MathFlows 2022

The Vlasov-Navier-Stokes system with absorption : viscous penalization ?

A. Moussa

LJLL, SU & DMA, ENS

Outline

Fluid/kinetic systems

Dispersed phased moving within a continuous one

- $f(t, \mathbf{x}, \mathbf{v})$: kinetic equation for the particles
- $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$: fluid mechanics

The Vlasov-Navier-Stokes system

- Thin spray : no fraction volume, no internal interaction
- Fluid : viscous, homogeneous, incompressible
- Interaction : **drag** force and **Brinkman** force

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = -\int_{\mathbf{v}} (\mathbf{u} - \mathbf{v}) \ f \ \mathrm{d}\mathbf{v}, \\ \mathrm{div}\ \mathbf{u} = 0, \\ \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot (f(\mathbf{u} - \mathbf{v})) = 0. \end{cases}$$

Boundary conditions

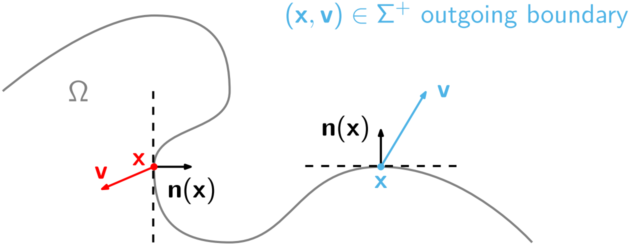
- Physical setting : aerosoltherapy
- f : aeorosol
- $\mathbf{u}$: air
- Ω : (part of) pulmonary duct
- $\partial\Omega$: mucus

$\rightarrow$ homogeneous Dirichlet + absorption

Boundary conditions

→ $\mathbf{u}(t,\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$

→ $f(t,\mathbf{x},\mathbf{v}) = 0$ for $\mathbf{x} \in \partial\Omega$ *and* $\mathbf{v}\cdot\mathbf{n}(\mathbf{x}) < 0$



$(\mathbf{x},\mathbf{v}) \in \Sigma^-$ ingoing boundary

Outline

1. The Vlasov-Navier-Stokes system
2. Mathematical analysis of VNS
3. Trace theory for Vlasov/transport equation
4. Viscous penalization

The energy-dissipation identity

Defining

$$\mathsf{E}(t) = \frac{1}{2}\int_{\mathbf{x}} |\mathbf{u}(t)|^2 + \frac{1}{2}\int_{\mathbf{x},\mathbf{v}} f(t)|\mathbf{v}|^2,$$

$$\mathsf{D}_{\mathsf{diss}}(t) = \int_{\mathbf{x}} |\nabla \mathbf{u}(t)|^2 + \int_{\mathbf{x},\mathbf{v}} f(t)|\mathbf{u}(t) - \mathbf{v}|^2,$$

we have formally

$$\frac{\mathsf{d}}{\mathsf{dt}}\mathsf{E}(t) + \mathsf{D}_{\mathsf{diss}}(t) = 0.$$

Mathematical analysis, without boundary

- Existence : [BDGm '09]
  - Energy estimate + scheme
  - Leray **u** and renormalized f
- Uniqueness : [HKM^3 '20]
  - For the moment only $d = 2$ (Leray)
  - Expected for $d = 3$ (Kato)

Mathematical analysis, without boundary

Boudin, Desvillettes, Grandmont & M., DIE 2009

- Existence : [BDGm '09]
  - Energy estimate + scheme
  - Leray $\mathbf{u}$ and renormalized f
- Uniqueness : [HKM^3 '20]
  - For the moment only $d = 2$ (Leray)
  - Expected for $d = 3$ (Kato)

Mathematical analysis, without boundary

- Existence : [BDGm '09]
  - Energy estimate + scheme
  - Leray **u** and renormalized f
- Uniqueness : [HKM^3 '20]
  - For the moment only $d = 2$ (Leray)
  - Expected for $d = 3$ (Kato)

Han-Kwan, Miot, M. & Moyano, RMI 2020

Mathematical analysis, without boundary

- Large-time : [CK '15], [HKM^2 '20], [HK '22]
  - Monokinetic behavior $f(t) \simeq \rho_f(t) \otimes \delta_{\mathbf{u}(t)}$
- Hydrodynamic limits : [HKM '22]
  - Non-homogeneous Navier-Stokes
- Derivation : [DGR '08], [H '18], [HmS '19], [HC '20]
  - Brinkman force only
  - non-inertial

