On the design of global-in-time Newton-Multigrid-Pressure Schur complement solvers for incompressible flow problems

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Keywords: *incompressible Navier-Stokes equations, global-in-time, Pressure Schur complement, Newton's method, space-time multigrid, time-simultaneous*

The work to be presented in this talk focuses on a new global-in-time solution strategy for incompressible flow problems which highly exploits the Pressure Schur complement (PSC) approach [3] for the construction of a space-time multigrid algorithm. For linear problems like the incompressible Stokes equations discretized in space using an inf-sup-stable finite element pair, the fundamental idea is to block the individual linear systems of equations at each time step into a single all-at-once saddle point problem for all velocity and pressure unknowns. Then the Pressure Schur complement can be used to eliminate the velocity fields and set up a linear system for all pressure variables only. This algebraic manipulation allows the construction of parallel-in-time preconditioners for the corresponding all-at-once Picard iteration by extending frequently used sequential PSC preconditioners in a straightforward manner (cf. [1]). We show that those preconditioners can be applied very efficiently on modern high performance computing facilities and are asymptotically exact in the limit of vanishing time increments.

To accelerate the convergence of the proposed fixed-point iteration, this iterative solver is embedded as a smoother into a space-time multigrid algorithm, where the computational complexity of the coarse grid problem highly depends on the coarsening strategy in space and/or time. Although coarsening in space using commonly used FE intergrid transfer operators is possible, most promising results could be obtained by only coarsening in time using tailor-made prolongation and restriction operators. This procedure even allows the efficient solution of the nonlinear Navier-Stokes equations for many time steps by employing Newton's method for linearization.

At the end, the presented multigrid solution strategy only requires the efficient solution of time-dependent linear convection-diffusion-reaction equations and many (highly parallelizable) Poisson problems. The potential of this approach for CFD simulations with large time horizons on modern HPC architectures is illustrated in numerical examples.

References

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