Parallel Space-Time Finite Element Methods for Parabolic Optimal Control Problems

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Outline

1 Introduction

2 A localized space-time upwind scheme

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5 Conclusions & Outlook

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Space-time Methods

What do we mean by that?

- every numerical method solving time-dependent problems is a space-time method !
- however, usually the discretization of space and time are considered "seperate", and time-stepping is used!
- for us, time is *just another variable* $t = x_{d+1}$!
- we focus on (completely) unstructured decompositions of the space-time cylinder

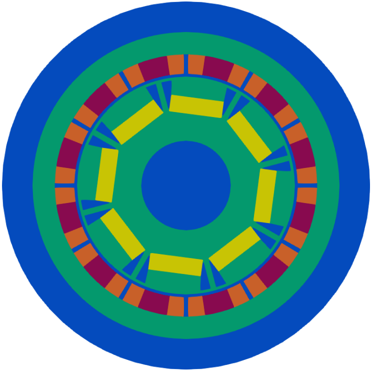
Space-time Methods (contd.)

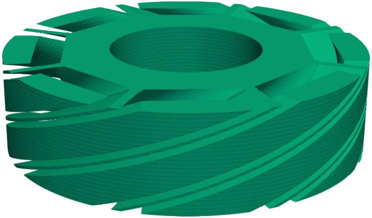
Why are we doing this?

- full space-time adaptivity
- full space-time parallelization
- "easier" treatment of moving domains
- optimization problems, e.g. optimal control !

What are the disadvantages?

- memory
- ...?





Why use a space-time approach for OC?

Optimality system:

- forward-in-time problem
- backward-in-time-problem
- both are coupled

Space-time approach: time is just another variable

- only *one* problem
  - however: one dimension higher
- full space-time adaptivity
- space-time parallelization at once

Space-time tracking problem

Given y_d (desired state) and $\varrho > 0$, find the state y and the control u minimizing

$$J(y,u) = \frac{1}{2}\int_Q |y - y_d|^2 \, \mathrm{d}Q + \frac{\varrho}{2}\|u\|^2_{L_2(Q)}$$

subject to

$$\partial_t y - \Delta_x y = u \text{ in } Q = \Omega \times (0,T),$$
$$y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), \quad y = 0 \text{ on } \Sigma_0 = \Omega \times \{0\},$$

where $\Omega \subset \mathbb{R}^d$ is the spatial domain and $T > 0$ the final time.

First Order Optimality Conditions

The reduced optimality system: find $(y, p) \in Y_0 \times P_T$ such that

$$\int_Q \varrho\,(\partial_t y\, v + \nabla_x y \cdot \nabla_x v) + p\, v \,\mathrm{d}Q = 0,$$
$$\int_Q (-\partial_t p\, q) + \nabla_x p \cdot \nabla_x q - y\, q \,\mathrm{d}Q = -\int_Q y_d\, q \,\mathrm{d}Q,$$

for all $v, q \in V = L_2(0,T; H_0^1(\Omega))$, where
$Y_0 = \{y \in W(0,T) : y = 0 \text{ on } \Sigma_0\}$, P_T analogously,
and the control u was eliminated via the gradient equation

$$u + \varrho\, p = 0.$$

First Order Optimality Conditions (contd.)

We observe *maximal parabolic regularity* (m.p.r.) holds for y and p, i.e, $\partial_t y \in L_2(Q)$ and $\Delta_x y \in L_2(Q)$, etc.

Then: the solution $(y, p) \in (Y_0 \cap H^{L,1}(Q)) \times (P_T \cap H^{L,1}(Q))$ satisfies

$$\left.\begin{aligned} \varrho(\partial_t y - \Delta_x y) + p &= 0, \\ -\partial_t p - \Delta_x p &= y_d - y, \end{aligned}\right\} \text{ in } L_2(Q),$$

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The main idea

Recall:

- time is just another variable
- the solution $(y, p) \in (Y_0 \cap H^{L,1}(Q)) \times (P_T \cap H^{L,1}(Q))$ satisfies

$$\left.\begin{aligned} \varrho(\partial_t y - \Delta_x y) + p = 0, \\ -\partial_t p - \Delta_x p = y_d - y, \end{aligned}\right\} \text{ in } L_2(Q),$$

- treat $\partial_t y$ as a (strong) convection in time-direction

The main idea (contd.)