Mathematical analysis, without boundary

Choi & Kwon, Nonlinearity 2015

- Large-time : [CK '15], [HKM^2 '20], [HK '22]
  - Monokinetic behavior $f(t) \simeq \rho_f(t) \otimes \delta_{\mathbf{u}(t)}$
- Hydrodynamic limits : [HKM '22]
  - Non-homogeneous Navier-Stokes
- Derivation : [DGR '08], [H '18], [HmS '19], [HC '20]
  - Brinkman force only
  - non-inertial

Mathematical analysis, without boundary

- Large-time : [CK '15], [HKM^2 '20], [HK '22]
  - Monokinetic behavior $f(t) \simeq \rho_f(t) \otimes$ ...

Han-Kwan, M. & Moyano, ARMA 2020

- Hydrodynamic limits : [HKM '22]
  - Non-homogeneous Navier-Stokes
- Derivation : [DGR '08], [H '18], [HmS '19], [HC '20]
  - Brinkman force only
  - non-inertial

Mathematical analysis, without boundary

- Large-time : [CK '15], [HKM^2 '20], [HK '22]

  Han-Kwan, PMP 2022

  - Monokinetic behavior $f(t) \simeq \rho_f(t) \otimes \delta_{\mathbf{u}(t)}$
- Hydrodynamic limits : [HKM '22]
  - Non-homogeneous Navier-Stokes
- Derivation : [DGR '08], [H '18], [HmS '19], [HC '20]
  - Brinkman force only
  - non-inertial

Mathematical analysis, without boundary

- Large-time : [CK '15], [HKM^2 '20], [HK '22]
  - Monokinetic behavior $f(t) \simeq \rho_f(t) \otimes \delta_{\mathbf{u}(t)}$
- Hydrodynamic limits : [HKM '22]
  - Non-homogeneous Navier-Stokes
- Derivation : [DGR '08], [H '18], [HmS '19], [HC '20]
  - Brinkman force only
  - non-inertial

Mathematical analysis, without boundary

- Large-time : [CK '15], [HKM^2 '20], [HK '22]
  - Monokinetic behavior $f(t) \simeq \rho_f(t) \otimes \delta_{\mathbf{u}(t)}$
- Hydrodynamic limits : [HKM '22]
  - Non-homogeneous Navier-Stokes

Han-Kwan & Michel
Mem. AMS 2022

- Derivation : [DGR '08], [H '18], [HmS '19], [HC '20]
  - Brinkman force only
  - non-inertial

Mathematical analysis, without boundary

- Large-time : [CK '15], [HKM^2 '20], [HK '22]
  - Monokinetic behavior $f(t) \simeq \rho_f(t) \otimes \delta_{\mathbf{u}(t)}$
- Hydrodynamic limits : [HKM '22]
  - Non-homogeneous Navier-Stokes
- Derivation : [DGR '08], [H '18], [HmS '19], [HC '20]
  - Brinkman force only
  - non-inertial

Mathematical analysis, without boundary

- Large-time : [CK '15], [HKM^2 '20], [HK '22]
  - Monokinetic behavior $f(t) \simeq \rho_f(t) \otimes \delta_{\mathbf{u}(t)}$
- Hydrodynamic limits : [HKM '22]
  - Non-homogeneous Navier-Stokes
- Derivation : [DGR '08], [H '18], [HmS '19], [HC '20]
  - Brinkman force only
  - non-inertial

Desvillettes, Golse & Ricci JSP 2008

Mathematical analysis, without boundary

- Large-time : [CK '15], [HKM^2 '20], [HK '22]
  - Monokinetic behavior $f(t) \simeq \rho_f(t) \otimes \delta_{\mathbf{u}(t)}$
- Hydrodynamic limits : [HKM '22]
  - Non-homogeneous Navier-Stokes
- Derivation : [DGR '08], [H '18], [HmS '19], [HC '20]
  - Brinkman force only
  - non-inertial

Hillairet, ARMA 2018

And with boundary ?

- Existence : [BGm '17], [BMm '20]
  - Non-cylindrical domain or polydispersed aerosol
- Uniqueness : $\varnothing$
- Large-time : [GHKm '18], [EHKm '21], [E '21]
  - Monokinetic behavior ... or not
- Hydrodynamic limits : [E '22]
  - Boussinesq-Navier-Stokes on half-space
- Derivation : $\varnothing$

And with boundary ?

