Boundary vorticity estimate for the Navier-Stokes equation and control of layer separation in the inviscid limit

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The equation

Consider the incompressible Navier-Stokes equation in a periodic tunnel $\Omega = [0,1] \times \mathbb{T}^2$:

$$\begin{cases} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla P^\nu = \nu \Delta u^\nu & \text{in } (0,T) \times \Omega \\ \text{div } u^\nu = 0 & \text{in } (0,T) \times \Omega \\ u^\nu = 0 & \text{on } (0,T) \times \partial\Omega \\ u_\nu(0,\cdot) = u_\nu^0 \ \varepsilon\text{-perturbation of } Ae_1 & \text{in } \Omega. \end{cases} \qquad (\mathrm{NSE}_\nu)$$

We are interesting in the inviscid limit $\nu \to 0$ under the condition that u_ν^0 converges to Ae_1 in $L^2(\Omega)$.

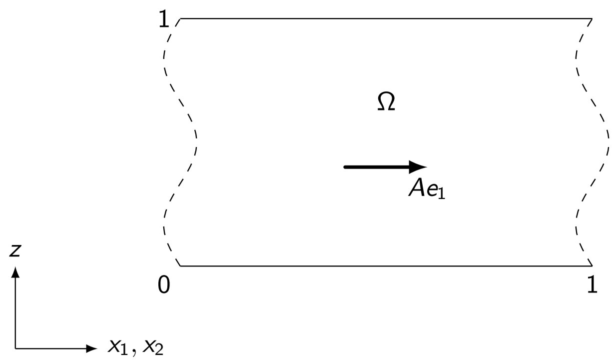

Figure: 3D Periodic Channel

Asymptotic limit

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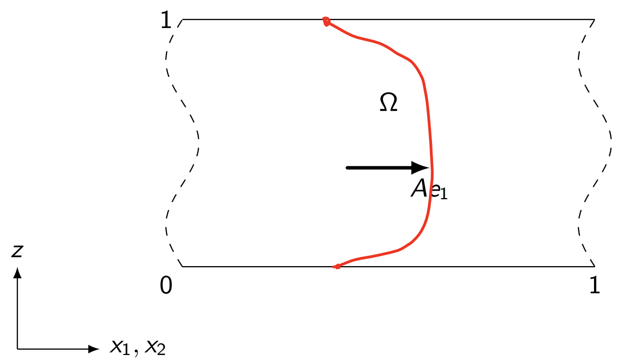
- If $u_\nu^0 = Ae_1$, then u^ν corresponds to the Prandtl Layer.



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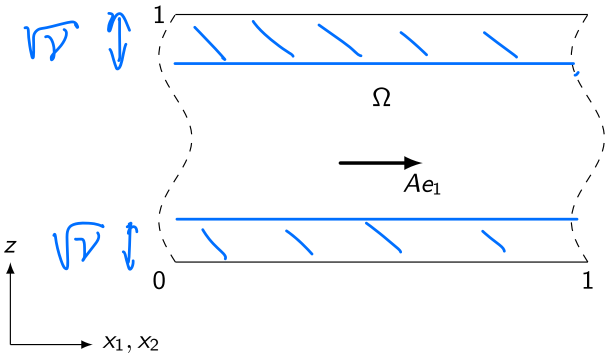

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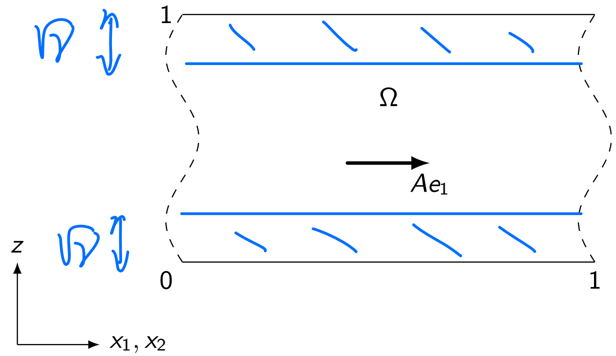