- decompose Q into shape-regular finite elements $K \in \mathcal{T}_h$,
- define conforming finite element spaces $Y_{0h} \subset Y_0, P_{Th} \subset P_T$,
- global upwind (downwind) test functions

$$v_h(x,t) + \theta\, \lambda_h^2 \partial_t v_h(x,t), \text{ for all } (x,t) \in K,$$
$$q_h(x,t) - \theta\, \lambda_h^2 \partial_t q_h(x,t), \text{ for all } (x,t) \in K,$$

  with θ positive parameter, and $\lambda_h \in W^{1,\infty}(Q)$ globally cont.
- multiply the KKT system in $L_2(Q)$ by the resp. test functions and integrate,
- integration by parts and boundary conditions

The (discretized) first order optimality system

The solution (y, p) satisfies the consistency identity

$$a_h(y,p;v_h,q_h) = \ell_h(v_h,q_h) \quad \forall (v_h,q_h) \in Y_{0h} \times P_{Th},$$

with the combined bilinear and linear forms

$$\begin{aligned}
a_h(y,p;v,q) \;&=\; \sum_{K\in\mathcal{T}_h} \int_K \Big[\varrho\big(\partial_t y\, v + \theta\,\lambda_h^2 \partial_t y \partial_t v + \nabla_x y \cdot \nabla_x v - \theta\,\lambda_h^2 \Delta_x y\, \partial_t v\big) \\
&\qquad + p(v + \theta\,\lambda_h^2 \partial_t v) - \partial_t p\, q + \theta\,\lambda_h^2 \partial_t p \partial_t q + \nabla_x p \cdot \nabla_x q \\
&\qquad + \theta\,\lambda_h^2 \Delta_x p\, \partial_t q - y(q - \theta\,\lambda_h^2 \partial_t q)\Big]\,\mathrm{d}K \quad \text{and} \\
\ell_h(v,q) \;&=\; -\sum_{K\in\mathcal{T}_h} \int_K y_d(q - \theta\,\lambda_h^2 \partial_t q)\,\mathrm{d}K,
\end{aligned}$$

The (discretized) first order optimality system

Find $(y_h, p_h) \in Y_{0h} \times P_{Th}$ such that

$$a_h(y_h, p_h; v_h, q_h) = \ell_h(v_h, q_h) \quad \forall (v_h, q_h) \in Y_{0h} \times P_{Th},$$

with the combined bilinear and linear forms

$$\begin{aligned} a_h(y,p;v,q) \quad &= \quad \sum_{K \in \mathcal{T}_h} \int_K \Big[\varrho \big(\partial_t y\, v + \theta\, \lambda_h^2 \partial_t y \partial_t v + \nabla_x y \cdot \nabla_x v - \theta\, \lambda_h^2 \Delta_x y\, \partial_t v \big) \\ &\qquad + p(v + \theta\, \lambda_h^2 \partial_t v) - \partial_t p\, q + \theta\, \lambda_h^2 \partial_t p \partial_t q + \nabla_x p \cdot \nabla_x q \\ &\qquad + \theta\, \lambda_h^2 \Delta_x p\, \partial_t q - y(q - \theta\, \lambda_h^2 \partial_t q) \Big] \,\mathrm{d}K \quad \text{and} \\ \ell_h(v,q) \quad &= \quad - \sum_{K \in \mathcal{T}_h} \int_K y_d (q - \theta\, \lambda_h^2 \partial_t q) \,\mathrm{d}K, \end{aligned}$$

Existence & Uniqueness

$$\|(v_h,q_h)\|_h^2 := \varrho\left(\sum_{K\in\mathcal{T}_h}\left[\|\nabla_x v_h\|_K^2+\theta\|\lambda_h\partial_t v_h\|_K^2\right]+\frac{1}{2}\|v_h\|_{L_2(\Sigma_T)}^2\right)$$
$$+\sum_{K\in\mathcal{T}_h}\left[\|\nabla_x q_h\|_K^2+\theta\|\lambda_h\partial_t q_h\|_K^2\right]+\frac{1}{2}\|q_h\|_{L_2(\Sigma_0)}^2.$$

Lemma (Coercivity on the FE space)

There exits a constant μ_c *such that*

$$a_h(v_h,q_h;v_h,q_h)\ge\mu_c\|(v_h,q_h)\|_h^2,\quad\forall(v_h,q_h)\in Y_{0h}\times P_{Th},$$

with $\mu_c=1/2$ *for* $0<\theta\le\frac{1}{2}\min\{\frac{\varrho}{\varrho a+b},\frac{1}{a+b}\}$, *with* $a=\frac{1}{2}\overline{\lambda}_0^{-2}c_{\mathrm{div}}^2$, *and* $b=\overline{\lambda}_0\lambda_1 hC_{\mathrm{F}\Omega}(\nu)^2$, *where* $c_{\mathrm{div}}=\max_{K\in\mathcal{T}_h}c_{\mathrm{div},K}$.

Existence & Uniqueness

Coercivity $\Rightarrow$ Uniqueness $\Rightarrow$ Existence

Solve

$$\mathbf{K}_h \begin{pmatrix} \mathbf{y}_h \\ \mathbf{p}_h \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{f}_h \end{pmatrix},$$

with

$$\mathbf{K}_h = \begin{pmatrix} \mathbf{K}_{yy} & \mathbf{K}_{yp} \\ \mathbf{K}_{py} & \mathbf{K}_{pp} \end{pmatrix},$$

where $\mathbf{K}_h \neq \mathbf{K}_h^\top$, but $\mathbf{K}_h$ **positive definite**!

