Preconditioner for block lower triangular Toeplitz matrices

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Preconditioning a Toeplitz matrix B by a circulant (or α-circulant) matrix C is a very old idea (dating back to the 1980s and the work by Gilbert Strang), and the properties of the preconditioned matrix $C^{-1}B$ are well studied. This idea works in general because C is an optimal approximation of B in a certain sense. The reason for preconditioning B by C lies in the fact that we can diagonalize C as $C = \mathrm{F}^*D\mathrm{F}$ with F being the discrete Fourier matrix, and thus the preconditioning step $C^{-1}r$ can be solved efficiently via FFT. However, when B is embedded in a larger matrix $\boldsymbol{K}$, as is encountered in many places, replacing B by C does not necessarily produce a good preconditioner of $\boldsymbol{K}$. There is no guarantee for neither fast implementation of the preconditioning step nor rapid convergence of the preconditioned solver. Preconditioning a block Toeplitz matrix (with *non*-Toeplitz blocks) is a very recent area of research, and only few results are available so far. In this talk, we will introduce our most recent progress for preconditioning $\boldsymbol{K}$ and $\boldsymbol{K}\boldsymbol{K}^\top$, when $\boldsymbol{K}$ is a block *lower triangular* matrix arising from the all-at-once formulation of evolution problems. The preconditioners can be used in a PinT mode and block FFT is applicable. We prove discretization- and problem-independent eigenvalue bounds for the preconditioned matrix, which are sharp and hold for *any* stable one-step time-integrator.