Figure: Prandtl Layer

- With perturbations on the initial values, only conditional results exist. The Kato criterion (1984) states that if, when $\nu \to 0$:

$$\int_0^T \int_{\{|z|<R\nu\}\cup\{|1-z|<R\nu\}} \nu |\nabla u^\nu|^2 \, dx_1 \, dx_2 \, dz \longrightarrow 0, \qquad \|u_\nu^0 - Ae_1\|_{L^2(\Omega)} \longrightarrow 0,$$

then

$$u^\nu \longrightarrow Ae_1, \text{ in } L^\infty(0,T;L^2(\Omega)).$$

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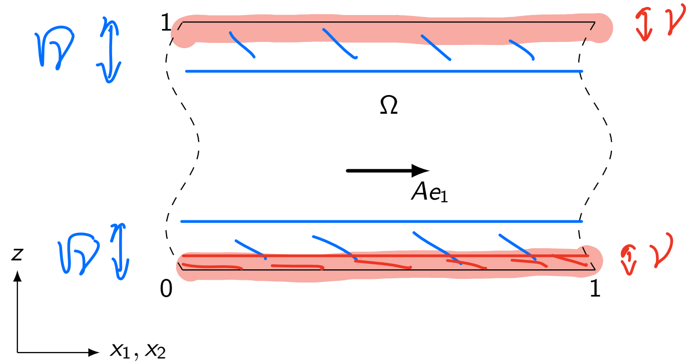

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Turbulence and layer separation

What if the limit does not hold ?

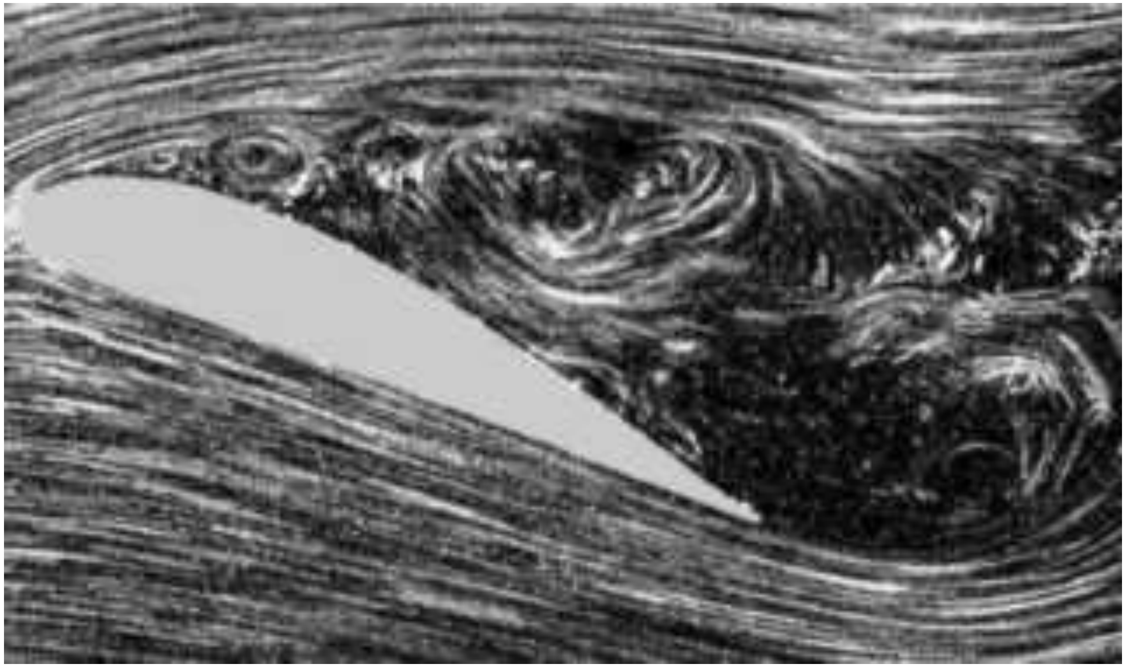

Figure: Turbulence and layer separation
the case of an airfoil

Prediction of layer separation

- Formally, the asymptotic system for $\nu = 0$ is the Euler system:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = 0 & \text{in } (0,T) \times \Omega \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega \\ u \cdot \mathrm{n} = 0 & \text{on } (0,T) \times \partial\Omega \\ u(0,\cdot) = Ae_1 & \text{in } \Omega. \end{cases} \tag{E}$$

- The method of convex integration shows that the solution $u(t,x) = Ae_1$ of (E) is not unique (see Székelyhidi, CRAS, 2011).
- For every constant $C < 2$, there exists a solution with layer separation for $T < 1/A$:

$$\|u(T) - Ae_1\|^2_{L^2(\Omega)} = CA^3 T.$$

- Is it the biggest separation possible ?
  Can we get some control of the layer separation as the level of the Navier-Stokes equation ?

