# Introducing time parallelisation within data assimilation using parareal

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**Motivation**

- Numerical weather prediction (NWP) heavily relies on the use of high performance architectures to give forecasts in reasonable time
- Example: A climate simulation problem with $50,000$ unknowns given $10,000$ cores. Using $10 \times 10$ unknowns for the horizontal domain would use 500 cores.
  - More cores $\implies$ performance loss
- Data assimilation (DA) is a highly sequential algorithm, why not exploit PinT methods...?

**In this talk**

- Coupling parareal with the data assimilation framework
- Tuning the precision levels in minimisation for running parareal adaptively
- results with a 1D shallow-water model and a theoretical speedup check
- Some of the possible improvements and future work

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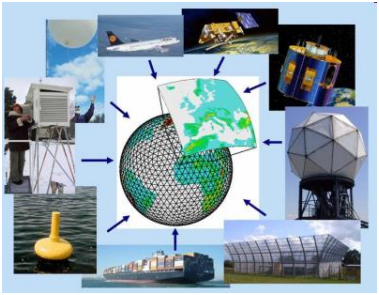
In terms of NWP, the aim is to retrieve the *optimal* initial condition which describes the true state of the atmosphere to give an improved forecast



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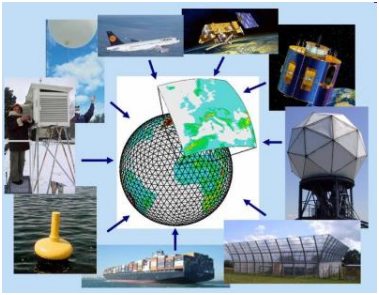
Available information (pressure, temperature, wind speed,...)

- numerical model describing the governing laws/physics
  - state vector of order $10^8 - 10^9$
- in-situ measurements, satellite data etc
  - observation vector of order $10^6 - 10^7$



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Discrete nonlinear model $\mathcal{F}$ with state variable $\mathbf{x}$

$$\begin{aligned} \mathbf{x}(t_0) &= \mathbf{x}_0 \\ \mathbf{x}_i &= \mathcal{F}_{t_{i-1} \to t_i}(\mathbf{x}_{i-1}) \quad i = 1, ..., N \qquad N \text{ - no. of time windows} \end{aligned} \tag{1}$$

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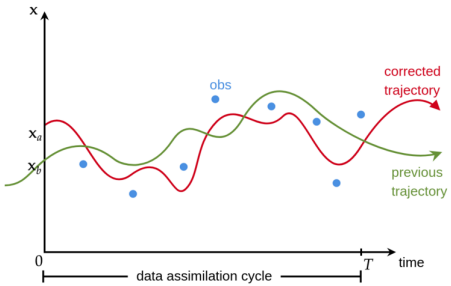
**Variational DA** $\longrightarrow$ solving a least squares problem of the discrepancies between what model gives and what we observe!



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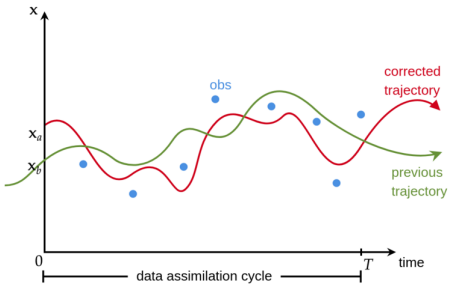
- observation operator $\mathcal{H}$
- covariance matrices $\mathbf{B}$ and $\mathbf{R}$

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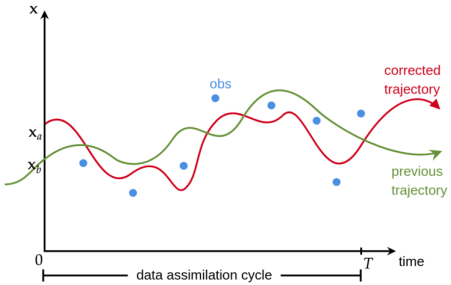


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$$\mathcal{J}(\mathbf{x}_0) \rightarrow \text{one model run}$$