A localized space-time upwind scheme

A priori discretization error estimates

Generalized boundedness

$$\|(v,q)\|_{h,*}^2 := \|(v,q)\|_h^2 + \varrho \sum_{K\in\mathcal{T}_h} \theta\|\lambda_h \Delta_x v\|_K^2 + 3\varrho\,\theta^{-1}\|\lambda_h^{-1} v\|_Q^2 + \sum_{K\in\mathcal{T}_h} \theta\|\lambda_h \Delta_x q\|_K^2 + 3\,\theta^{-1}\|\lambda_h^{-1} q\|_Q^2$$

Lemma

Let $(y,p) \in (Y_{0h} + (Y_0 \cap H^{L,1}(Q))) \times P_{Th} + (P_T \cap H^{L,1}(Q))$*.*
There exits a constant μ_b *such that*

$$|a_h(y,p;v_h,q_h)| \le \mu_b \|(y,p)\|_{h,*}\|(v_h,q_h)\|_h, \ \forall (v_h,q_h) \in Y_{0h} \times P_{Th}.$$

A best approximation estimate

Lemma (Céa-like)

Let θ be chosen such that the f.e. scheme is coercive and bounded. Then the best approximation estimate

$$\|(y-y_h,p-p_h)\|_h \leq \inf_{(v_h,q_h)\in Y_{0h}\times P_{Th}} \Big(\|(y-v_h,p-q_h)\|_h + \frac{\mu_b}{\mu_c}\|(y-v_h,p-q_h)\|_{h,*}\Big),$$

holds, with (y_h,p_h) the solution of the finite element scheme.

A priori discretization error estimate

Theorem

Let the solution y and the adjoint solution p belong to $H^\ell(Q)$, with $\ell > (d+1)/2$. Let $(y_h, p_h) \in Y_{0h} \times P_{Th}$ be the solution of the f.e. scheme. Then we have the a priori discretization error estimate

$$\|(y - y_h, p - p_h)\|_h \leq \left(\sum_{K \in \mathcal{T}_h} c_K(y,p) h_K^{2(s-1)} \right)^{1/2}$$

with $s = \min\{k+1, \ell\}$, where k is the polynomial degree of the finite element functions.

Note: Can be extended to $\ell \leq (d+1)/2$.

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Marking algorithm

Dörfler marking:

- Determine set $\mathcal{M}$ of (almost) minimal cardinality such that

$$\Xi \sum_{K \in \mathcal{T}_h} \eta_K(y_h, p_h)^2 \leq \sum_{K \in \mathcal{M}} \eta_K(y_h, p_h)^2,$$

  for given $\Xi \in (0,1)$.
- efficient realization as a search[1]

[1] Pfeiler, C.-M., and Praetorius, D. Dörfler marking with minimal cardinality is a linear complexity problem. *Mathematics of Computation 89*, 326 (2020), 2735–2752.

A posteriori error estimation

Error localization

Residual error indicator

Langer, Steinbach, Tröltzsch and Yang ('21) proposed the *error indicator*

$$(\eta_K)^2 := h_K^2 \|p_h + \varrho(\partial_t y_h - \Delta_x y_h)\|_K^2 + h_K \|[\![\nabla_x y_h]\!]\|_{\partial K}^2 \\ + h_K^2 \|y_d - y_h - \partial_t p_h - \Delta_x p_h\|_K^2 + h_K \|[\![\nabla_x p_h]\!]\|_{\partial K}^2$$

BUT!

$$\underline{c}\|(y - y_h, p - p_h)\|^2 \overset{?}{\leq} \sum_{K \in \mathcal{T}_h} (\eta_K)^2 \overset{?}{\leq} \overline{C}\|(y - y_h, p - p_h)\|^2$$

New Functional Error Estimator

Using the continuous inf-sup condition[2], we get the guaranteed upper bound

$$
\begin{aligned}
\frac{1}{\sqrt{2}}\|(y-\tilde{y},p-\tilde{p})\|_X &\leq \sup_{(v,q)\in V\times V} \frac{a((y-\tilde{y},p-\tilde{p});(v,q))}{\sqrt{\varrho\|v\|_V^2+\|q\|_V^2}} \\
&\leq \cdots \leq \mathfrak{M}_{\oplus}(\boldsymbol{\tau},\boldsymbol{\sigma},\tilde{y},\tilde{p}),
\end{aligned}
$$

with $\boldsymbol{\tau},\boldsymbol{\sigma}\in H(\mathrm{div}_x,Q)$ and $\tilde{y},\tilde{p}\in H^1(Q)$, and $\mathfrak{M}_{\oplus}(\boldsymbol{\tau},\boldsymbol{\sigma},\tilde{y},\tilde{p})$ *computable*.

