Tent space maximal regularity for the Stokes operator on the half-space

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MathFlows
Luminy, December 6, 2022

Outline

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Motivation

Consider incompressible Navier-Stokes in $\mathbb{R}_+ \times \mathbb{R}^3$

$$\begin{aligned} u_t + (u \cdot \nabla)u - \Delta u + \nabla p &= 0 \\ \operatorname{div}(u) &= 0 \\ u(0) &= u_0 \end{aligned}$$

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$$u_0 \in \underbrace{\dot{\mathrm{H}}^{1/2}(\mathbb{R}^3)}_{\text{Fujita, Kato '64}} \subset \underbrace{\mathrm{L}^3(\mathbb{R}^3)}_{\text{Kato and Giga '84/'86}} \subset \underbrace{\dot{\mathrm{B}}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)}_{\text{Canonne '97}} \subset \underbrace{\mathrm{BMO}^{-1}}_{\text{Koch, Tataru '01}}$$

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Auscher, Frey '17: different proof for BMO^{-1}

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The maximal regularity operator for the half-space Stokes is bounded on $\mathrm{T}^{\infty,2}$.

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The maximal regularity operator for the half-space Stokes is bounded on $\mathrm{T}^{\infty,2}$.

Remark: Auscher-Monniaux-Portal '12: result for elliptic operators

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Tent spaces

$\mathrm{T}^{\infty,2}((0,\infty)\times\mathbb{R}^d_+)$: f measurable with

$$\|f\|_{\mathrm{T}^{\infty,2}} := \sup_{x\in\mathbb{R}^d_+}\sup_{\tau>0}\left(\int_0^\tau\int_{B(x,\sqrt{\tau})\cap\mathbb{R}^d_+}\tau^{-d/2}|f(s,y)|^2\,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{2}}<\infty$$

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Stokes semigroup on $\mathrm{L}^2(\mathbb{R}^d_+)$

- via Helmholtz projection $\mathrm{e}^{-tA}\mathbb{P}$: no off-diagonal decay

  $\|\mathrm{e}^{-tA}\mathbb{P}f\|_{\mathrm{L}^2(F)} \leq g(\frac{\operatorname{dist}(E,F)^2}{t})\|f\|_{\mathrm{L}^2(E)}, \operatorname{supp}(f) \subset E, g(x) \to 0 \text{ for } x \to \infty$

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  $\Rightarrow \|\mathbb{P}f\|_{\mathrm{L}^2(F)} = \lim_{t\to 0} \|\mathrm{e}^{-tA}\mathbb{P}f\|_{\mathrm{L}^2(F)} \to 0$

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- Stokes resolvent problem + Cauchy integral

Consider Stokes resolvent problem, $\lambda \in \mathbb{C}$ with $|\arg(\lambda)| < \pi - \varepsilon$

$$\lambda u - \Delta u + \nabla p = f, \text{ in } \mathbb{R}^d_+$$
$$u = 0, \text{ on } \partial\mathbb{R}^d_+$$
$$\operatorname{div}(u) = 0, f \in \mathrm{L}^2_\sigma(\mathbb{R}^d_+)$$

Desch-Hieber-Prüss '01, Maekawa-Miura-Prange '20 $\rightsquigarrow$

$$R^u(\lambda)\colon \mathrm{L}^2_\sigma(\mathbb{R}^d_+) \to \mathrm{L}^2_\sigma(\mathbb{R}^d_+) \qquad (\sim \frac{1}{\lambda})$$
$$R^p(\lambda)\colon \mathrm{L}^2_\sigma(\mathbb{R}^d_+) \to \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d_+)$$

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- $\lambda R^u(\lambda)f - \Delta R^u(\lambda)f + \nabla R^p(\lambda)f = f \quad \forall f \in \mathrm{L}^2(\mathbb{R}^d_+)$
- $R^u(\cdot)$ satisfies $R^u(\lambda) - R^u(\mu) = (\mu - \lambda)R^u(\lambda)R^u(\mu)$ on $\mathrm{L}^2(\mathbb{R}^d_+)$