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**Tangent linear model** of $\mathcal{F}$ (for initial perturbation $\delta\mathbf{x}_0$)

$$\delta\mathbf{x}_{i+1} = F_i\delta\mathbf{x}_i$$

**Corresponding adjoint equations:**

$$\begin{aligned} \lambda_N &= H_N^T\mathbf{R}_N^{-1}[\mathbf{y}_N - \mathcal{H}_N(\mathbf{x}_N)] \\ \lambda_i &= F_i^T\lambda_{i+1} + H_i^T\mathbf{R}_i^{-1}[\mathbf{y}_i - \mathcal{H}_i(\mathbf{x}_i)] \qquad i = N-1,\ldots,1 \\ \lambda_0 &= F_0^T\lambda_1 + \mathbf{B}^{-1}(\mathbf{x}_0 - \mathbf{x}_0^b) + H_0^T\mathbf{R}_0^{-1}[\mathbf{y}_0 - \mathcal{H}_0(\mathbf{x}_0)] \end{aligned} \tag{3}$$

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- Solve small sub-minimisations by sucessive approximations
- Gives a quadratic $J$

**Tangent linear approximations** (around $\mathbf{x}^{(k)}$)

$$\mathcal{F}(\mathbf{x}^{(k)} + \delta\mathbf{x}^{(k)}) \simeq \mathcal{F}(\mathbf{x}^{(k)}) + F|_{\mathbf{x}^{(k)}}\, \delta\mathbf{x}^{(k)}$$
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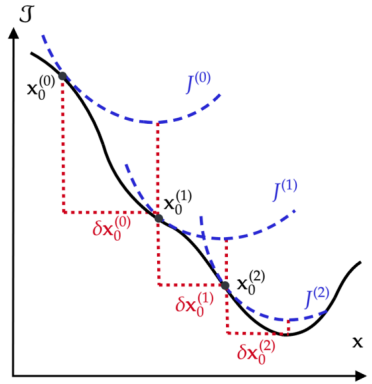


- At $k$th minimisation solve

$$J^k(\delta\mathbf{x}_0^{(k)}) = \frac{1}{2}\|\delta\mathbf{x}_0^{(k)} - (\mathbf{x}_0^b - \mathbf{x}_0^{(k)})\|^2_{\mathbf{B}^{-1}} + \frac{1}{2}\sum_{i=0}^{N}\|\mathbf{y}_i - \mathcal{H}_i \circ \mathcal{F}(\mathbf{x}_0^{(k)} + \delta\mathbf{x}_0^{(k)})\|^2_{\mathbf{R}^{-1}} \quad (5)$$

  - Innovation vector : $\mathbf{d}_i = \mathbf{y}_i - \mathcal{H}_i \circ \mathcal{F}(\mathbf{x}_0^{(k)})$
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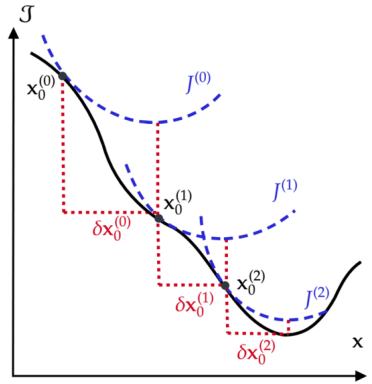


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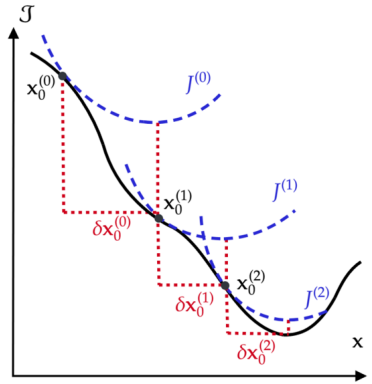


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**Description:**

$$\mathbf{x}_0^{k+1} = \mathbf{x}^0$$

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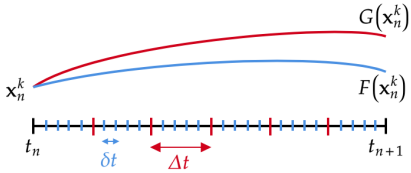