[2] U. Langer, O. Steinbach, F. Tröltzsch and H. Yang, Unstructured space-time finite element methods for optimal control of parabolic equation, *SIAM Journal on Scientific Computing* **43** (2021), A744–A771.

New Functional Error Estimator (contd.)

Observe

$$\mathfrak{M}_{\oplus}(\boldsymbol{\tau},\boldsymbol{\sigma},\tilde{y},\tilde{p}) \leq \sqrt{2}\,\mathfrak{M}_{+}(\boldsymbol{\tau},\boldsymbol{\sigma},\tilde{y},\tilde{p}),$$

with

$$\begin{aligned}\mathfrak{M}_{+}^2(\boldsymbol{\tau},\boldsymbol{\sigma},v,q) = \varrho\left[\|\boldsymbol{\tau}-\nabla_x v\|_Q^2 + C_{\mathrm{F}\Omega}^2\|-\partial_t v + \mathrm{div}_x\,\boldsymbol{\tau} - \frac{1}{\varrho}q\|_Q^2\right] \\ + \left[\|\boldsymbol{\sigma}-\nabla_x q\|_Q^2 + C_{\mathrm{F}\Omega}^2\|-y_d + \partial_t q + v + \mathrm{div}_x\,\boldsymbol{\sigma}\|_Q^2\right]\end{aligned}$$

New Functional Error Estimator (contd.)

Realization: Compute

$$(\boldsymbol{\tau}_h^{(1)}, \boldsymbol{\sigma}_h^{(1)}) = \underset{(\boldsymbol{\tau}_h, \boldsymbol{\sigma}_h) \in [V_h]^d \times [V_h]^d}{\operatorname{argmin}} \mathfrak{M}_+^2(\boldsymbol{\tau}_h, \boldsymbol{\sigma}_h, y_h, p_h)$$

Error *estimator*

$$\eta_K^2 = \varrho \left[\|\boldsymbol{\tau}_h^{(1)} - \nabla_x y_h\|_K^2 + C_{\mathrm{F}\Omega}^2 \| -\partial_t y_h + \operatorname{div}_x \boldsymbol{\tau}_h^{(1)} - \frac{1}{\varrho} p_h \|_K^2 \right]$$
$$+ \left[\|\boldsymbol{\sigma}_h^{(1)} - \nabla_x p_h\|_K^2 + C_{\mathrm{F}\Omega}^2 \| -y_d + \partial_t p_h + y_g + \operatorname{div}_x \boldsymbol{\sigma}_h^{(1)} \|_K^2 \right]$$

since

$$\|(y - y_h, p - p_h)\|_X \leq \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2} = \mathfrak{M}_+(\boldsymbol{\tau}_h^{(1)}, \boldsymbol{\sigma}_h^{(1)}, y_h, p_h)$$

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Key information

- Space-time FEM implemented in MFEM,
- Linear system solved by means of flexible GMRES (FGMRES),
  - **STOP:** residual reduction by 10^{-8},
- Block-diagonal AMG preconditioner, i.e.,

$$\mathbf{C}_h = \mathsf{blockdiag}(\mathbf{C}_{yy}, \mathbf{C}_{pp}),$$

  with $\mathbf{C}_{yy} = AMG(\mathbf{K}_{yy})$ and $\mathbf{C}_{pp} = AMG(\mathbf{K}_{pp})$,
- AMG provided by *hypre*
- compare *efficiency index*

$$\mathrm{I}_{\text{eff}} = \frac{\left(\sum_{K\in\mathcal{T}_h} \eta_K^2\right)^{1/2}}{\|(y - y_h, p - p_h)\|}$$

- parallel tests run on `Quartz` (LLNL)

Key information (contd.)

Standard V-cycle AMG

- **A**lgebraic **M**ulti**G**rid
- black box, easy to apply
- but does not perform "optimally"

Non-linear variable step AMLI-cycle AMG
(together with P.S. Vassilevski)

- **A**lgebraic **M**ulti**L**evel **I**terations
- generalization of the MG W-cycle
- main difference to V-cycle multigrid
  - instead of applying MG recursively, apply MG-preconditioned (F)GMRES

Numerical Results

Smooth problem with manufactured solution

Problem description

Exact state, co-state & control (L_2-regularization)

