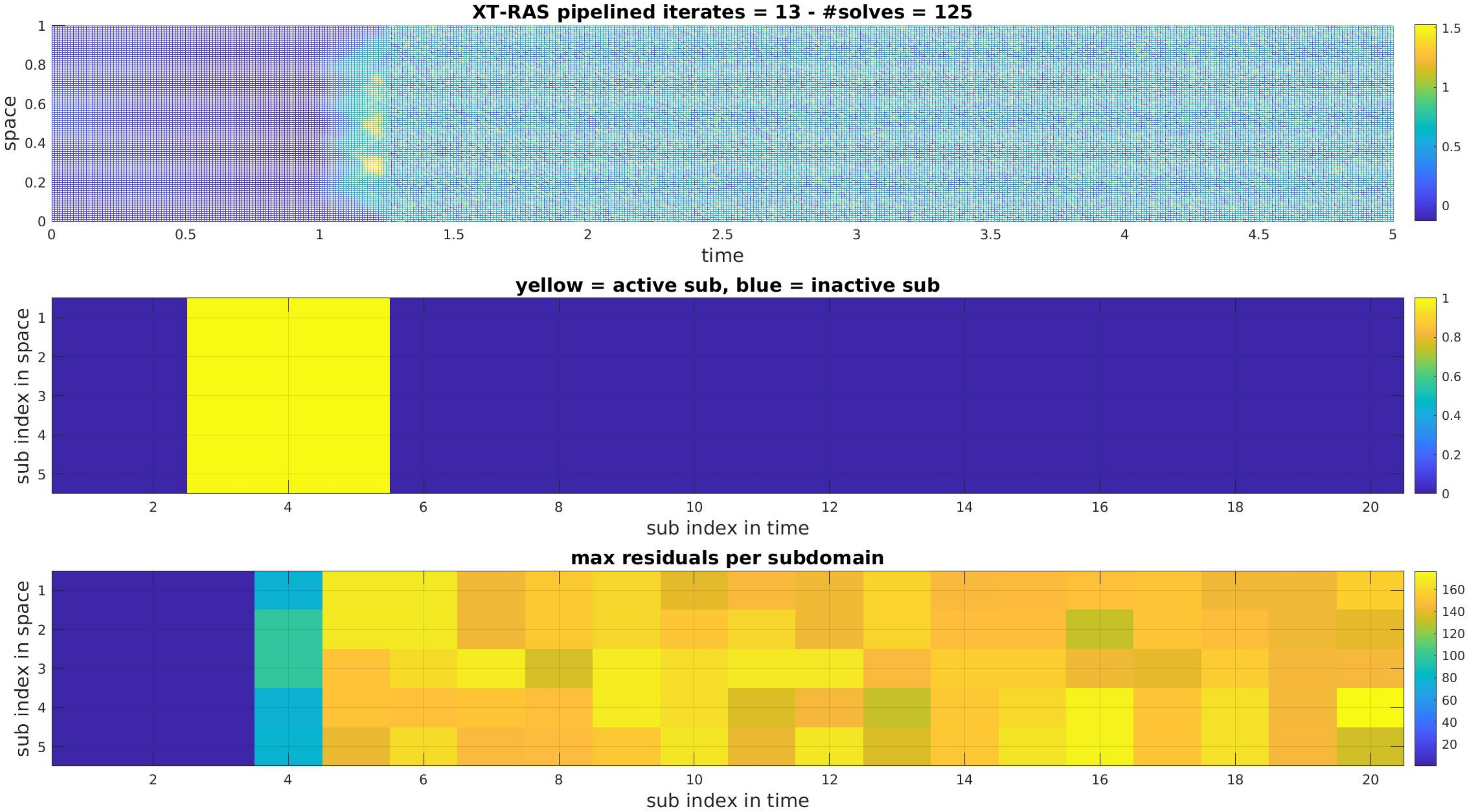
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



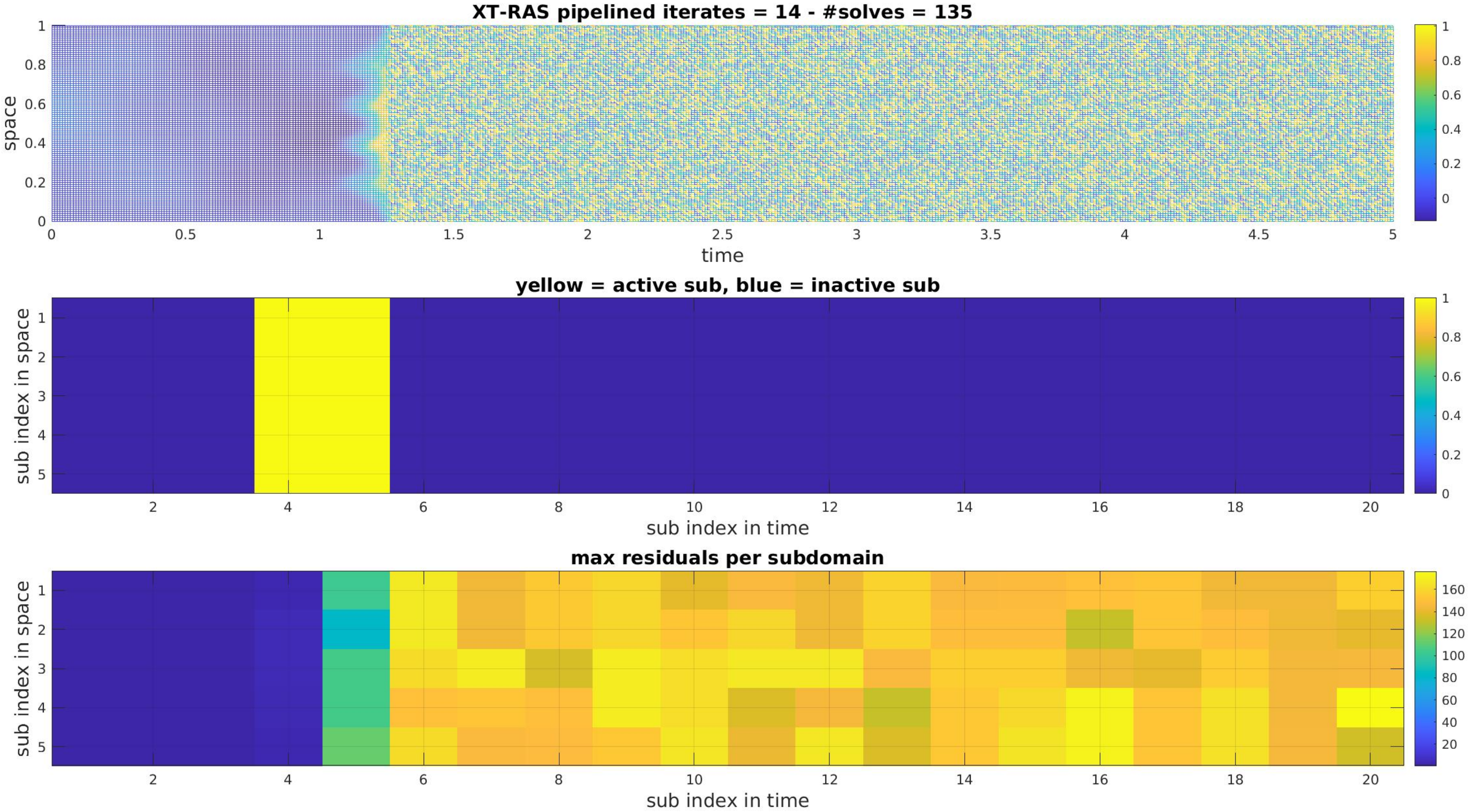