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$\rightsquigarrow E^u(t) := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_t} \mathrm{e}^{\lambda t} R^u(\lambda) \, \mathrm{d}\lambda$ is a semigroup on $\mathrm{L}^2(\mathbb{R}^d_+)$

L^2-estimate

For $F\colon [0,\infty) \to \mathrm{L}^2_\sigma(\mathbb{R}^d_+)$:

$$A \int_0^t \mathrm{e}^{-(t-s)A} F(s,\cdot)\,\mathrm{d}s = [\mathcal{F}^{-1}[\mathrm{i}\xi(\mathrm{i}\xi - A)^{-1} - \mathsf{Id}]\mathcal{F}F\chi_{(0,\infty)}](t)$$

where A is the Stokes operator on $\mathrm{L}^2_\sigma(\mathbb{R}^d_+)$.

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$E(t) \coloneqq (E^u(t), E^p(t))$ $\qquad\qquad \tilde{A}(f,g) = -\Delta f + \nabla g$

For $F\colon [0,\infty) \to \mathrm{L}^2_c(\mathbb{R}^d_+)$:

$$\tilde{A}\int_0^t E(t-s)F(s,\cdot)\,\mathrm{d}s = [\mathcal{F}^{-1}[\mathrm{i}\xi R^u(\mathrm{i}\xi) - \mathsf{Id}]\mathcal{F}F\chi_{(0,\infty)}](t)$$

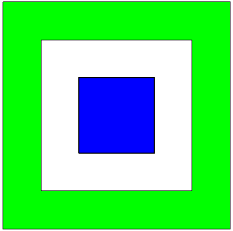
$\rightsquigarrow \|\tilde{A}\int_0^t E(t-s)F(s,\cdot)\,\mathrm{d}s\|_{\mathrm{L}^2(\mathbb{R}^d_+)} \lesssim \|F\|_{\mathrm{L}^2(\mathbb{R}^d_+)}$

Off-diagonal estimates

For $F \in \mathrm{T}^{\infty,2}$ we have

$$\|x \mapsto [\tilde{A}E(t-s)\chi_{C_l \cap \mathbb{R}^d_+} F(s,\cdot)](x)\|_{\mathrm{L}^2(Q(x_0,\sqrt{\tau}))} \lesssim 2^{-l\frac{d+1}{2}} \tau^{-\frac{1}{4}} (t-s)^{-\frac{3}{4}} \|F(s,\cdot)\|_{\mathrm{L}^2(C_l \cap \mathbb{R}^d_+)}$$

with $C_l := Q(x_0, 2^{l+1}\sqrt{\tau}) \setminus Q(x_0, 2^l\sqrt{\tau})$.



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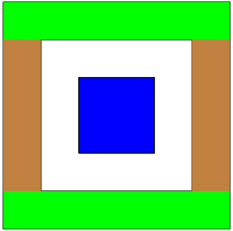

Proof of the Theorem

For $x_0 \in \mathbb{R}^d_+$ and $\tau > 0$:

$$\left(\frac{1}{\tau^{\frac{d}{2}}} \int_0^\tau \int_{Q(x_0,\sqrt{\tau})\cap\mathbb{R}^d_+} |\tilde{A} \int_0^t [E(t-s)F(s,\cdot)](x) \,\mathrm{d}s|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}$$

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$$\leq \mathrm{I} + \sum_{l=1}^{\infty} \mathrm{II}_l$$

$$\mathrm{II}_l \lesssim \tau^{-\frac{d}{4}} 2^{-l\frac{d+1}{2}} \left(\int_0^\tau \int_s^\tau (t-s)^{-\frac{3}{4}} \|F(s,\cdot)\|^2_{\mathrm{L}^2(C_l\cap\mathbb{R}^d_+)} \, \mathrm{d}t \, \mathrm{d}s \right)^{\frac{1}{2}}$$

$$\lesssim 2^{-\frac{l}{2}} \|F\|_{\mathrm{T}^{\infty,2}}$$

Outlook

- time weights t^β for $\beta \in (-\infty, 1)$
- Case $p < \infty$, how small can p be? Auscher-Monniaux-Portal: $p > \frac{2d}{d+2}$