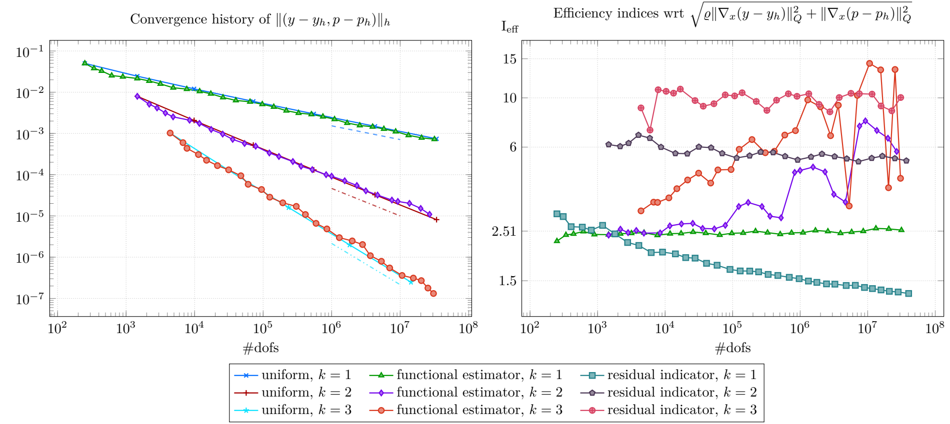
$$y(x,t) = (a\,t^2 + b\,t)\left(\sum_{i=1}^{d} \sin(x_i\,\pi)\right),$$

$$p(x,t) = -\varrho\left(2\,\pi^2\,a\,t^2 + (2\,\pi^2\,b + 2\,a)t + b\right)\left(\sum_{i=1}^{d} \sin(x_i\,\pi)\right),$$

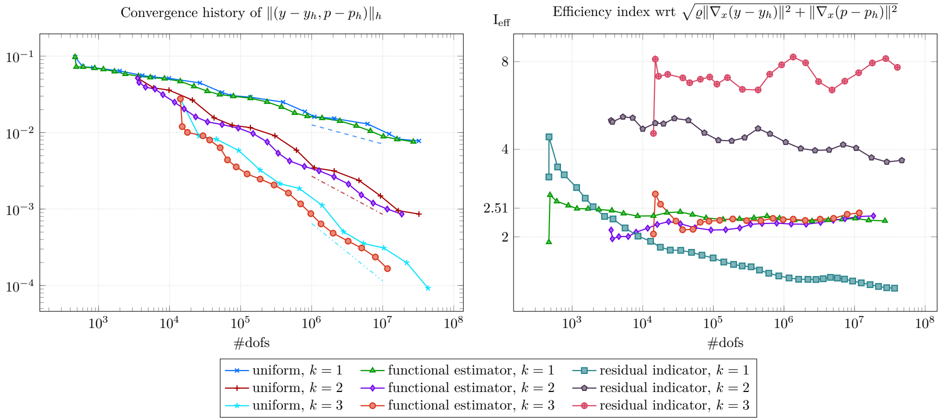
$$u(x,t) = \left(2\,\pi^2\,a\,t^2 + (2\,\pi^2\,b + 2\,a)t + b\right)\left(\sum_{i=1}^{d} \sin(x_i\,\pi)\right),$$

with $a = \frac{2\,\pi^2+1}{2\,\pi^2+2}$ and $b = 1$.

Convergence history & Efficiency indices $(d+1=3)$



Convergence history & Efficiency indices $(d+1=4)$

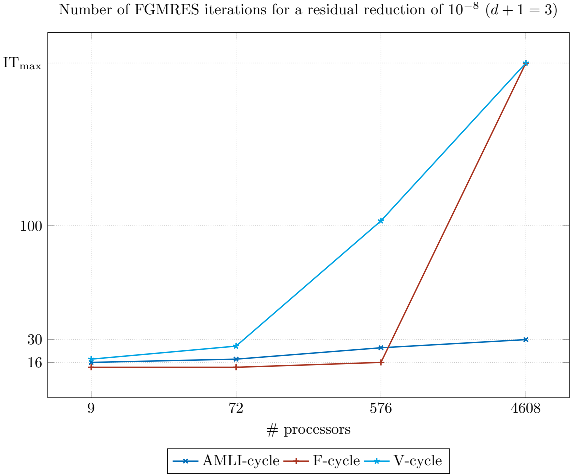


Solver performance ($d+1=3$)

Outer (F)GMRES iterations for $d+1=3$, with solution times in parantheses, using 1152 cores on `Quartz`; for $k=1$ and $\varrho=0.01$.