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



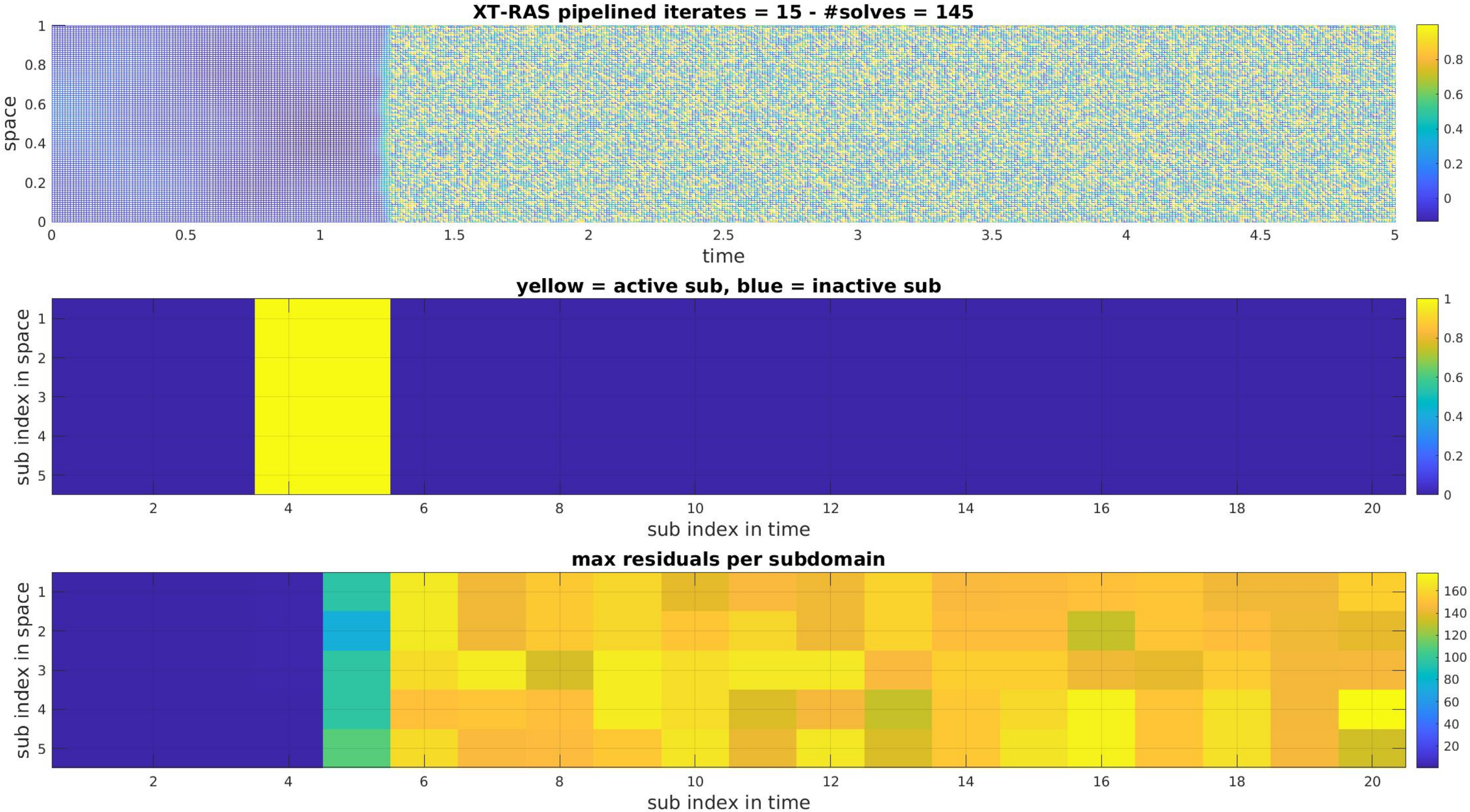
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



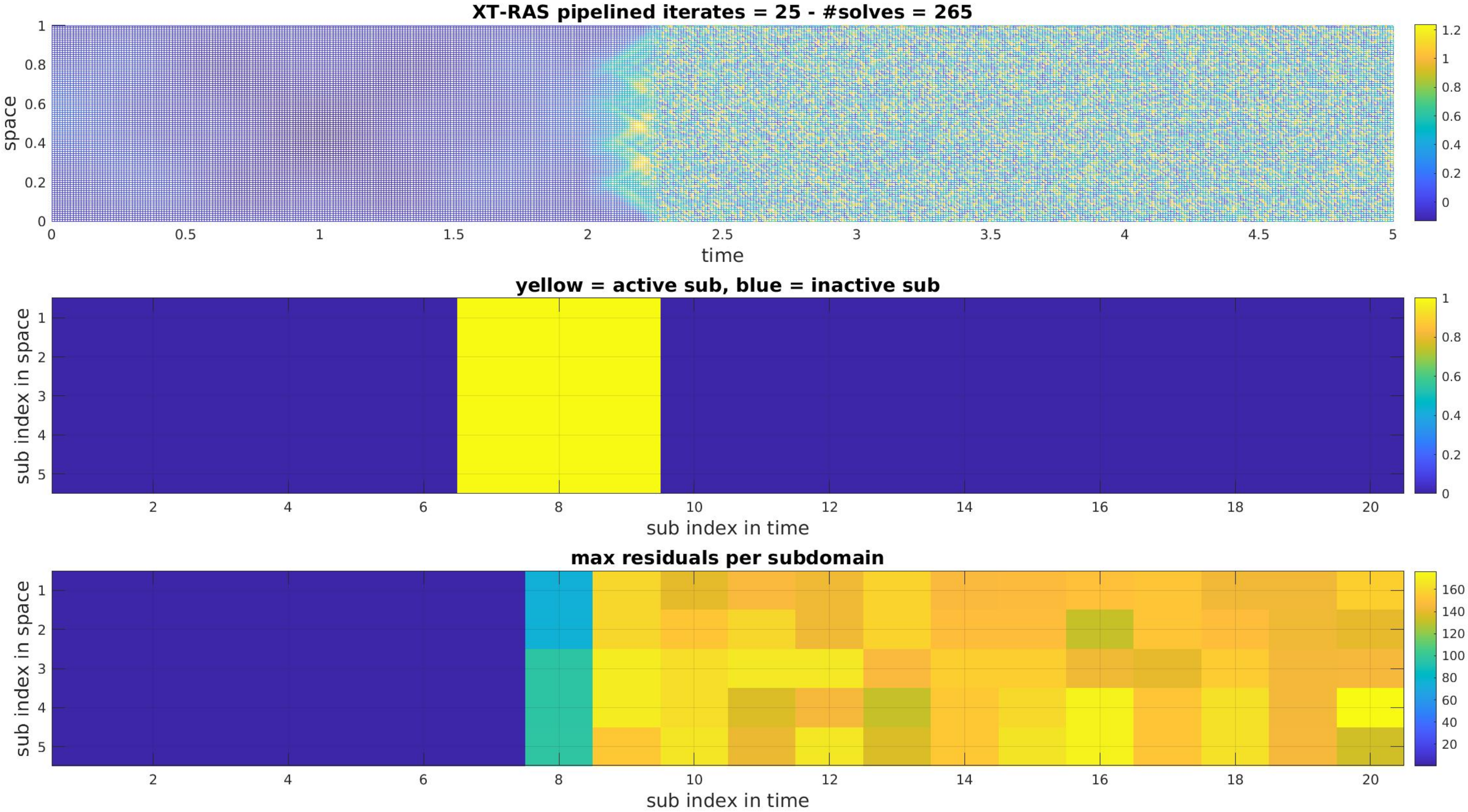