- Fine solver computations in parallel
- Nice convergence for parabolic problems, not so good for hyperbolic

**Error:**

With parareal integrator $P(k)$ at iteration $k$ such that $P(k)\mathbf{x}_0 = \mathbf{x}_N^k$, we can write [2]

$$F^N - P(k) = \sum_{p=k+1}^{N} C_p^N (F-G)^p G^{N-p} \tag{7}$$

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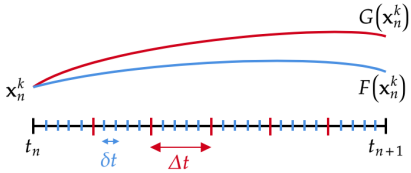


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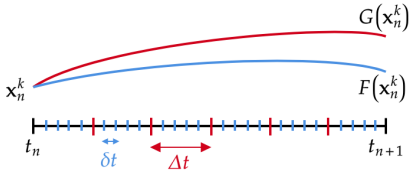


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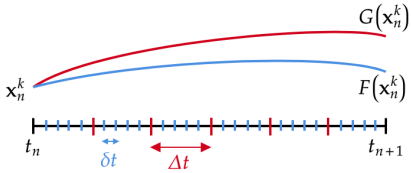


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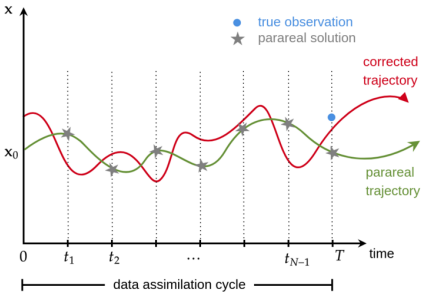

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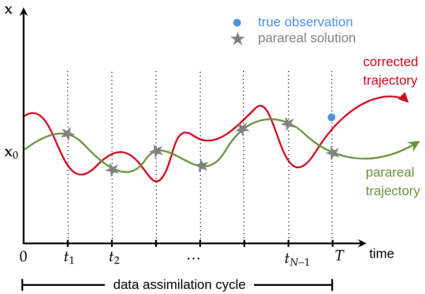


$\longrightarrow$ Conjugate gradient for minimisation as $A^*$ is symmetric

# Minimisation: Inexact conjugate gradient[4]

- *Inexact* matrix-vector multiplication (a costly operation!)
- By gradually increasing the inexactness of the mat-vec product, one could still produce the exact version's level of accuracy.
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If additionally

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**Result**

If $F$ is invertible, $\|E_j(k)\mathbf{p}_j\|_{A^{-1}}$ and $\|P(k)\mathbf{p}_j - F^N\mathbf{p}_j\|_2$ are rigorously the same whatever the parareal approximation. $F^N\mathbf{p}_j$ is the corresponding exact solution.

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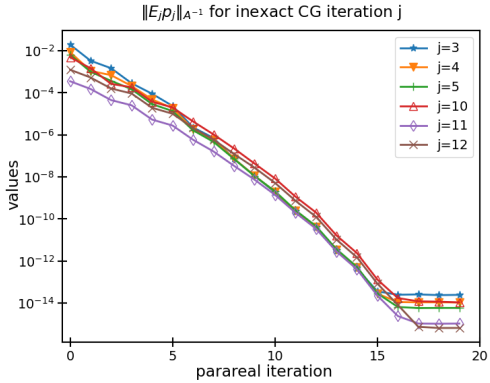


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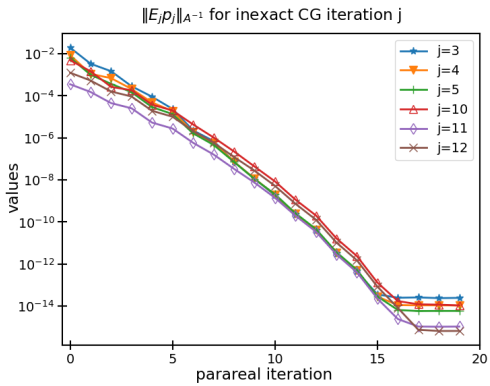