ℓ	#dofs	**AMLI-cycle**		**V-cycle**		**F-cycle**	
		0	**NI**	**0**	**NI**	**0**	**NI**
0	71 874	17 (0.33 s)	21 (0.41 s)	17 (0.88 s)	20 (0.93 s)	14 (1.03 s)	17 (1.07 s)
1	549 250	18 (0.79 s)	2 (0.09 s)	19 (0.63 s)	3 (0.51 s)	14 (0.96 s)	2 (0.57 s)
2	4 293 378	18 (2.24 s)	3 (0.43 s)	23 (5.84 s)	3 (5.14 s)	14 (6.12 s)	2 (4.29 s)
3	33 949 186	22 (5.51 s)	2 (0.52 s)	35 (9.01 s)	2 (8.70 s)	14 (8.96 s)	2 (7.07 s)
4	270 011 394	27 (25.43 s)	2 (1.78 s)	173 (91.71 s)	2 (12.51 s)	34 (36.90 s)	4 (13.99 s)

Solver performance $(d+1=3)$



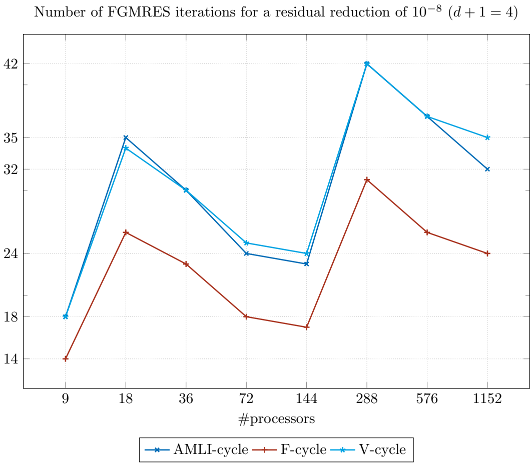
Local problem size $\sim 470\,000$ dofs.

Solver performance ($d + 1 = 4$)

Outer (F)GMRES iterations for $d + 1 = 4$, with solution times in parantheses, using 1152 cores on `Quartz`; for $k = 1$ and $\varrho = 0.01$.

ℓ	#dofs	**AMLI-cycle**		**V-cycle**		**F-cycle**	
		0	**NI**	**0**	**NI**	**0**	**NI**
0	79 458	18 (0.38 s)	23 (0.49 s)	17 (0.97 s)	22 (0.99 s)	14 (1.13 s)	18 (1.18 s)
4	832 834	21 (0.95 s)	4 (0.18 s)	20 (0.66 s)	4 (0.45 s)	16 (1.28 s)	3 (0.52 s)
8	12 422 274	26 (2.49 s)	5 (0.45 s)	28 (5.89 s)	5 (5.42 s)	20 (8.93 s)	3 (6.75 s)
12	193 208 578	32 (14.98 s)	5 (2.09 s)	35 (18.45 s)	6 (12.28 s)	24 (23.97 s)	4 (10.79 s)

Solver performance ($d+1=4$)



Local problem size $\sim 110\,000$ – $180\,000$ dofs.

Numerical Results

Discontinous target state

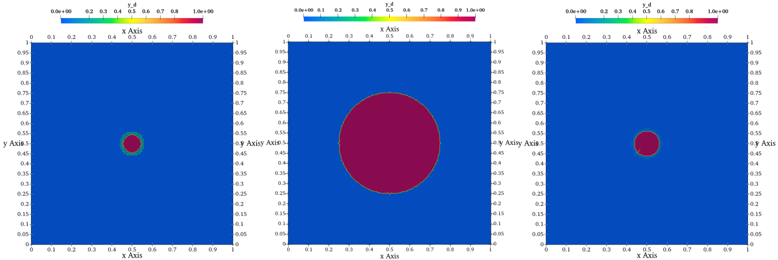
Problem description

Target state

$$y_d(x,t) = \begin{cases} 1 & \sqrt{|x-0.5|^2 + (t-0.5)^2} \leq \frac{1}{4} \\ 0 & \text{otherwise.} \end{cases}$$

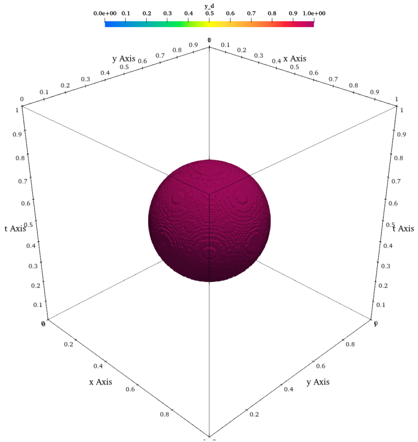
Exact solution and optimal control

Not known!



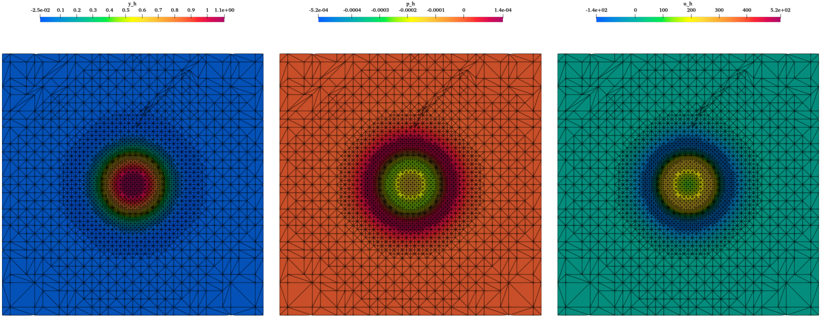
t = 0.254 t = 0.5 t = 0.742

Projection of the discontinuous target function y_d onto a finite element function on a refined mesh, for $d+1=3$; cuts through the space-time cylinder at $t = 0.3, 0.5, 0.7$.