Boudin, Grandmont & M., JDE 2017

- Existence : [BGm '17], [BMm '20]
  - Non-cylindrical domain or polydispersed aerosol
- Uniqueness : $\varnothing$
- Large-time : [GHKm '18], [EHKm '21], [E '21]
  - Monokinetic behavior ... or not
- Hydrodynamic limits : [E '22]
  - Boussinesq-Navier-Stokes on half-space
- Derivation : $\varnothing$

And with boundary ?

- Existence : [BGm '17], [BMm '20]
  - Non-cylindrical domain or polydispersed aerosol
- Uniqueness : $\varnothing$
- Large-time : [GHKm '18], [EHKm '21], [E '21]
  - Monokinetic behavior ... or not
- Hydrodynamic limits : [E '22]
  - Boussinesq-Navier-Stokes on half-space
- Derivation : $\varnothing$

And with boundary ?

- Existence : [BGm '17], [BMm '20]
  - Non-cylindrical domain or polydispersed aerosol
- Uniqueness : $\varnothing$
- Large-time : [GHKm '18], [EHKm '21], [E '21]
  - Monokinetic behavior ... or not
- Hydrodynamic limits : [E '22]
  - Boussinesq-Navier-Stokes on half-space
- Derivation : $\varnothing$

Glass, Han-Kwan & M. ARMA 2018

And with boundary ?

- Existence : [BGm '17], [BMm '20]
  - Non-cylindrical domain or polydispersed aerosol
- Uniqueness : $\varnothing$
- Large-time : [GHKm '18], [EHKm '21], [E '21]
  - Monokinetic behavior ... or not
- Hydrodynamic limits : [E '22]
  - Boussinesq-Navier-Stokes on half-space
- Derivation : $\varnothing$

And with boundary ?

- Existence : [BGm '17], [BMm '20]
  - Non-cylindrical domain or polydispersed aerosol
- Uniqueness : ∅
- Large-time : [GHKm '18], [EHKm '21], [E '21]
  - Monokinetic behavior ... or not
- Hydrodynamic limits : [E '22]
  - Boussinesq-Navier-Stokes on half-space

Ertzbischoff, 2022

- Derivation : ∅

And with boundary ?

- Existence : [BGm '17], [BMm '20]
  - Non-cylindrical domain or polydispersed aerosol
- Uniqueness : $\varnothing$
- Large-time : [GHKm '18], [EHKm '21], [E '21]
  - Monokinetic behavior ... or not
- Hydrodynamic limits : [E '22]
  - Boussinesq-Navier-Stokes on half-space
- Derivation : $\varnothing$

Outline

Vlasov = transport

Vlasov equation

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot [(\mathbf{u} - \mathbf{v}) f)] = 0,$$

for $\mathbf{b} = (\mathbf{v}, \mathbf{u} - \mathbf{v})$ is a transport equation on $\mathbb{R}^d \times \mathbb{R}^d$

$$\partial_t f + \mathbf{b} \cdot \nabla f = df,$$

with $\mathbf{b}$ having Sobolev regularity

Approximation procedure

- On $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, DiPerna-Lions : well-posedness, stability ✓
- Characteristics method, $f(t,\mathbf{x},\mathbf{v}) = f_0(\mathbf{Z}_{t,0}(\mathbf{x},\mathbf{v}))e^{dt}$
- Solution f_{abs} on Ω vanishing on Σ^- then given

$$f_{\text{abs}}(t) = f(t)\mathbf{1}_{A_t},$$

where $\mathrm{A}_t := \{(\mathbf{x},\mathbf{v}) \,:\, \forall s \in [0,t], \mathbf{Z}_{t,s}(\mathbf{x},\mathbf{v}) \in \Omega \times \mathbb{R}^d\}$.

→ well-posedness, stability ?

Trace theory

$\Omega \subset \mathbb{R}^d$ bounded, Lipschitz, $\Omega_T := (0,T) \times \Omega$, $\Gamma_T := (0,T) \times \partial\Omega$.

Theorem (Boyer '05 - Mischler '00)

$\mathbf{b} \in \mathrm{L}^1_t(\mathrm{W}^{1,1}_{\mathbf{x}})$, $\operatorname{div} \mathbf{b} \in \mathrm{L}^\infty(\Omega_T)$ *and* $f, g \in \mathrm{L}^\infty(\Omega_T)$.

$$\partial_t f + \mathbf{b} \cdot \nabla f = g \text{ in } \mathscr{D}'(\Omega_T).$$

Then $f \in \mathscr{C}^0_t(\mathrm{L}^p_{\mathbf{x}})$ *and has a trace* $\gamma_f \in \mathrm{L}^\infty(\Gamma_T, |\mathbf{b} \cdot \mathbf{n}| \mathrm{d}\mathbf{x}\, \mathrm{d}t)$.