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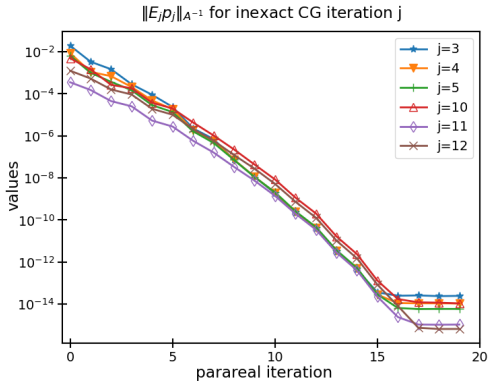


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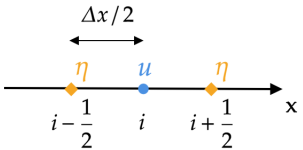
- Check that $\xi_j$ doesn't lie between last two norm values

1d linear shallow water model

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}$$
$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} \tag{12}$$

**Setup:** Arakawa C-grid discretisation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = C\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix}\eta \\ u\end{pmatrix}$$




Parameters

- $N = 20$
- no. of fine time steps per time window, $N_f = 100$
- no. of coarse time steps per time window, $N_g = 20$
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- spatial grid-step, $\Delta x = 1$
- diffusion constant, $\mu = 0.15$
- average height, $H = 0.9$

Coarse and fine propagators are defined using theta scheme ($\theta = 0.51$)

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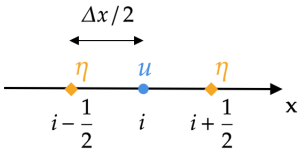


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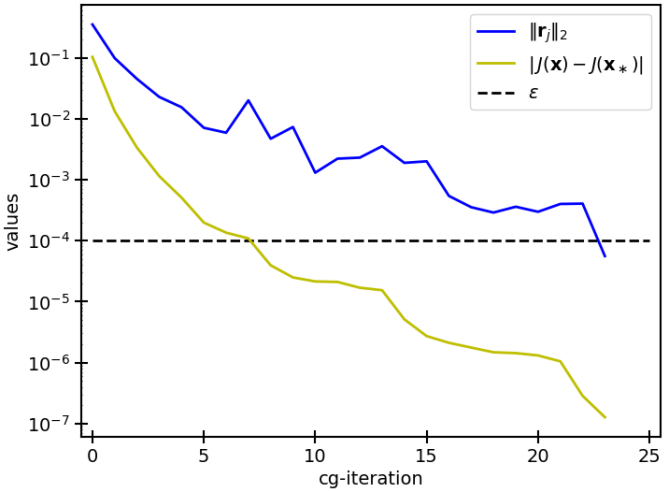
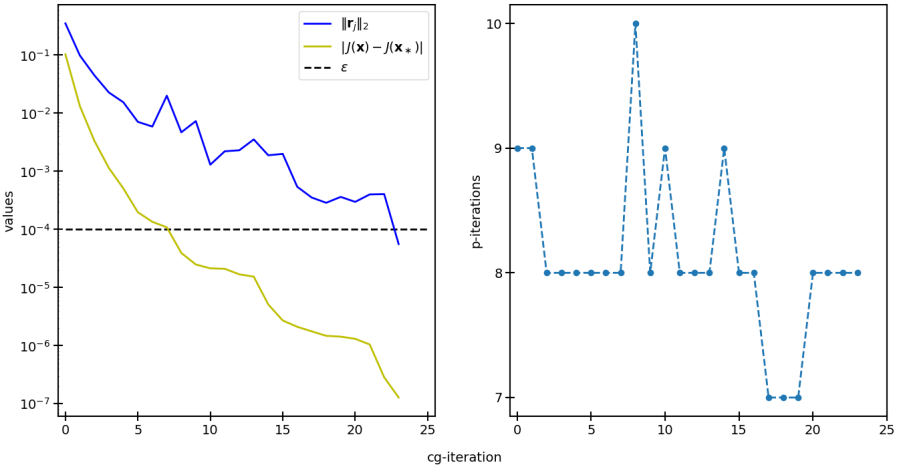
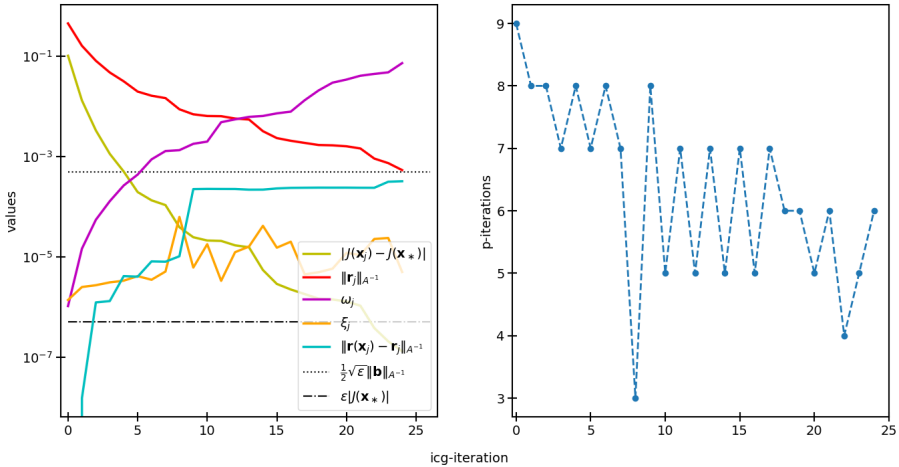
- Initial condition - a gaussian at the middle of the basin/domain
- Using exact CG with $\epsilon_{cg} = 10^{-4}$, takes 24 minimisation iterations



