# Global unique solutions for the inhomogeneous Navier-Stokes equation with only bounded density, in critical regularity spaces

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# System

- We focus on the following inhomogeneous incompressible Navier-Stokes system:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mu \Delta u + \nabla P = 0, \\ \mathrm{div}\, u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases} \tag{INS}$$

where $u = u(t,x) \in \mathbb{R}^d$ with $t \geq 0$ and $x \in \mathbb{R}^d$ denotes the velocity field, $P = P(t,x) \in \mathbb{R}$, the pressure and $\rho = \rho(t,x) \in \mathbb{R}^+$, the density. $\mu$ is the viscosity constant.

# Introduction

- Initial data: $\rho_0$ is in $L_\infty(\mathbb{R}^d)$, $u_0$ is divergence free satisfy

$$\|\rho_0 - 1\|_{L_\infty(\mathbb{R}^d)} < c, u_0 \in \dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d) \quad \text{with} \quad 1 < p < d \qquad (1)$$

- Energy balance:

$$\frac{1}{2}\|\sqrt{\rho(t)}u(t)\|^2_{L_2} + \mu \int_0^t \|\nabla u(t)\|^2_{L_2}\, d\tau = \frac{1}{2}\|\sqrt{\rho_0}u_0\|^2_{L_2}. \qquad (2)$$

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- Previous results:
  - Initial density in Besov space $\dot{B}_{p,1}^{\frac{d}{p}}(\mathbb{R}^d)$: R.Danchin (2003), H.Abidi(2007), H.Abidi and M.Paicu(2007), H.Abidi and G.Gui(2021)...
  - Initial density in $L_\infty(\mathbb{R}^d)$:J.Huang, M.Paicu and P.Zhang (2013), M.Paicu, P.Zhang and Z.Zhang (2013), D.Chen, Z.Zhang and W.Zhao (2016), R.Danchin and P.B.Mucha (2019), P.Zhang(2020)...

# Introduction

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For System (INS), it can be rewritten as

$$u_t - \mu\Delta u + \nabla P = -(\rho - 1)u_t + \rho u \cdot \nabla u \cdot$$

So, to achieve our goals, we start with the following Stokes system:

$$\begin{cases} u_t - \mu\Delta u + \nabla P = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{S}$$

# Tools and Approach

The following maximal regularity property of the Stokes system has been pointed out by R. Danchin, P.B. Mucha and P.Tolksdorf (2021).

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The following maximal regularity property of the Stokes system has been pointed out by R. Danchin, P.B. Mucha and P.Tolksdorf (2021).

## Proposition (Maximal regularity property)

*Let $1 < p, q < \infty$ and $1 \leq r \leq \infty$. Then, for any $u_0 \in \dot{B}^{2-2/q}_{p,r}(\mathbb{R}^d)$ with $\operatorname{div} u_0 = 0$, and any $f \in L_{q,r}(\mathbb{R}_+; L_p(\mathbb{R}^d))$, the system (S) has a unique solution $(u, \nabla P)$ with*

$$u \in \mathcal{C}(\mathbb{R}_+; \dot{B}^{2-2/q}_{p,r}(\mathbb{R}^d)) \quad \textit{and} \quad u_t, \nabla^2 u, \nabla P \in L_{q,r}(\mathbb{R}_+; L_p(\mathbb{R}^d)),$$

*and the following inequality holds true for some constant $C$ independent of $T$:*

$$\mu^{1-1/q}\|u\|_{L_\infty(\mathbb{R}_+;\dot{B}^{2-2/q}_{p,r}(\mathbb{R}^d))} + \|u_t, \mu\nabla^2 u, \nabla P\|_{L_{q,r}(\mathbb{R}_+;L_p(\mathbb{R}^d))}$$
$$\leq C\big(\mu^{1-1/q}\|u_0\|_{\dot{B}^{2-2/q}_{p,r}(\mathbb{R}^d)} + \|f\|_{L_{q,r}(\mathbb{R}_+ T;L_p(\mathbb{R}^d))}\big).$$

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# Tools and Approach

Furthermore, if $2/q + d/p > 2$, then for all $s \in (q, \infty)$ and $m \in (p, \infty)$ such that $1 + \frac{d}{2}(\frac{1}{m} - \frac{1}{p}) > 0$ and

$$\frac{d}{2m} + \frac{1}{s} = \frac{1}{q} + \frac{d}{2p} - 1,$$

it holds that

$$\mu^{1+\frac{1}{s}-\frac{1}{q}} \|u\|_{L_{s,r}(\mathbb{R}_+;L_m(\mathbb{R}^d))} \leq C\big(\mu^{1-1/q}\|u\|_{L_\infty(\mathbb{R}_+;\dot{B}^{2-2/q}_{p,r}(\mathbb{R}^d))} + \|u_t, \mu\nabla^2 u\|_{L_{q,r}(\mathbb{R}_+;L_p(\mathbb{R}^d))}\big).$$

