The anisotropic Navier-Stokes system with small unidirectional derivative on the torus

Xingyu LI (BCAM)

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In progress, joint work with Yanlin Liu (Beijing Normal University)

The model and functional setting

In $\mathbb{R}^3$, we consider the incompressible Navier-Stokes equation

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \ \operatorname{div} u = 0, \ u(0, \cdot) = u_0, \quad \text{where} \quad t \geq 0, x \in \mathbb{R}^3 \quad \text{(NS)}$$

and the anisotropic Navier-Stokes equation with only horizontal dissipation

$$\partial_t u + u \cdot \nabla u - \Delta_h u + \nabla p = 0, \ \operatorname{div} u = 0, \ u(0, \cdot) = u_0, \ \text{where } t \geq 0, x \in \mathbb{R}^3 \quad \text{(ANS)}$$

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For homogenous tempered distributions a, u, we define the anisotropic Sobolev space $H^{s,s'}(\mathbb{R}^3)$ as

$$||a||^2_{H^{s,s'}} := \int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_3|^{2s'} |\hat{a}(\xi)|^2 d\xi, \quad \text{with} \quad \xi_h = (\xi_1, \xi_2), \quad \xi = (\xi_h, \xi_3)$$

and the anisotropic Besov spaces $\mathcal{B}, B^{s_1,s_2}_{p,q}$ as

$$||a||_{\mathcal{B}} := \sum_{l\in\mathbb{Z}} 2^{\frac{l}{2}} ||\Delta^v_l a||_{L^2(\mathbb{R}^3)}, \quad ||u||_{B^{s_1,s_2}_{p,q}} := (\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} 2^{qks_1} 2^{qls_2} ||\Delta^h_k \Delta^v_l u||^q_{L^p})^{1/q}$$

with Δ^h_k, Δ^v_l being the anisotropic dyadic operators

$$\Delta^h_k a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\hat{a}(\xi)), \quad \Delta^v_l a = \mathcal{F}^{-1}(\varphi(2^{-l}|\xi_3|)\hat{a}(\xi))$$

$\varphi(\tau)$ smooth, satisfies $\operatorname{Supp}\varphi \subset \{3/4 \leq |\tau| \leq 8/3\}, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1.$

Global well-posedness of (NS) with small unidirectional deriative

Theorem

(Y.Liu, P.Zhang, Arch. Ration. Mech. Anal. 2020) Let $\delta \in (0,1)$*, and* $u_0 = (u_0^h, u_0^3) \in H^{-\delta,0} \cap H^2(\mathbb{R}^3)$ *with* $\partial_3 u_0^h \in H^{-\delta,0} \cap H^{-\frac{1}{2},0}$*. There exists a universal small positive constant* ε_0 *such that if*

$$||\partial_3 u_0||^2_{H^{-\frac{1}{2},0}} \exp(CA_\delta(u_0^h) + CB_\delta(u_0)) \leq \varepsilon_0$$

where

$$A_\delta(u_0^h) := \left(\frac{||\nabla_h u_0^h||^2_{L_v^\infty(L_h^2)} ||u_0^h||^{2/\delta}_{L_v^\infty(B^{-\delta}_{2,\infty})_h}}{||u_0^h||^{2/\delta}_{L_v^\infty(L_h^2)}} + ||u_0^h||^2_{L_v^\infty(L_h^2)} \right) \cdot \exp(C_\delta(1+||u_0^h||^4_{L_v^\infty(L_h^2)})),$$

$$B_\delta(u_0) := ||u_0||^{\frac{1}{2}}_{H^{-\delta,0}} ||u_0||^{\frac{1}{2}}_{H^{\delta,0}} ||\partial_3 u_0||^{\frac{1}{2}}_{H^{-\delta,0}} ||\partial_3 u_0||^{\frac{1}{2}}_{H^{\delta,0}} \exp(CA_\delta(u_0^h))$$

then (NS) has a unique global solution $u \in C(\mathbb{R}^+, H^{\frac{1}{2}}) \cap L^2(\mathbb{R}^+, H^{\frac{3}{2}})$*.*

Global well-posedness of (ANS) with small unidirectional deriative

Theorem

(M.Paicu, P.Zhang, Comm. Math. Phys. 2011) There exist universal constants $L > 0$ *and a sufficiently small constant* $\varepsilon_0 > 0$ *such that*

$$||u_0^h||_{\mathcal{B}} \exp(L||u_0^3||_{\mathcal{B}}^4) \leq \varepsilon$$

then (ANS) has a unique global solution $u \in C([0, \infty, \mathcal{B})$ *with* $\nabla_h u \in L^2([0, \infty), \mathcal{B})$.

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Theorem

(Y.Liu, M.Paicu and P.Zhang, Arch. Ration. Mech. Anal. 2020) Let $u_0, \partial_3 u_0$ *satisfy* $u_0, \partial_3 u_0 \in L^2 \cap H^{-1,1}$*, and there exist universal constants* $N_1, N_2 > 0$ *and a sufficiently small constant* $\varepsilon_0 > 0$ *such that*

$$||\partial_3 u_0||_{H^{-1,0}} ||\partial_3^2 u_0||_{H^{-1,0}} \times \exp(N_1(1 + ||u_0^3||_{L^2}^2 ||\partial_3 u_0^3||_{L^2}^2) \exp(\exp(N_2 ||u_0^h||_{L^2}^2 ||\partial_3 u_0^h||_{L^2}^2))) \leq \varepsilon_0$$

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The anisotropic Navier-Stokes equation on the torus

We want to give a more delicate description for the smallness of the deriative on vertical variable, and we start from the torus $\mathbb{T}_h^2 \times \mathbb{R}_v$. For the solution u, we do the Fourier expansion

$$u = \sum_{k \in \mathbb{Z}^2} \hat{u}_k e^{ikx_h}, \quad u_k = \bar{u}_{-k}$$

then we have

$$\begin{cases} \partial_t u_0 = -\begin{pmatrix} 0 \\ \partial_3 p_0 \end{pmatrix} - \partial_3 \left(\sum_{m \neq 0} u_m^3 u_{-m} \right) \\ \partial_t u_k + k^2 u_k = -\begin{pmatrix} ik \\ \partial_3 \end{pmatrix} p_k - ik_j \sum_{m+n=k} u_m^j u_n - \partial_3 \left(\sum_{m+n=k} u_m^3 u_n \right), \ k \neq 0 \end{cases} \tag{ANST}$$

Main result

Theorem

There exist universal constants $N_1, N_2 > 0$ and a constant $\varepsilon_0 > 0$ small enough, such that if the initial data $u_{k,int}$ satisfies

$$\left(||\partial_3 \hat{u}_{0,int}||_{L^2} + \sum_{k\neq 0} \frac{1}{|k|}||\partial_3 \hat{u}_{k,int}||_{L^2}\right)\left(||\partial_3^2 \hat{u}_{0,int}||_{L^2} + \sum_{k\neq 0} \frac{1}{|k|}||\partial_3^2 \hat{u}_{k,int}||_{L^2}\right)$$

$$\times \exp((N_1 + N_1||u^3_{0,int}||^2_{L^2}||\partial_3 u^3_{0,int}||^2_{L^2} + N_1 \sum_{k\neq 0} ||\partial_3 u^3_{k,int}||^4_{L^2})$$

$$\times \exp(\exp(N_2||u^h_{0,int}||^2_{L^2}||\partial_3 u^h_{0,int}||^2_{L^2} + N_2 \sum_{k\neq 0} ||\partial_3 u^h_{k,int}||^4_{L^2}))) \leq \varepsilon_0$$

then (ANST) has a unique global solution.

Strategies of the proof

We approximate the solution by

$$\partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \Delta_h \bar{u}^h = -\nabla_h \bar{p}, \quad \mathrm{div}_h \bar{u}^h = 0$$

We introduce Helmholtz decomposition for the initial data. For any vector field u^h in dimension 2, we decompose u^h as

$$u^h = \nabla_h^\perp \Delta_h^{-1}(\mathrm{curl}_h u^h) + \nabla_h \Delta_h^{-1}(\mathrm{div}_h u^h) := u^h_{\mathrm{curl}} + u^h_{\mathrm{div}}$$

where $\mathrm{div}_h u^h_{\mathrm{curl}} = 0$. Thus we can choose $\bar{u}^h_{t=0} := \nabla_h^\perp \Delta_h^{-1}(\mathrm{curl}_h u_0^h)$ and the reminder term $v = u - \bar{u}$ satisfies

$$\begin{cases} \partial_t v^h + v \cdot \nabla v^h + \bar{u}^h \cdot \nabla_h v^h + v \cdot \nabla \bar{u}^h - \Delta_h v^h = -\nabla_h p + \nabla_h \bar{p}, \\ \partial_t v^3 + v \cdot \nabla v^3 + \bar{u}^h \cdot \nabla_h v^3 - \Delta_h v^3 = -\partial_3 p, \\ \mathrm{div} v = 0, v|_{t=0} = \left(-\nabla_h \Delta_h^{-1}(\partial_3 u_0^3), u_0^3\right). \end{cases}$$

Strategies of the proof

Let the Fourier multiplier $\Lambda_h := (-\Delta_h)^{\frac{1}{2}}$ in (ANS). The key step is to give the following energy estimates

$$||\bar{u}^h(t)||_{L^2}^2 + 2||\nabla_h \bar{u}^h(t)||_{L_t^2(L^2)}^2 = ||\bar{u}_0^h||_{L^2}^2$$

and

$$||\Lambda_h^{-1}\partial_3^\alpha \bar{u}^h(t)||_{L^2}^2 + ||\partial_3^\alpha \bar{u}^h||_{L_t^2(L^2)}^2 \leq ||\Lambda_h^{-1}\partial_3^\alpha \bar{u}_0^h||_{L^2}^2 \underbrace{\exp(\dots \exp}_{\alpha \text{ times}}(C||\bar{u}_0^h||_{L^2}^2||\partial_3 \bar{u}_0^h||_{L^2}^2))$$

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for $\alpha = 1, 2$. For the equation of reminder v, we study the time weighted Chemin-Lerner space as follows. For some $\lambda > 0$, define

$$v_\lambda(t) := v(t)\exp\left(-\lambda \int_0^t f(t')dt\right)$$

where f needs to be chosen differently for v^h and v^3. And we multiply the equation of v^h, v^3 by $\exp(-\lambda \int_0^t f(t')dt')$ to give the estimates in some Hilbert space $\mathcal{H}$.

Strategies of the proof

Finally, we prove the main theorem by a continuity argument. For the maximal solution of (ANST) with lifespan T^*, consider the following continuity argument:

$$||v^h||_{\tilde{L}^\infty_t(\mathcal{H})} + ||v^h||_{\tilde{L}^2_{t,f}(\mathcal{H})} + ||\nabla_h v^h||_{\tilde{L}^2_t(\mathcal{H})} \leq A(u_0), \tag{A1}$$

$$||v^3||_{\tilde{L}^\infty_t(\mathcal{H})} + ||\nabla_h v^3||_{\tilde{L}^2_t(\mathcal{H})} \leq B(u_0), \tag{A2}$$

here $B(u_0) << A(u_0)$. Denote

$$T^{**} := \sup\{T \in (0, T^*) : \text{(A1)}, \text{(A2)} \quad \text{hold for any} \quad t \in [0, T]\}$$

we can prove that if $A(u_0)$ is small enough, then $T^{**} = T^*$. In view of the blow-up criterion, we have $T^* = \infty$. (see M. Paicu, Equation anisotrope de Navier-Stokes dans des espaces critiques, Rev. Mat. Iberoamericana 2005) And the global well-posedness is proved.

Future work

- A natural question is that for general $\partial_X u_0$ small, where X is a variable direction (which can depend on time and space), could we obtain global well-posedness result?
- What could we say about the anisotropic Navier-Stokes equation with only vertical dissipation? As far as we have known, there are many open problems about it.

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Thanks for your attention!