The result

Theorem (V.-Yang, 2021)

For $d = 2, 3$, there exists a universal constant $C > 0$ such that for any $\bar{u}$ inviscid weak limit of sequences of Leray-Hopf solutions u^ν to (NSE_ν) with u_0^ν converging to Ae_1 in $L^2(\Omega)$, we have for almost every $T > 0$:

$$\|\bar{u}(T) - Ae_1\|_{L^2(\Omega)}^2 \leq CA^3 T.$$

- This corresponds to the layer separation predicted by the convex integration.

Non-uniqueness and pattern predictability

- In general, non uniqueness result by convex integration raised the question of predictability: Why can we observe patterns ?
- The shear flow $u = Ae_1$ has an energy of
$$\int_\Omega |u|^2\,dx = A^2,$$
while we prove that any *inviscid asymptotic* obtained by *double limit* has an energy at time T of at most CA^3T.
- Therefore, the perturbation always stays negligible on a time span $T \ll 1/A$. This is a large time for A small (small pattern).
- It predicts the lapse of time where the pattern stays predictable.

Previous work

- **Prandtl layer:** existence, stability, instability: Prandtl (1904),... W.E Engquist (97) Grenier (00), Gerard-Varet, dormy (10), Kukavica, Vicol (13), Grenier-Nguyen (18),...., Guo, Masmoudi Iyer (21)
- **Extensions of the Kato criterion:** Kato (84), Kelliher (08,09,17), Bardos Titi (07, 13), Temam Wang (98), Maekawa (14), Lopes Filho Mazzucato, Nussenzveig (08), Mazzucato taylor (08), Constantin Elgindi Ignatova Vicol (17) Constantin Vicol (18)...
- Our result is the **first** non conditional result in the turbulent regime.
- An important question is whether non-unique solution can be reached as limit of Navier-Stokes solutions.

  Note that the solutions constructed by Buckmaster-Vicol (Annals of Math 19) do not apply to this situation because:
  - we consider a bounded domain with boundary,
  - The Navier-Stokes solutions are *suitable*.

General idea

- Maekawa and Mazzucato (The inviscid limit and boundary layers for Navier-Stokes flows ,2018):
  "Mathematically, the main difficulty in the case of the no-slip boundary condition is the lack of a priori estimates on strong enough norms to pass to the limit, which in turn is due to the lack of a useful boundary condition for vorticity or pressure."
- We show a boundary vorticity control for the unscaled Navier-Stokes equation ($\nu = 1$) that is SCALABLE through the inviscid limit ($\nu \to 0$).

Why vorticity on the boundary ?

We have

$$\begin{aligned}\frac{d}{dt}\|u^\nu - Ae_1\|_{L^2}^2 &= \frac{d}{dt}\|u^\nu\|_{L^2}^2 - 2A\frac{d}{dt}\int_\Omega u_1^\nu \, dx\, dz \\ &\le -\nu\|\nabla u^\nu\|_{L^2}^2 + 2A\int_\Omega (\mathrm{div}(u^\nu u_1^\nu) + \partial_1 P - \nu\Delta u_1^\nu)\, dx \\ &\le -\nu\|\nabla u^\nu\|_{L^2}^2 + 2A\int_{\partial\Omega} (u_3^\nu u_1^\nu) - \nu\partial_3 u_1^\nu)\, dx \\ &\le -\nu\|\nabla u^\nu\|_{L^2}^2 - 2A\int_{\partial\Omega} \nu(\partial_3 u_1^\nu - \partial_1 u_3^\nu)\, dx \\ &\le -\nu\|\nabla u^\nu\|_{L^2}^2 - 2A\int_{\partial\Omega} \nu\omega_2^\nu \, dx,\end{aligned}$$

where $\omega^\nu = \mathrm{curl}\, u^\nu$ is the vorticity of u^ν.

