Why Multigrid Algorithms in Time are Different

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The parabolic multigrid algorithm of Hackbusch from 1984 was the first multigrid algorithm for time dependent problems. It uses a sequential Jacobi smoother in time and has excellent convergence properties when coarsening in space only is used. Time coarsening however degrades performance rapidly. A continuous variant of the parabolic multigrid algorithm is the multigrid waveform relaxation method by Lubich and Osterman (1987). The first to address the problem with coarsening in time were Horton and Vandewalle (1995): they identified the strong anisotropy of the space-time operator as the critical issue, and proposed to use special forward interpolation operators in the time direction. A new space-time multigrid method (STMG) for parabolic problems was introduced by Gander and Neumüller (2016) for general Discontinuous Galerkin discretizations in time. This method has excellent strong and weak scaling properties up to hundred-thousands of cores, and is already faster than the best parallelized spatial multigrid method when using only 2 processors for such parabolic problems. What are the essential ingredients for this performance?

For a simple one dimensional heat equation, discretized by centered differences in space and Backward Euler in time, I will first show why time coarsening in the parabolic multigrid method fails. I will then apply the method to the Dahlquist equation, where time coarsening is successful. This reveals that the essential ingredient necessary for coarsening in time is a block Jacobi smoother. A simple Fourier local analysis finally shows how to choose the relaxation parameter in the block Jacobi smoother, and how to perform the space and time coarsening in STMG, in order to obtain its performance.