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



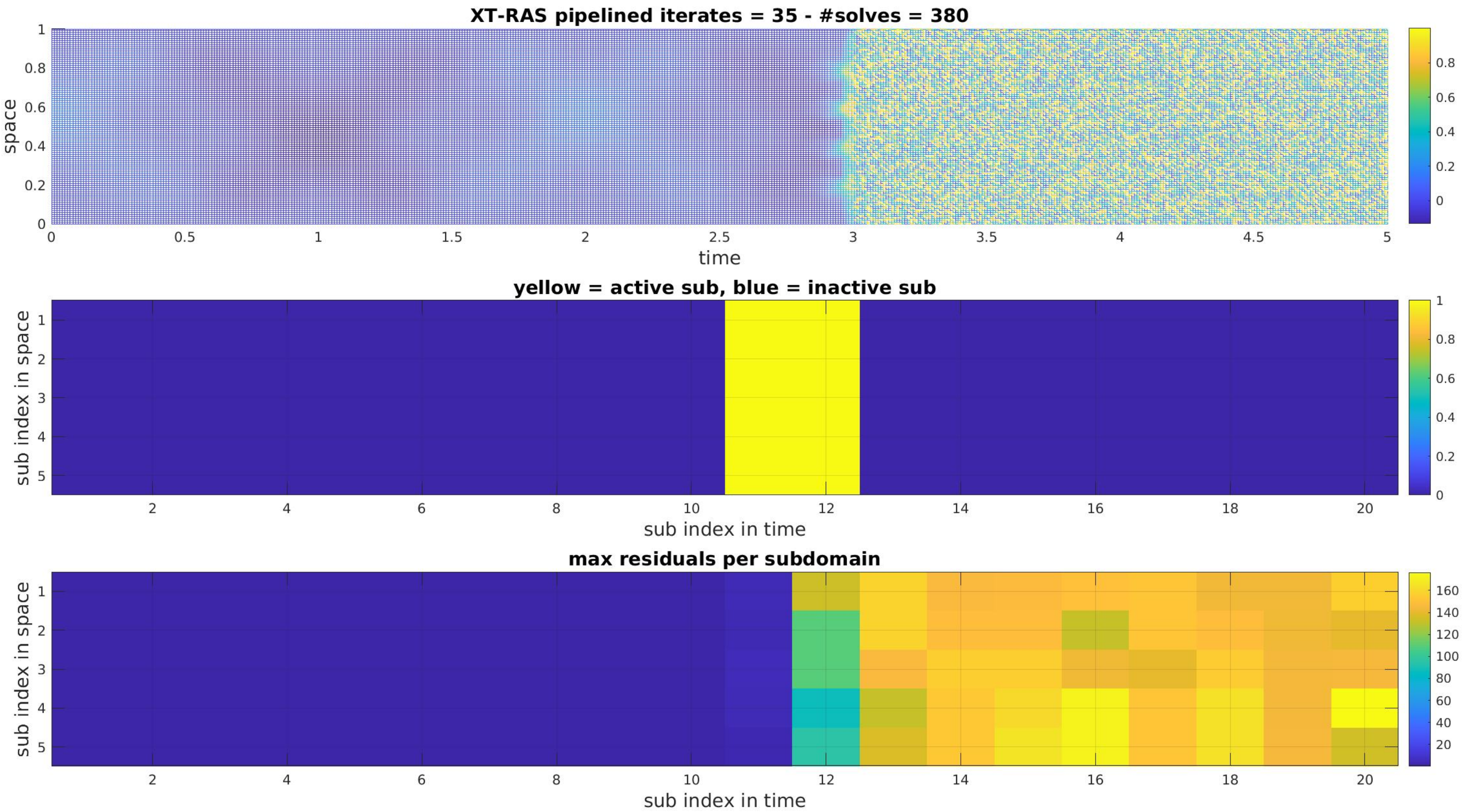
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