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$$
\begin{aligned}
\frac{1}{2}\|u^\nu(T) - Ae_1\|_{L^2(\Omega)}^2 \leq\ & \frac{1}{2}\|u_0^\nu - Ae_1\|_{L^2(\Omega)}^2 - \int_{(0,T)\times\partial\Omega} \nu\, |\nabla u^\nu|^2\, dx\, dt \\
& - A\int_{(0,T)\times\partial\Omega} \nu\omega_2^\nu\, dx\, dt.
\end{aligned}
$$

Boundary vorticity estimate for Navier-Stokes

Theorem (Boundary Regularity)

Let Ω be a periodic channel of period W and height H. There exists a universal constant constant C depending only on the ratio W/H, such that the following holds. For any Leray-Hopf solution u to (NSE_1) in $(0,T)\times\Omega$, there exists a parabolic dyadic decomposition

$$(0,T)\times\partial\Omega = \bigcup_i \bar{Q}^i,$$

such that the following is true. Define the piecewise constant function $\tilde{\omega}:(0,T)\times\partial\Omega\to\mathbb{R}$ by

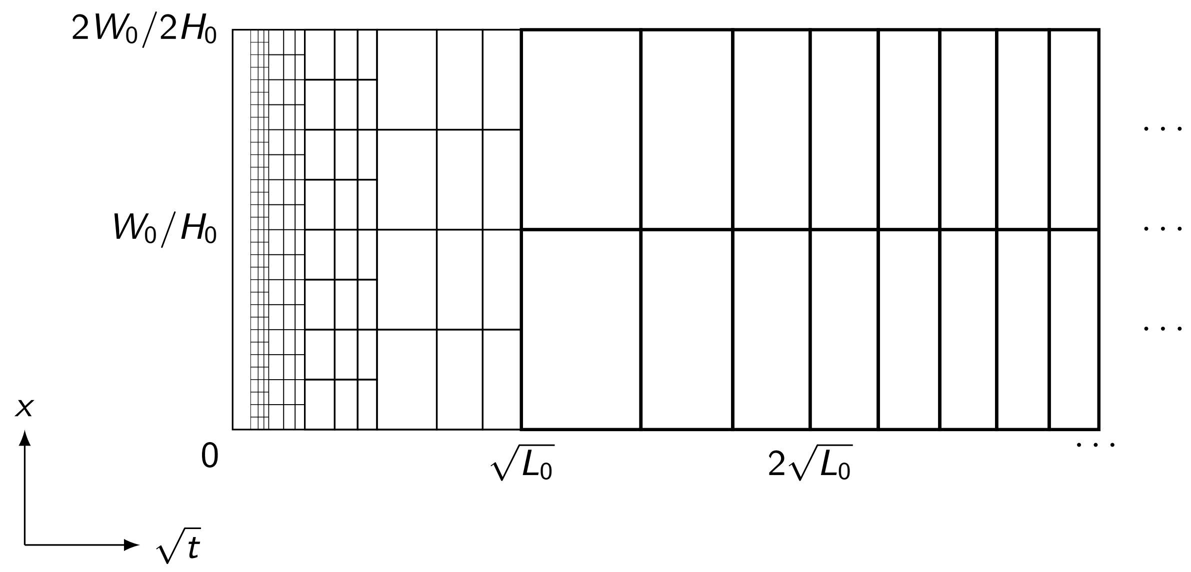
$$\tilde{\omega}(t,x) = \fint_{\bar{B}^i}\left|\fint_{s_i}^{t_i}\omega\,\mathrm{d}t\right|\mathrm{d}x', \qquad \text{for } (t,x)\in\bar{Q}^i = (s_i,t_i)\times\bar{B}^i.$$

Then we have:

$$\left\|\tilde{\omega}1_{\left\{|\tilde{\omega}|>\max\left\{\frac{1}{t},\frac{1}{W^2},\frac{1}{H^2}\right\}\right\}}\right\|^{3/2}_{L^{3/2,\infty}((0,T)\times\partial\Omega)} \le C\|\nabla u\|^2_{L^2((0,T)\times\Omega)}.$$

Smoothing local oscillations of the vorticity

Parabolic partition of $\partial\Omega \times [0, T]$:



$\tilde{\omega}$ **is the average of** ω **on each parabolic cylinder.**

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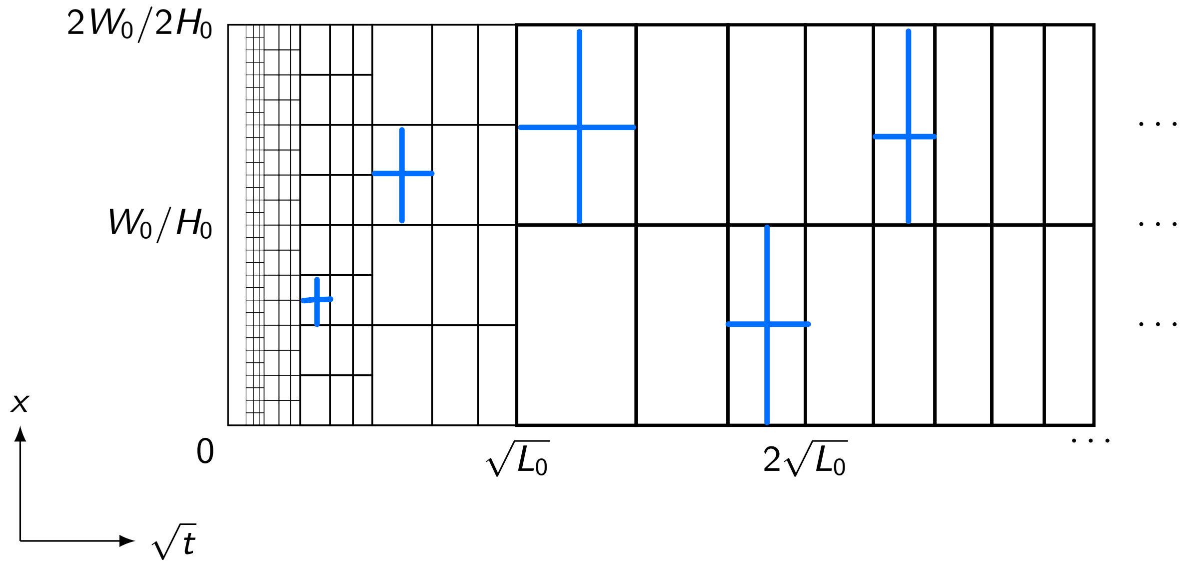

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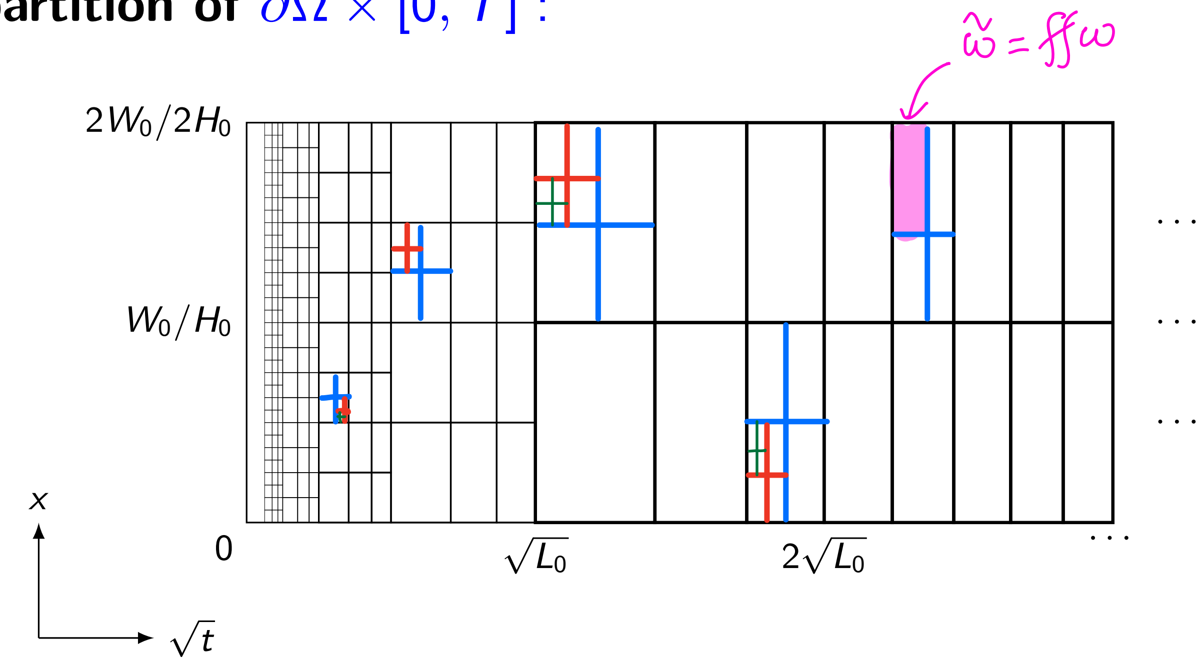