- analogy with Sobolev trace theory
- in line with [DiPerna-Lions '89]
- well-posedness, stability of f and γ_f

Trace theory

Precise meaning of γ_f ?

Only element of $L^\infty(\partial\Omega, |\mathbf{b}\cdot\mathbf{n}|d\mathbf{x})$ such that

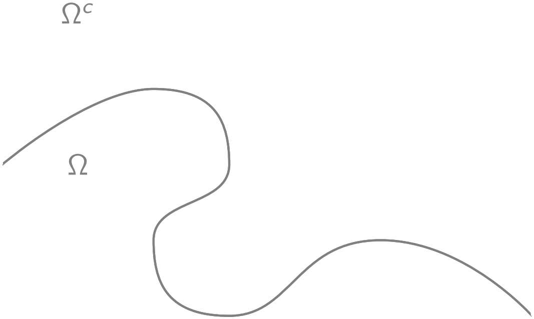
$$\int_{\mathbb{R}_+}\int_\Omega \beta(f)(\partial_t\varphi + \mathrm{div}\,(\mathbf{b}\varphi)) + \int_{\mathbb{R}_+}\int_\Omega \varphi\beta'(f)g = \int_{\partial\Omega}\beta(\gamma_f)\,\mathbf{b}\cdot\mathbf{n},$$

for arbitrary $\varphi \in \mathscr{D}(\mathbb{R}_+^\star \times \mathbb{R}^d \times \mathbb{R}^d)$ and $\beta \in \mathscr{C}^1(\mathbb{R})$.

Outline

The program

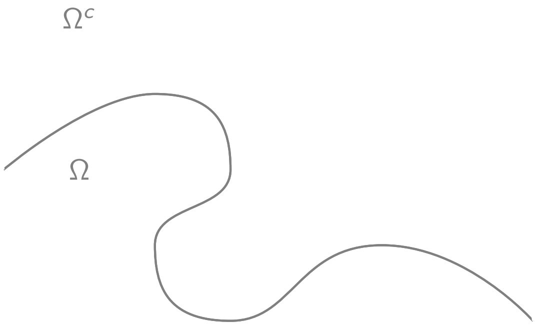
Recall that $f(t,\mathbf{x},\mathbf{v}) = f_0(\mathbf{Z}_{t,0}(\mathbf{x},\mathbf{v}))$



The program

Recall that $f(t, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{Z}_{t,0}(\mathbf{x}, \mathbf{v}))$

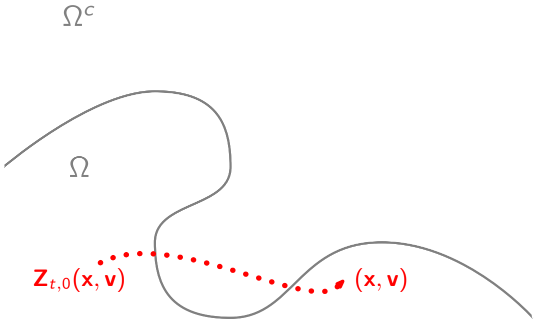
→ $\mathrm{Supp}(f_0) \subset \Omega$ not sufficient to avoid entering trajectories



The program

Recall that $f(t, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{Z}_{t,0}(\mathbf{x}, \mathbf{v}))$

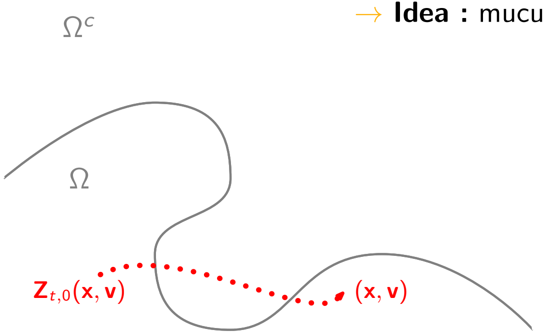
$\rightarrow$ $\mathrm{Supp}(f_0) \subset \Omega$ not sufficient to avoid entering trajectories



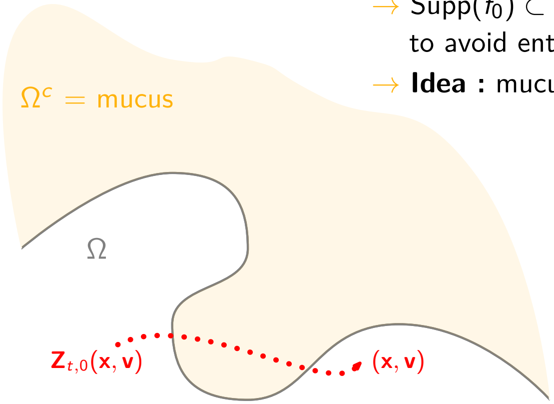
The program

Recall that $f(t,\mathbf{x},\mathbf{v}) = f_0(\mathbf{Z}_{t,0}(\mathbf{x},\mathbf{v}))$

- $\rightarrow$ $\mathrm{Supp}(f_0) \subset \Omega$ not sufficient to avoid entering trajectories
- $\rightarrow$ **Idea :** mucus slows down



The program

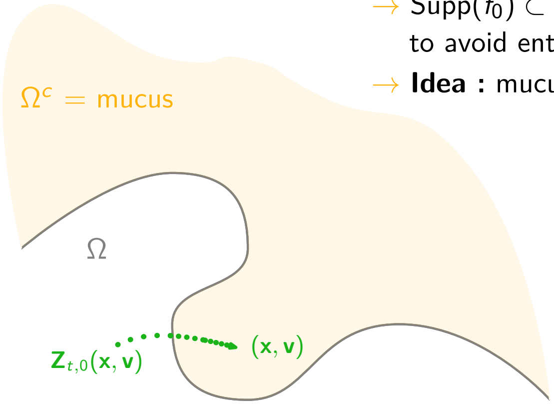


Recall that $f(t,\mathbf{x},\mathbf{v}) = f_0(\mathbf{Z}_{t,0}(\mathbf{x},\mathbf{v}))$

→ $\mathrm{Supp}(f_0) \subset \Omega$ not sufficient to avoid entering trajectories

→ **Idea :** mucus slows down

The program



Recall that $f(t,\mathbf{x},\mathbf{v}) = f_0(\mathbf{Z}_{t,0}(\mathbf{x},\mathbf{v}))$

→ Supp(f_0) $\subset \Omega$ not sufficient to avoid entering trajectories

→ **Idea :** mucus slows down

Penalization strategies

- For Navier-Stokes (homogeneous Dirichlet)
  - L^2-penalization : $-\varepsilon^{-1}\mathbf{u}$
  - H^1-penalization
    - [Angot-Bruneau-Fabrie '99]
    - $\mathrm{div}\,(D(\mathbf{u})) \leftrightarrow \mathrm{div}\,(\varepsilon^{-1}D(\mathbf{u}))$
- For Vlasov (specular reflection)
  - [Mischler '00]
  - $-\varepsilon^{-1}\mathbf{n}$

Penalized VNS

Inhomogeneous viscosity $\mu_\varepsilon := \mathbf{1}_\Omega + \varepsilon^{-1}\mathbf{1}_{\Omega^c}$

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \operatorname{div}\left(\mu_\varepsilon \mathrm{D}(\mathbf{u})\right) + \nabla p = -\mu_\varepsilon \displaystyle\int_{\mathbf{v}} (\mathbf{u}-\mathbf{v})\, f\, \mathrm{d}\mathbf{v}, \\ \operatorname{div}\, \mathbf{u} = 0, \\ \partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}}\cdot\left(\mu_\varepsilon f(\mathbf{u}-\mathbf{v})\right) = 0. \end{cases}$$

- Drag and Brinkman forces depend on viscosity
- Rough : mucus not newtonian and $\partial_t \partial\Omega \neq 0$

Energy estimate

$$\mathrm{D}^{\varepsilon}_{\mathrm{diss}}(t) = \int_{\mathbf{x}} \mu_\varepsilon |\mathrm{D}(\mathbf{u}_\varepsilon)(t)|^2 + \int_{\mathbf{x},\mathbf{v}} \mu_\varepsilon f_\varepsilon(t) |\mathbf{u}_\varepsilon(t) - \mathbf{v}|^2,$$

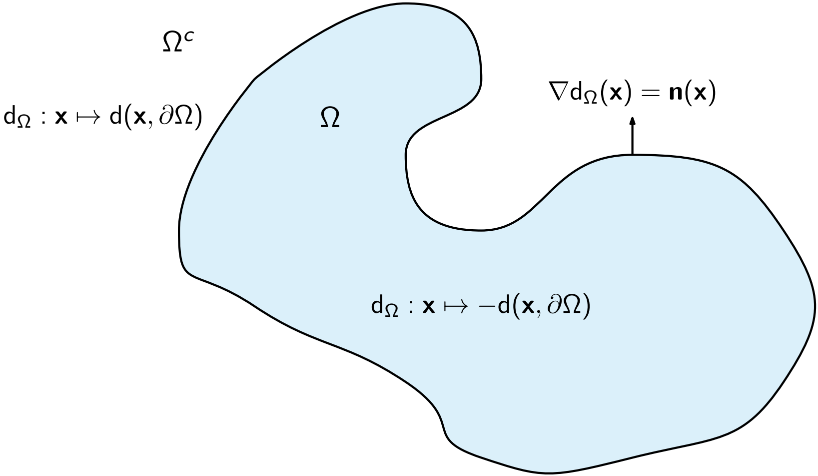
- Existence theory ✓
- As $\varepsilon \to 0$, expected :
  - $\mathrm{D}(\mathbf{u}_\varepsilon)$ and $f_\varepsilon^{1/2}|\mathbf{u}_\varepsilon - \mathbf{v}|$ are $\mathrm{O}_{\mathrm{L}^2}(\varepsilon^{1/2})$ in Ω^c
  - $\mathbf{u}_\varepsilon \simeq$ rigid outside Ω, even $\mathbf{u}_\varepsilon \simeq 0$ for adequate BC
  - $f_\varepsilon \simeq \rho_f \otimes \delta_0$

⚠ High friction limit ($\rightsquigarrow$ Daniel's talk)

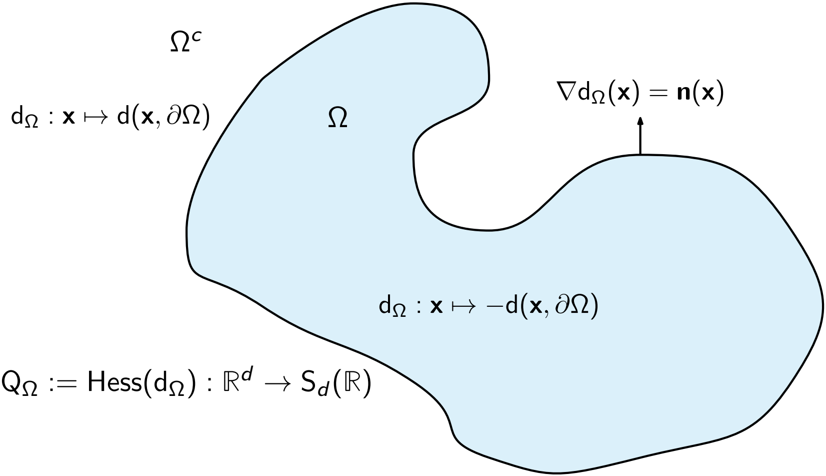
Simplified case

- Ω bounded smooth domain
- Penalization when $\mathbf{u}$ given with vanishing trace on $\partial\Omega$?

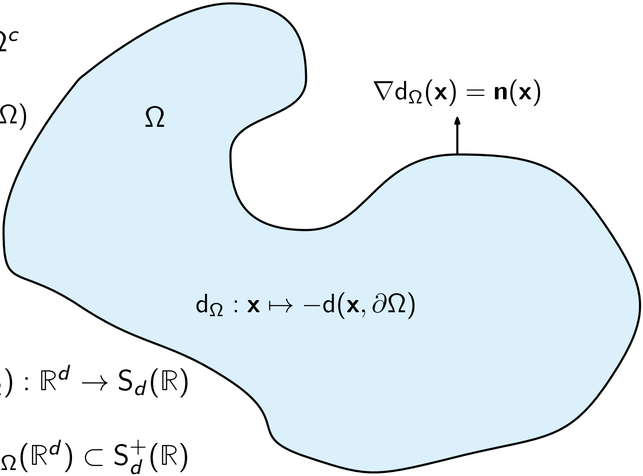
Signed distance function



Signed distance function



Signed distance function



$$\mathrm{Q}_\Omega := \mathrm{Hess}(\mathrm{d}_\Omega) : \mathbb{R}^d \to \mathrm{S}_d(\mathbb{R})$$

$$\Omega \text{ convex} \Rightarrow \mathrm{Q}_\Omega(\mathbb{R}^d) \subset \mathrm{S}_d^+(\mathbb{R})$$