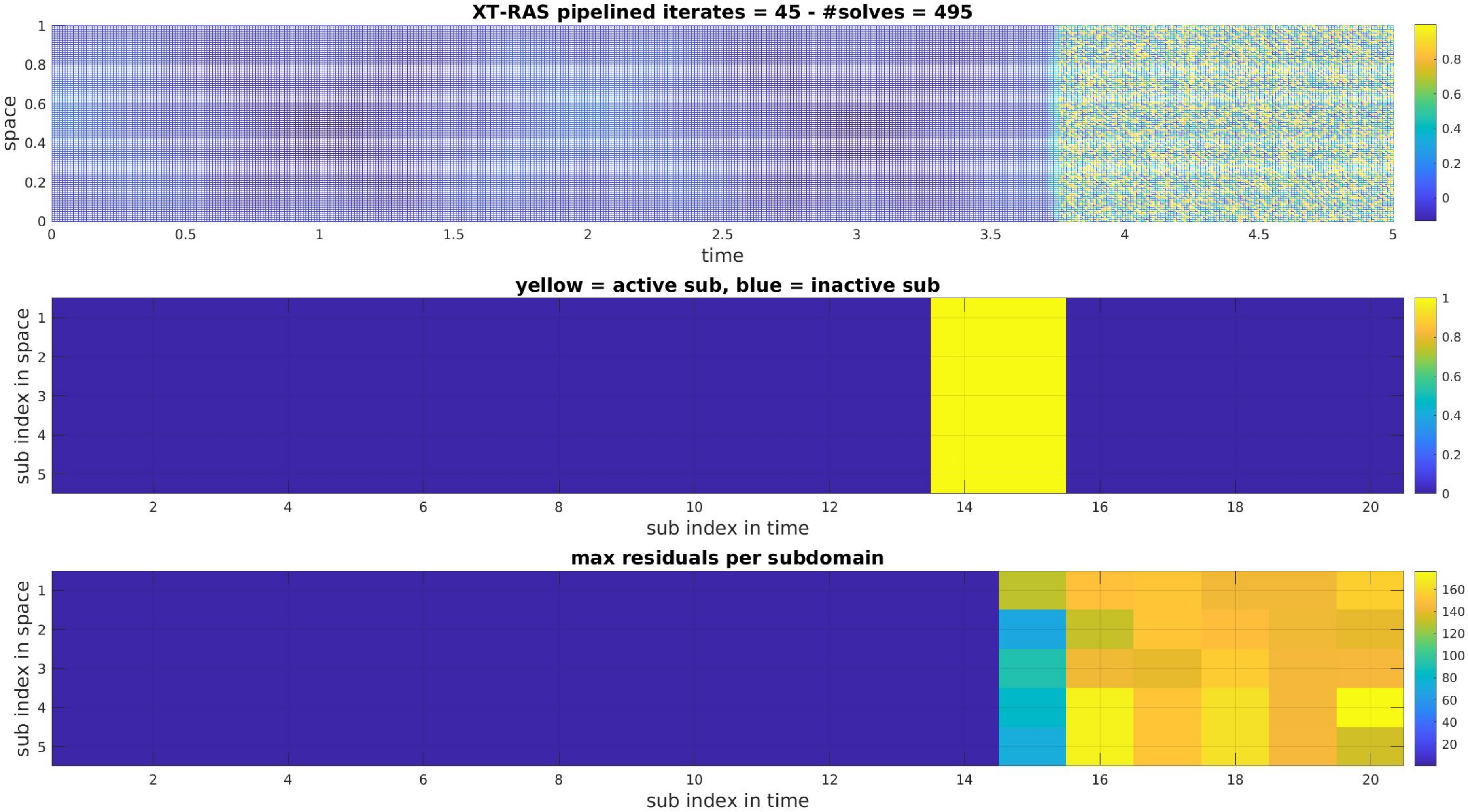
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



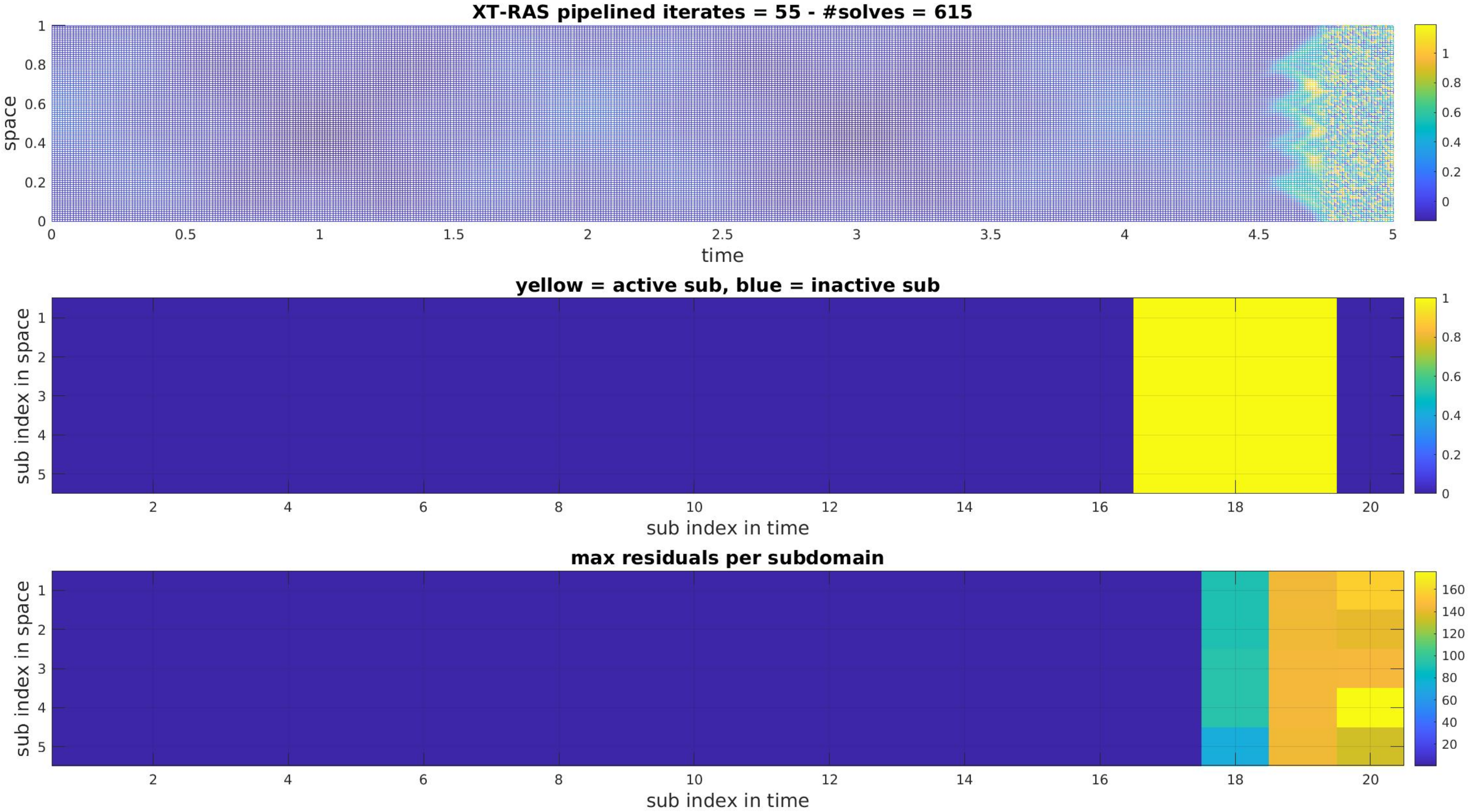