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Boundary vorticity estimate for Navier-Stokes (2)

- Up to the limit case, the theorem (almost) says that for u solution to Navier-Stokes with $\nu = 1$ in $(0, T/\nu) \times \Omega/\nu = (0, T_\nu) \times \Omega_\nu$:

$$\int_0^{T_\nu} \int_{\partial\Omega_\nu} |\tilde{\omega}|^{3/2} \, dx \, dt \leq C \int_0^{T_\nu} \int_{\Omega_\nu} |\nabla u|^2 \, dx \, dz \, dt.$$

- Considering $u^\nu(t, x) = u(t/\nu, x/\nu)$, this gives the estimates on solutions to(NSE$_\nu$):

$$\int_0^{T} \int_{\partial\Omega} |\nu\tilde{\omega}^\nu|^{3/2} \, dx \, dt \leq C \int_0^{T} \int_{\Omega} \nu |\nabla u^\nu|^2 \, dx \, dz \, dt.$$

- Therefore the boundary estimate is SCALABLE.

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- Therefore the boundary estimate is SCALABLE.
- This can be seen as an extention of the *a*-contraction theory first introduced for the stability of 1-D fluid mechanics (See for instance [Kang-V., *Inventiones*: 2021]).

How to conclude using the boundary estimate

- The main theorem can then be obtained as follows (up to a small time layer at $t = 0$):

$$
\begin{aligned}
&\frac{1}{2}\|u^\nu(T) - Ae_1\|^2_{L^2(\Omega)} - \frac{1}{2}\|u^\nu_0 - Ae_1\|^2_{L^2(\Omega)} + \int_{(0,T)\times\partial\Omega} |\nabla u^\nu|^2 \, dx\, dt \\
&\quad \leq -A \int_{(0,T)\times\partial\Omega} \nu\omega_2^\nu \, dx\, dt \\
&\quad \leq - \int_{(0,T)\times\partial\Omega} (\nu\tilde{\omega}_2^\nu) A \, dx\, dt \\
&\quad \leq \varepsilon \|\tilde{\omega}\|^{3/2}_{L^{3/2,\infty}((0,T)\times\partial\Omega)} + C_\varepsilon A^3 T \\
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Blow-up method

We use a blow up method introduced in V. (Annales IHP 10) [see also Choi-V. (14)] to control higher derivatives, following the flow at the scale of the blow-up.

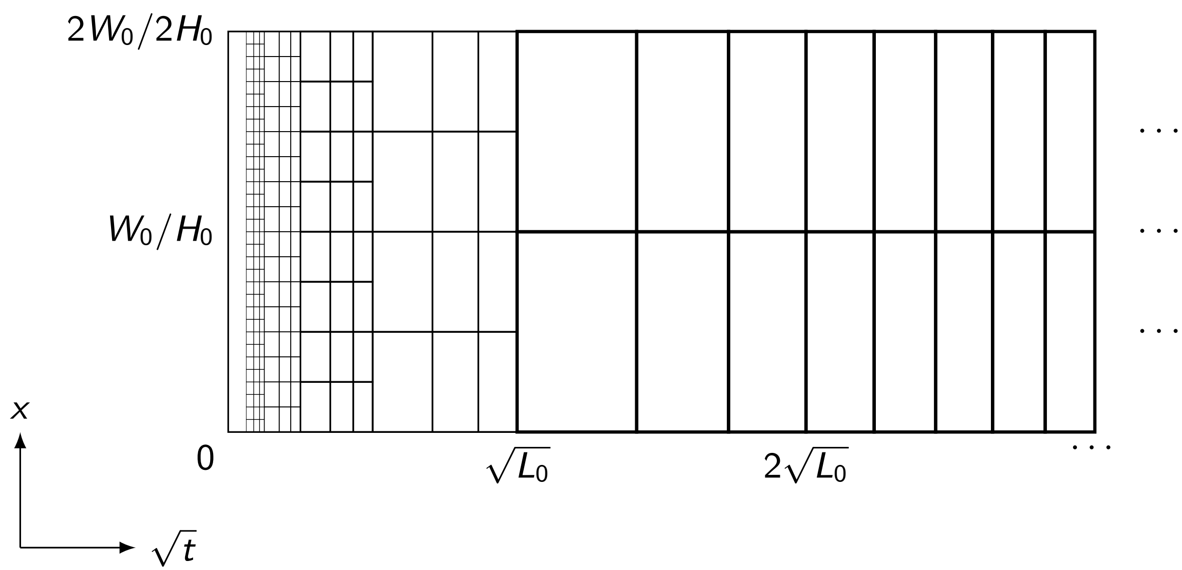
Relies on:

- a local regularity result at the boundary, under smallness condition on the local dissipation $\int |\nabla u|^2 dx\, dz\, dt < \eta$,
- and rescaling of the local regularity result through the universal scaling for Navier-Stokes $u_\varepsilon(t, x) = \varepsilon u(\varepsilon^2 t, \varepsilon x)$.

Problem of boundary: without control on the pressure, the local Stokes regularity does no hold at the boundary.

but it holds AFTER taking local mean value $\tilde{\omega}$.

The parabolic partition of the boundary



We continue to decompose this grid of cubes based on the following property: a parabolic cube Q with dimension $4^{-k}L_0 \times 2^{-k}W_0 \times 2^{-k}H_0$ is said to be suitable if it satisfies

$$\fint_Q \mathcal{M}(|\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}t \leq c_0(2^{-k}R_0)^{-2p} \tag{S}$$

for some c_0 to be determined. For each parabolic cube in the initial partition $\mathcal{Q}^\circ$ that is not suitable, we dyadically dissect it into 4×2^d smaller parabolic cubes. For each smaller cube, we continue to dissect it unless it is suitable. This process will finish in finitely many steps, so all sufficiently small cubes are suitable.

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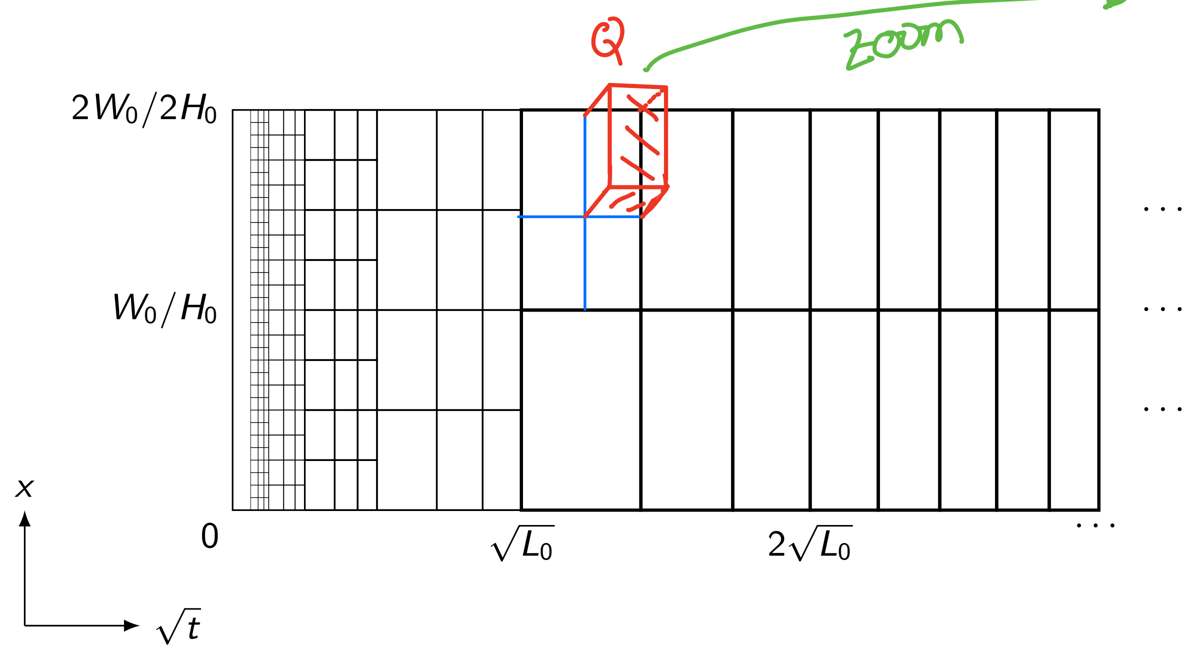