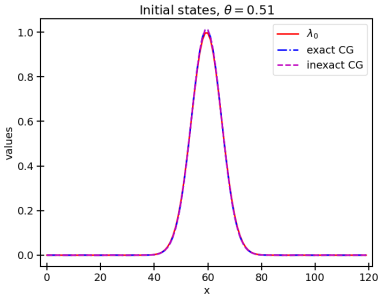
- Fixed stopping tolerance for parareal $\epsilon_{\mathrm{p}} = 10^{-6},\ \epsilon_{\mathrm{cg}} = 10^{-4}$
- On average 8.125 p-iterations per icg-iteration

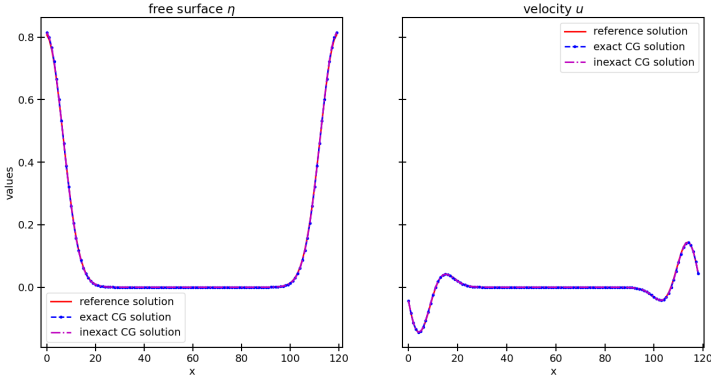


- On average 6.36 p-iterations per icg-iteration, nice!
- Theoretical speedup: $\frac{N}{\text{p-iterations}} = \frac{20}{6.36} = 3.1$
- $\omega_j$ is the tolerance for the error norm $\|E_j(k)\|_{A^{-1},A}$



# Retreived initial condition and final results



- Reducing the cost of the coarse solver.
  - Using the Krylov enhanced parareal[5] version. Choosing the right vectors for the convergence of Krylov enhanced method because of storage issues.
- Running a parallel version of adjoint - running parareal but in reverse direction leading to double degree of parallelism. Similar error analysis for adjoint.
- Current processes work on synchronicity. Introduce asynchronous methods which don't wait for neighbouring processes and are more linked to the spatial discretisation.
- Other ways to couple DA and PinT algorithms - solving the forward and backward model together as a system as done in ParaOpt[6].
- Extension to 2d model

[5] Gander and Petcu, 2008
[6] Gander, Kwok and Salomon, 2020