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



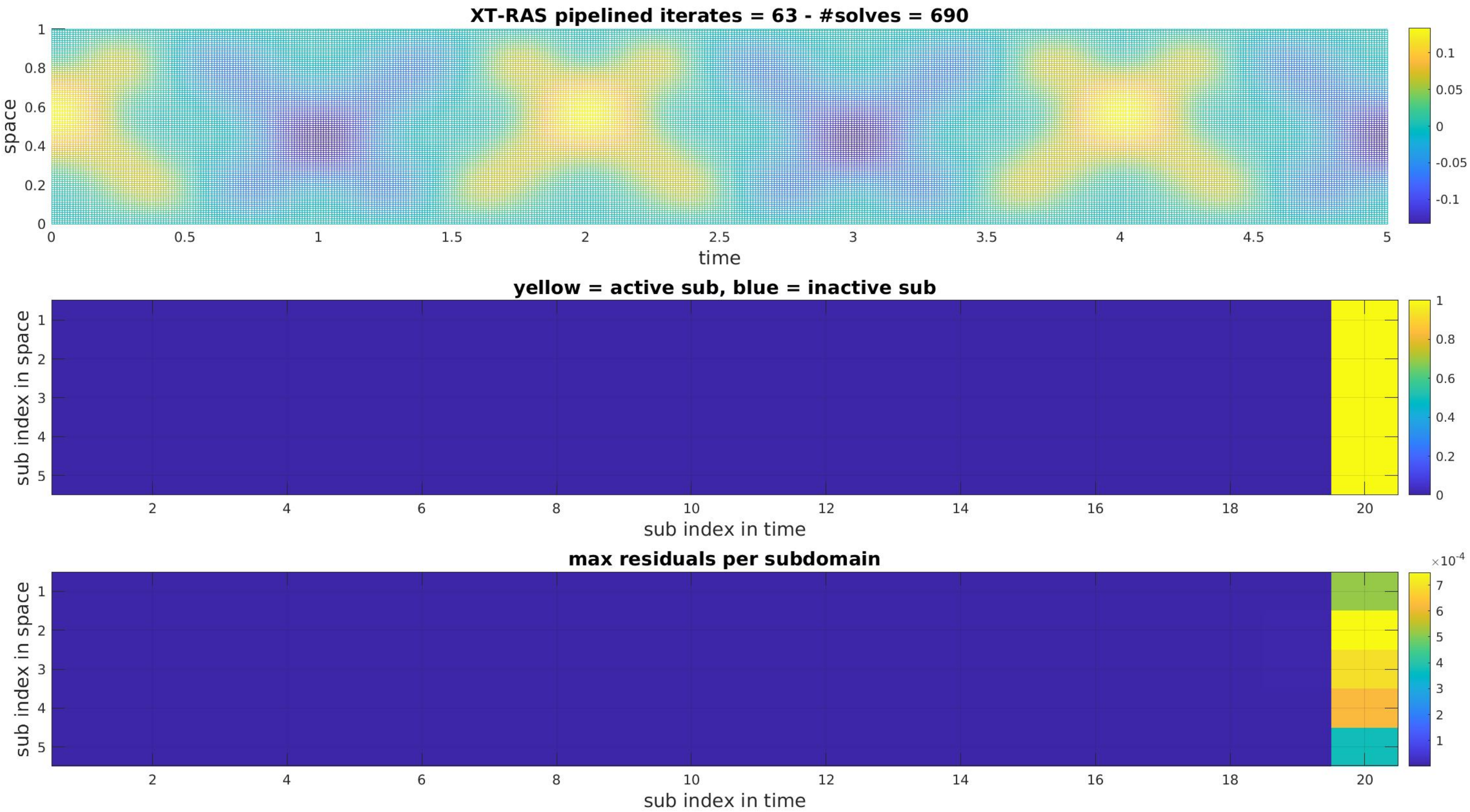
Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



Numerical experiments ($T = 5$, $N_X = 5$, $N_T = 20$, $L \approx 0.1$)

(XT-RAS with no Pipeline: Iterations=62 and Solves=6200)



Conclusions:
- Convergence analysis of the SWR method for the solution of wave equations (including damping).
- A new parallel XT-RAS framework.

Future work:
- Relations and comparison with the tent-pitching algorithm.
- Extension to 2D and 3D problems.
- Extension to true XT FE discretization on polygonal meshes.
- Test on concrete application (acoustics, geosciences, etc.)

Thank you!