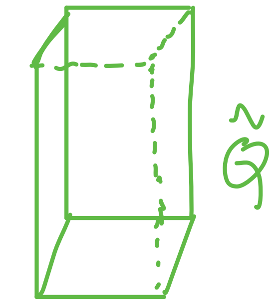

$\int_{\tilde{Q}_1} |\nabla u|^2 \, dx dt \sim h$

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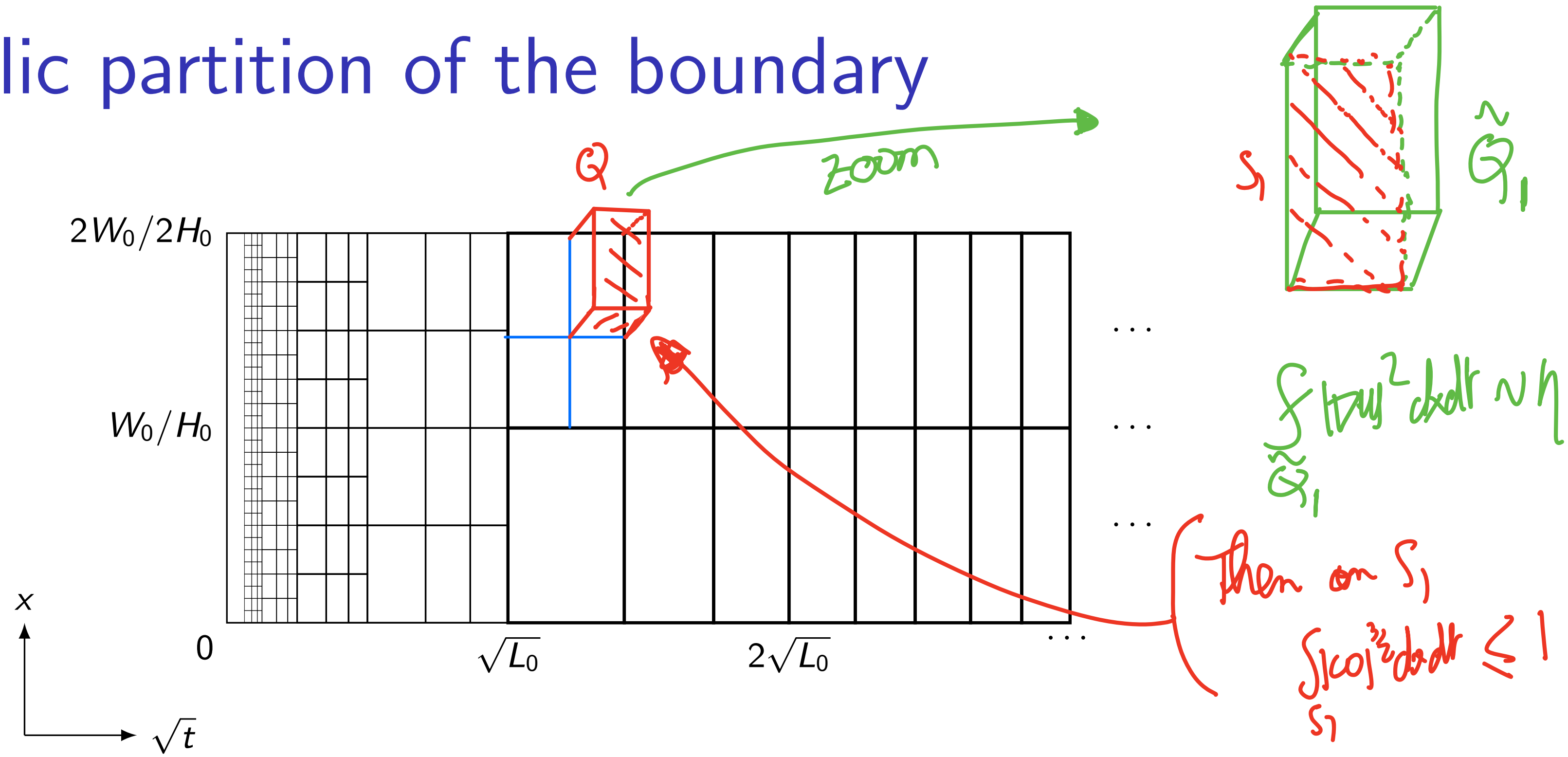

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