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# Results in the 2D case

Denote

$$\dot{W}^{2,1}_{p,(q,r)}(\mathbb{R}_+ \times \mathbb{R}^d) := \left\{u \in \mathcal{C}(\mathbb{R}_+; \dot{B}^{2-2/q}_{p,r}(\mathbb{R}^d)) : u_t, \nabla^2 u \in L_{q,r}(\mathbb{R}_+; L_p(\mathbb{R}^d))\right\}. \quad (3)$$

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## Theorem (2D case)

*Let $p \in (1,2)$ and $q$ be defined by $1/q + 1/p = 3/2$. Denote by $s$ and $m$ the conjugate Lebesgue exponents of $p$ and $q$, respectively. Assume that the initial density is in $L_\infty(\mathbb{R}^2)$ and the initial divergence-free velocity $u_0$ is in $\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)$, and satisfy (1), then (INS) has a unique global-in-time solution $(\rho, u, \nabla P)$ satisfying the energy balance (2), $u \in \dot{W}^{2,1}_{p,(q,1)}(\mathbb{R}_+ \times \mathbb{R}^2)$, $\nabla P \in L_{q,1}(\mathbb{R}_+; L_p(\mathbb{R}^2))$,*

$$\|\rho - 1\|_{L_\infty(\mathbb{R}_+ \times \mathbb{R}^2)} = \|\rho_0 - 1\|_{L_\infty(\mathbb{R}^2)} < c.$$

# Results in the 2D case

Moreover, we obtained the following properties:

- $\nabla u \in L_1(\mathbb{R}_+; L_\infty(\mathbb{R}^2))$ and $u \in L_2(\mathbb{R}_+; L_\infty(\mathbb{R}^2))$;
- $tu \in L_\infty(\mathbb{R}_+; \dot{B}^{1+2/m}_{m,1}(\mathbb{R}^2))$;
- $\big(u, (tu)_t, \nabla^2(tu), \nabla(tP)\big) \in L_{s,1}(\mathbb{R}_+; L_m(\mathbb{R}^2))$;
- $t\dot{u} \in \dot{W}^{2,1}_{p,(q,1)}(\mathbb{R}_+ \times \mathbb{R}^2)$ and $t\dot{u} \in L_2(\mathbb{R}_+; L_\infty(\mathbb{R}^2))$;
- $t^{\frac{k}{2}} \nabla^k u \in L_\infty(\mathbb{R}_+; L_2(\mathbb{R}^2))$ and $t^{\frac{k}{2}} \nabla^{k+1} u \in L_2(\mathbb{R}_+ \times \mathbb{R}^2)$ for $k = 0, 1, 2$,
- $t^{\frac{k+2}{2}} \nabla^k \dot{u} \in L_\infty(\mathbb{R}_+; L_2(\mathbb{R}^2))$ for $k = 0, 1$ and $t^{\frac{k+1}{2}} \nabla^k \dot{u} \in L_2(\mathbb{R}_+ \times \mathbb{R}^2)$ if $k = 0, 1, 2$,
- $t^{\frac{1}{2}} \nabla P \in L_2(\mathbb{R}_+ \times \mathbb{R}^2)$ and $t\nabla P \in L_\infty(\mathbb{R}_+; L_2(\mathbb{R}^2))$.

# Results in the 3D case

## Theorem (3D case)

*Let* $p \in (1,3)$ *and* $q \in (1,\infty)$ *such that* $3/p + 2/q = 3$. *There exists a positive constant* $c$ *such that if the initial density* $\rho_0$ *is in* $L_\infty(\mathbb{R}^d)$, $u_0$ *is divergence free satisfy* (1) *and* $u_0 \in L_2(\mathbb{R}^3)$ $(2 < p < 3)$ *with* $\|u_0\|_{\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)} < c\mu$, *then (INS) has a unique global-in-time solution* $(\rho, u, \nabla P)$ *with*

$$\nabla P \in L_{q,1}(\mathbb{R}_+; L_p(\mathbb{R}^3)) \quad \text{and} \quad u \in \dot{W}^{2,1}_{p,(q,1)}(\mathbb{R}_+ \times \mathbb{R}^3),$$

*satisfying the energy balance* (2) *if* $p > 2$,

$$\|\rho - 1\|_{L_\infty(\mathbb{R}_+ \times \mathbb{R}^3)} = \|\rho_0 - 1\|_{L_\infty(\mathbb{R}^3)} < c.$$

# Results in the 3D case

Furthermore, the following properties holds ture:

- $\nabla u \in L_1(\mathbb{R}_+; L_\infty(\mathbb{R}^3))$ and $u \in L_2(\mathbb{R}_+; L_\infty(\mathbb{R}^3))$;
- $(tu) \in W^{2,1}_{m,(s,1)}(\mathbb{R}_+ \times \mathbb{R}^3)$ and $t\nabla P \in L_{s,1}(\mathbb{R}_+; L_m(\mathbb{R}^3))$ for all $3 < m < \infty$ and $q < s < \infty$ such that $3/m + 2/s = 1$;
- $t\dot{u} \in \dot{W}^{2,1}_{p,(q,1)}(\mathbb{R}_+ \times \mathbb{R}^3)$;
- $(u, t\dot{u}) \in L_{s,1}(\mathbb{R}_+; L_m(\mathbb{R}^3))$.

# Thank you!