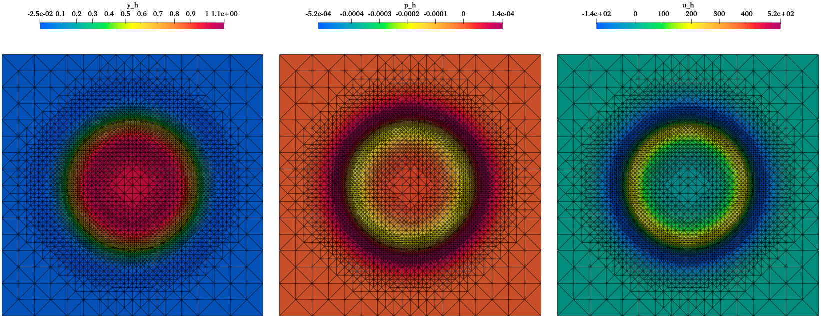
Projection of the discontinuous target function y_d onto a finite element function on a refined mesh, for $d + 1 = 3$; a contour plot of y_d in the space-time cylinder.

L_2-regularization $(d+1=3)$



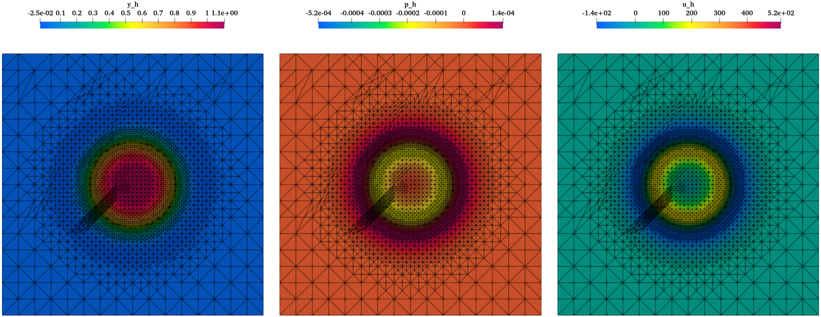
$t = 0.28125$

L_2-regularization $(d+1=3)$



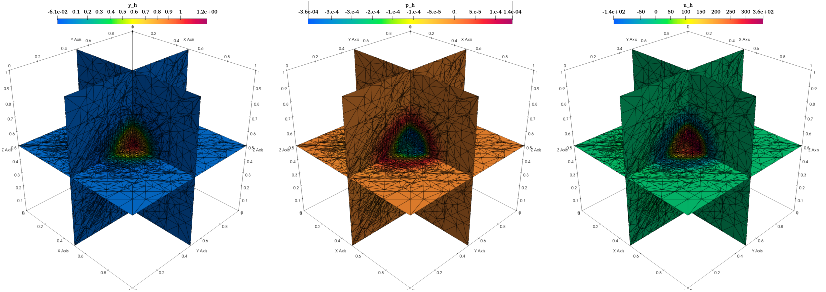
$t = 0.5$

L_2-regularization $(d+1=3)$



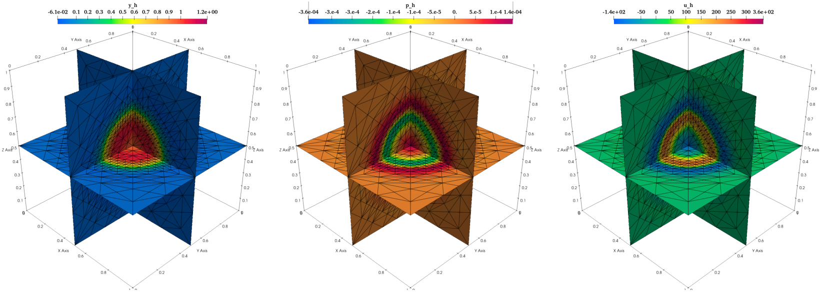
$$t = 0.6875$$

L_2-regularization $(d+1=4)$



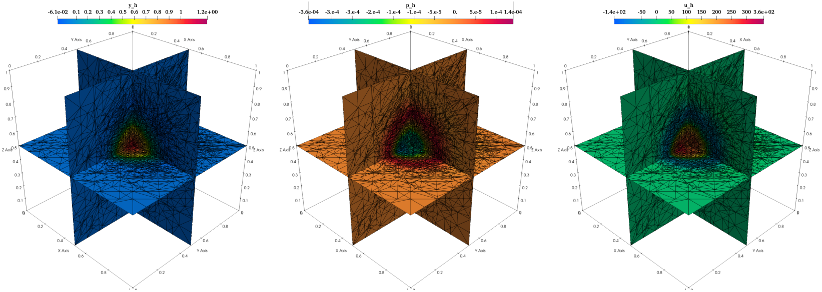
$t=0.3$

L_2-regularization $(d+1=4)$



$$t = 0.5$$

L_2-regularization $(d+1=4)$



$$t = 0.7$$

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Conclusions

Adaptive space-time Optimal Control:

- stable space-time FE scheme for all-at-once discretizations,
- localized a priori error estimates,
- a posteriori error techniques:
  - residual error **indicators**
  - functional error **estimators**
- numerical results up to $d + 1 = 4$,

Outlook & Work in progress

- Solvers
- Parallelization
- Energy regularization, i.e. $u \notin L_2(Q)$, but $u \in L_2(0,T;H^{-1}(\Omega))$
- Application to non-linear, (time-periodic) problems
- Goal-oriented a posteriori error estimates

References:

[1] Langer, U., and S., A. Adaptive space-time finite element methods for parabolic optimal control problems. *Journal of Numerical Mathematics* (2021). first published online by de Gruyter Nov. 3, 2021, https://doi.org/10.1515/jnma-2021-0059.

[2] Langer, U., and S., A. Simultaneous space-time finite element methods for parabolic optimal control problems. DK Report 2021-04, 2021. accepted for publication in the LSSC2021 proceedings, Lecture Notes in Computer Science (LNCS), v. 13127, Chapter 36.

[3] S., A. *Space-time Finite Element Methods*. PhD thesis, Johannes Kepler University Linz, 2021.

Thank you!

Appendix

Some remarks on θ

- on linear elements, i.e, $k = 1$: $a = 0$, i.e.,

$$\mu_c = \frac{1}{2} \quad \text{for} \quad 0 < \theta \leq \frac{1}{2} \min \left\{ \frac{\varrho}{\overline{\lambda}_0 \lambda_1 h C_{\mathrm{F}\Omega}(\nu)^2}, \frac{1}{\overline{\lambda}_0 \lambda_1 h C_{\mathrm{F}\Omega}(\nu)^2} \right\}$$

- on uniform meshes, i.e., $h_K \equiv h$ for all $K \in \mathcal{T}_h$: $b = 0$, i.e.

$$\mu_c = \frac{1}{2} \quad \text{for} \quad \theta = \frac{1}{\overline{\lambda}_0^2 c_{\mathrm{div}}^2},$$

and $\lambda_h \equiv h$ for all $K \in \mathcal{T}_h$

The mesh density function λ_h

Let $\lambda_h \in W^{1,\infty}(Q)$ be globally continuous, with

$$0 < \underline{\lambda}_0 \, h_K \leq \lambda_h(x,t) \leq \overline{\lambda}_0 \, h_K \quad \text{and} \quad |\partial_t \lambda(x,t)| \leq \lambda_1,$$

for a.e. $(x,t) \in K$, $K \in \mathcal{T}_h$, where $\underline{\lambda}_0, \overline{\lambda}_0, \lambda_1$ positive.

The mesh density function λ_h (contd.)

How to realize λ_h? Let

$$\lambda_h(x,t) = \sum_{i=1}^{n_h} h_i \, \varphi_i(x,t),$$

where

- n_h is the number of vertices of $\mathcal{T}_h$,
- φ_i is the linear f.e. function associated with vertex $\boldsymbol{x}_i$,
- h_i is the averaged edge length in the vertex $\boldsymbol{x}_i$, i.e.,

$$h_i = \frac{1}{|E(i)|} \sum_{e \in E(i)} h_e,$$

with $E(i) = \{e \subset \mathcal{T}_h : e \text{ is an edge} \wedge \overline{e} \cap \boldsymbol{x}_i \neq \emptyset\}$.

Work in progress

Energy regularization:

- $u \notin L_2(Q)$, but $u \in L_2(0,T;H^{-1}(\Omega))$, thus

$$J(y,u) = \int_Q |y - y_d|^2 \,\mathrm{d}Q + \frac{\varrho}{2}\|u\|^2_{L_2(0,T;H^{-1}(\Omega))}$$

- we know[3]

$$\|u\|^2_{L_2(0,T;H^{-1}(\Omega))} = \|\nabla_x w_u\|^2_Q = \langle u, w_u\rangle_Q$$

where $w_u \in V = L_2(0,T;H^1_0(\Omega))$ is the unique solution of

$$\int_Q \nabla_x w_u \cdot \nabla_x v \,\mathrm{d}Q = \langle u, v\rangle_Q \quad \forall v \in V.$$

Manufactured smooth solution: