On some regularized nonlinear hyperbolic equations

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MathFlows, CIRM

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Scalar conservation laws

- The transport equation

$$u_t \; + \; c\, u_x \; = \; 0, \qquad u(0,x) \; = \; u_0(x) \qquad \Longrightarrow \qquad u(t,x) \; = \; u_0(x - ct)$$

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- The Burgers equation

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can be written (for smooth solutions) as

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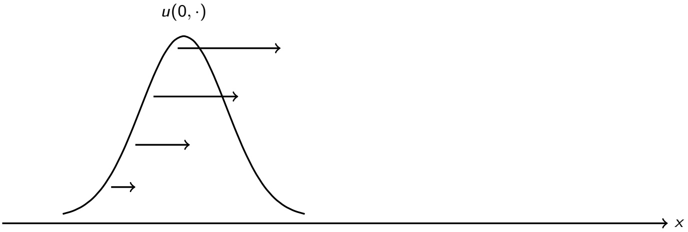

Figure: Propagation of the initial data over the time.

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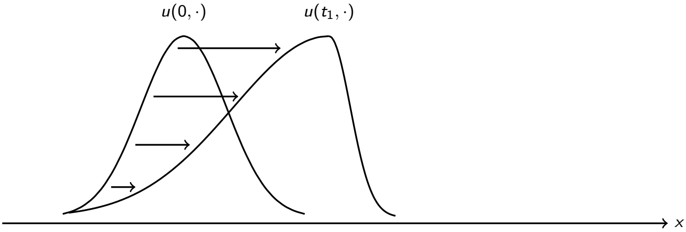

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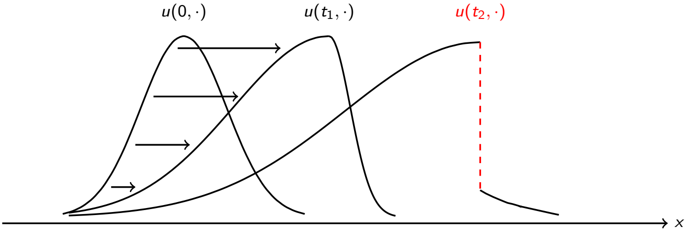

Figure: Propagation of the initial data over the time.

Theorem. If $\exists x_0 \in \mathbb{R}$ with $u_0'(x_0) < 0$, then $T_{max} = -1/\inf_x u_0'(x) < \infty$.

Plan

1. Introduction
2. A regularised Burgers equation
    - Some previous regularisations
    - A Hamiltonian regularisation of the Burgers equation
    - A Hamiltonian regularisation of the Burgers equation
    - The limiting cases $\ell \to 0$ and $\ell \to \infty$
3. A Hamiltonian regularisation of the Euler equations
    - A regularised barotropic Euler system
    - Local solutions
    - Blow-up and global solutions
4. Summary and perspectives

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4 Summary and perspectives

Some previous regularisations

- Adding diffusion

$$u_t + \frac{1}{2}[u^2]_x = \nu\, u_{xx},$$

Drawbacks: *Dissipation of the energy even for smooth solutions.*

- Adding dispersion

$$u_t + \frac{1}{2}[u^2]_x = \delta\, u_{xxx},$$

Drawbacks: *Spurious oscillations and not always sufficient for regularisation.*

- Adding diffusion and dispersion

$$u_t + \frac{1}{2}[u^2]_x = \nu\, u_{xx} + \delta\, u_{xxx},$$

Drawbacks: *Spurious oscillations and can not converge to the entropy solution.*

- Leray-like regularisation [Bhat and Fetecau 2006-2009]

$$u_t + \frac{1}{2}[u^2]_x = \ell^2\,[u_{txx} + u\, u_{xxx}],$$

Drawbacks: *No energy equation and can not converge to the entropy solution.*

A Hamiltonian regularisation of the Burgers equation

We consider the regularised Burgers equation

$$u_t + \mathbf{1}\, u\, u_x = \ell^2 \left[u_{txx} + u\, u_{xxx} + 2\, u_x\, u_{xx} \right],$$

$$\Big\downarrow (1 - \ell^2 \partial_x^2)^{-1}$$

$$u_t + u\, u_x = -\ell^2\, \partial_x\, (1 - \ell^2 \partial_x^2)^{-1} \left\{ \tfrac{1}{2}\, u_x^2 \right\}.$$

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$(1 - \ell^2 \partial_x^2)^{-1}$

Theorem (Local well-posedness)

Local (in time) existence and uniqueness of solutions in H^s, $s > 3/2$ satisfying the energy equation

$$\left[\tfrac{1}{2}\, u^2 + \tfrac{1}{2}\, \ell^2\, u_x^2\right]_t + \left[\tfrac{1}{3}\, u^3 + \tfrac{1}{2}\, \ell^2\, u\, u_x^2 + \ell^2\, u\, P\right]_x = 0.$$

Moreover, if $u_0 \not\equiv 0$, then $\frac{1}{\sup_x |u_0'(x)|} \leqslant T_{max} \leqslant \frac{-2}{\inf_x u_0'(x)}$.

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Moreover, if $u_0 \not\equiv 0$, then $\quad \dfrac{1}{\sup_x |u_0'(x)|} \leqslant T_{max} \leqslant \dfrac{-2}{\inf_x u_0'(x)}.$

Theorem (G et al. 2020. Global dissipative solutions)

There exist two types of global weak solutions in H^1, **conservative** *and* **dissipative** *solutions. The dissipative solutions satisfy the one-sided Oleinik inequality*

$$u_x(t, x) \;\leqslant\; 2/t.$$

Moreover, if $u_0 \in H^1 \cap BV$ and $u_0' \leqslant M$, then

$$TV\ u(t) \leqslant (M\, t/2 + 1)^2\, TV\ u_0.$$

The limiting cases $\ell \to 0$ and $\ell \to \infty$

$$u_t + \tfrac{1}{2}(u^2)_x + \ell^2 P_x = 0, \qquad P \overset{\text{def}}{=} (1 - \ell^2 \partial_x^2)^{-1} \left\{ \tfrac{1}{2} u_x^2 \right\}.$$

$\ell \to 0$

$$u_t + \tfrac{1}{2}(u^2)_x = 0.$$

The Burgers equation.

$\ell \to \infty$

$$\left[u_t + \tfrac{1}{2}(u^2)_x\right]_x = \tfrac{1}{2} u_x^2.$$

The Hunter–Saxton (HS) equation.

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Theorem (G, Junca, Clamond and Pego. 2020. Compactness $\ell \to 0$)

$u^\ell \to u^0$, where u^0 satisfies the Oleinik inequality and

$$u_t^0 + \tfrac{1}{2}\left[\left(u^0\right)^2\right]_x = -\mu_x,$$

with $0 \leqslant \mu$ is a Radon measure.

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$\ell \to 0$ (left) and $\ell \to \infty$ (right)

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Theorem (G, Junca, Clamond and Pego. 2020. Compactness $\ell \to \infty$)

$u^\ell \to u^\infty$, where u^∞ satisfies the Oleinik inequality and

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The limiting cases $\ell \to 0$ and $\ell \to \infty$

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$\ell \to 0$ | $\ell \to \infty$

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Hamiltonian regularisation of the barotropic Euler equations

Notations. $P = P(\rho)$, $P'(\rho) = \rho\,\varpi'(\rho)$, $\mathscr{V} \stackrel{\text{def}}{=} \int \varpi(\rho)\,\mathrm{d}\rho$

$$\mathscr{L}_{bE} \stackrel{\text{def}}{=} \tfrac{1}{2}\,\rho\,u^2 \;-\; \mathscr{V}(\rho) \;+\; \{\,\rho_t \;+[\,\rho\,u\,]_x\,\}\,\phi, \quad \longrightarrow \quad \begin{aligned} \rho_t \;+\; [\,\rho\,u\,]_x \;&=\; 0, \\ [\rho\,u]_t \;+\; \left[\rho\,u^2 \;+\; P\right]_x \;&=\; 0. \end{aligned}$$

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Let $\mathscr{A}$ be an increasing function of ρ, we modify the Lagrangian density as

$$\mathscr{L}_{rbE} \;=\; \left(\tfrac{1}{2}\,\rho\,u^2 \;+\; \varepsilon\,\mathscr{A}'\,\rho\,u_x^2\right) \;-\; \left(\mathscr{V} \;+\; \varepsilon\,\mathscr{A}'\,\mathscr{V}''\,\rho_x^2\right) \;+\; \{\rho_t \;+\; [\rho\,u]_x\}\,\phi.$$

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The Euler–Lagrange equations lead to the regularised barotropic Euler (rbE) system

$$\begin{aligned} \rho_t \;+\; [\,\rho\,u\,]_x \;&=\; 0, \\ [\,\rho\,u\,]_t \;+\; \left[\,\rho\,u^2 \;+\; P \;+\; \varepsilon\,\mathscr{R}\,\right]_x \;&=\; 0, \\ \mathscr{R} \;\stackrel{\text{def}}{=}\; \left(\rho^2\mathscr{A}'\right)'\,u_x^2 \;-\; 2\,\rho\,\mathscr{A}'\,[\,u_t \;+\; u\,u_x \;+\; \varpi_x\,]_x \;&+\; \left(\rho\mathscr{V}''/\mathscr{A}'\right)'\,\mathscr{A}_x^{\,2}. \end{aligned}$$

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Using the Sturm–Liouville operator $\mathcal{L}_\rho \stackrel{\text{def}}{=} \rho - 2\,\varepsilon\,\partial_x\,\rho\,\mathscr{A}'\,\partial_x$, rbE can be written as

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Smooth solutions conserve the energy $\mathscr{E}_t + \mathscr{D}_x = 0$ with

$$\mathscr{E} \stackrel{\text{def}}{=} \tfrac{1}{2}\,\rho\,u^2 \;+\; \varepsilon\,\mathscr{A}'\,\rho\,u_x^2 \;+\; \mathscr{V} \;+\; \varepsilon\,\mathscr{A}'\,\mathscr{V}''\,\rho_x^2.$$

Local solutions

$$\begin{aligned} \rho_t \;+\; [\,\rho\, u\,]_x \;&=\; 0, \\ u_t \;+\; u\,u_x \;+\; P_x/\rho \;&=\; -\,\varepsilon\,\big(\rho - 2\,\varepsilon\,\partial_x\,\rho\,\mathscr{A}'\,\partial_x\big)^{-1}\,\partial_x \Big\{ \big(\rho^2\mathscr{A}'\big)'\,u_x^2 + \big(\rho\mathscr{V}''/\mathscr{A}'\big)'\,\mathscr{A}'^2\,\rho_x^2 \Big\}. \end{aligned}$$

Theorem (G, Clamond & Junca. Nonlinear Anal. Real World Appl. 2022. Local existence)

- $\exists!$ *local solutions of rbE in* $H^s(\mathbb{R})$ *for* $s \geqslant 2$.
- $T_{\max} < \infty \implies \lim_{t\to T_{\max}} \|(\rho_x, u_x)\|_{L^\infty} = \infty$.

Local solutions

$$\rho_t + [\rho\, u]_x = 0,$$

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Proposition

- *There exists* $E^* > 0$, *such that if* $\int \mathscr{E}\, \mathrm{d}x < E^*$ *then* $\rho \geqslant \rho_{min} > 0$.
- *We say that the initial data have a small-energy if* $(\rho_0 - \bar{\rho}, u_0) \in \mathfrak{D}$ *where* $\mathfrak{D} \stackrel{def}{=} \left\{(\rho - \bar{\rho}, u) \in H^1,\ \int \mathscr{E}\, \mathrm{d}x < E^* \right\}$.

Blow-up and global solutions

Theorem (G. Nonlinear Anal. 2022)

- $T_{\max} < \infty \implies \lim_{t \to T_{\max}} \|u_x\|_{L^\infty} = \infty$.
- *If the initial data has a small-energy, then*

$$T_{max} < \infty \implies \liminf_{t \to T_{max}} \inf_{x \in \mathbb{R}} u_x(t,x) = -\infty \quad \text{AND} \quad \limsup_{t \to T_{max}} \|\rho_x\|_{L^\infty} = \infty.$$

- *There exists a C_c^∞ small-energy initial data, such that the corresponding solution blows-up in finite time.*

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Theorem (G. 2022. Global weak solutions)

For any small initial data, there exists a global weak solution to rbE satisfying

- $\rho_t, \rho_x, u_t, u_x \in L^p_{loc}, \ \forall p \in [2,3).$
- *The solution dissipates the energy* $\int_{\mathbb{R}} \mathscr{E}\,\mathrm{d}x \leqslant \int_{\mathbb{R}} \mathscr{E}_0\,\mathrm{d}x.$
- *There exists $C > 0$ such that the solution satisfies the Oleinik inequality*

$$u_x \pm \sqrt{\rho^{-1}\mathscr{V}''}\,\rho_x \leqslant C\left(1+\tfrac{1}{t}\right), \quad a.e.\ (t,x) \in (0,\infty)\times\mathbb{R}.$$

Summary and perspectives

- Regularised Burgers equation
  - Local and global solutions
  - The limiting cases $\ell \to 0$ and $\ell \to \infty$
- Regularised barotropic Euler equations
  - Local solutions
  - Global weak solutions for small data

- Uniqueness ? (scalar equation and system)
- Large data ?
- The limiting case $\varepsilon \to 0$ and $\varepsilon \to \infty$
- Other regularisations with an H^m-like energy ($m \in \mathbb{N}$).
- Regularisations in the multidimensional case.