Hodge decomposition and Maximal Regularity for the Hodge Laplacian on homogeneous Besov spaces on $\mathbb{R}^n_+$

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Laplacians and differential forms

For $\Omega \subset \mathbb{R}^3$, say connected,

- Dir/Neu Laplacian;
  $$-\Delta u = -\operatorname{div}(\nabla u)$$
  $$u_{|\partial\Omega} = 0 \text{ or } \nabla u \cdot \nu_{|\partial\Omega} = 0$$
- Laplacian on Vector Fields;
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$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$-Resolvent Estimates for $-\Delta_{\mathcal{H}}$, $1 < p < \infty$, $s \in (-1 + \frac{1}{p}, \frac{1}{p})$.

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  (similarly for Besov spaces $\dot{\mathrm{B}}^{s}_{p,q} = \dot{\mathrm{B}}^{s,\sigma}_{p,q,\mathfrak{t}} \bigoplus \ldots$).

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Consider an injective operator $\mathcal{L}$ with bounded analytic C_0-semigroup $(e^{-t\mathcal{L}})_{t\geqslant 0}$ on a UMD Banach space X, with $\mathcal{R}$-bounded resolvent.
Let $q \in (1,+\infty)$, $f \in \mathrm{L}^q(\mathbb{R}_+, X)$,

$$\begin{cases} \partial_t u + \mathcal{L}u = f, \text{ on } (0,+\infty), \\ \qquad u(0) = u_0 \in \mathring{\mathcal{D}}_{\mathcal{L}}(1/q', q). \end{cases} \tag{ACP}$$

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(ACP) admits a unique solution $u\in\mathrm{C}_b(\mathbb{R}_+,\mathring{\mathcal{D}}_{\mathcal{L}}(1/q',q))$, given by

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and with estimate

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_{\mathcal{L}}(1/q',q))}\lesssim_{\mathcal{L},q}\|(\partial_t u,\mathcal{L}u)\|_{\mathrm{L}^q(\mathbb{R}_+,X)}\lesssim_{\mathcal{L},q}\|f\|_{\mathrm{L}^q(\mathbb{R}_+,X)}+\|u_0\|_{\mathring{\mathcal{D}}_{\mathcal{L}}(1/q',q)}.$$

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Where, for $\theta\in(0,1)$, $r\in[1,+\infty)$,

$$\|v\|_{\mathring{\mathcal{D}}_{\mathcal{L}}(\theta,r)}:=\left(\int_0^{+\infty}\left(t^{1-\theta}\|\mathcal{L}e^{-t\mathcal{L}}v\|_X\right)^r\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}.$$

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Idea of **[Danchin,Hieber,Mucha,Tolksdorf;arXiv:2011.07918, Chapter 2]** :
Replace X by $\mathring{\mathcal{D}}_{\mathcal{L}}(\theta, q)$, provided $q \in [1, +\infty)$, $\theta \in (0, 1/q)$.

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Replace X by $\mathring{\mathcal{D}}_{\mathcal{L}}(\theta, q)$, provided $q \in [1, +\infty)$, $\theta \in (0, 1/q)$.
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The problem of Global-in-time L^q-Maximal Regularity

Problems of usual L^q-Maximal Regularity :

- $q = 1, \infty$ not allowed,
- X is UMD, then necessarily reflexive. Too restrictive.
- $\mathcal{R}$-boundedness of $\mathcal{L}$.

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(**Homogeneous** version of the Da Prato-Grisvard Theorem)

The problem of Global-in-time L^q-Maximal Regularity

Consider an injective operator $\mathcal{L}$ with bounded analytic C_0-semigroup $(e^{-t\mathcal{L}})_{t\geqslant 0}$ on a Banach space X.

Assumptions-Definitions [DHMT;arXiv:2011.07918, Chapter 2]

- $\exists(Y, \|\cdot\|_Y)$, such that for all $u \in \mathrm{D}(\mathcal{L})$, $\quad \|\mathcal{L}u\|_X \sim \|u\|_Y$.

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- $X \cap \mathrm{D}(\mathring{\mathcal{L}}) = \mathrm{D}(\mathcal{L})$.

Proposition [DHMT;arXiv:2011.07918, Prop.2.12]

For $q \in [1, +\infty]$, $\theta \in (0, 1)$, the functional $\|\cdot\|_{\mathring{\mathcal{D}}_{\mathcal{L}}(\theta,q)}$ provide an equivalent norm on the normed vector space

$$(X, \mathrm{D}(\mathring{\mathcal{L}}))_{\theta,q}.$$

Extrapolation on the Besov scale in higher regularity

Our goal here : Application for $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$ with

- $X = \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k)$;
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Theorem (G. 2022)

For $p \in (1,+\infty)$, $\theta \in (0,1)$, $q \in [1,+\infty]$, $k \in [\![0,n]\!]$ under condition

- $s + 2\theta \in (0, \frac{n}{p})$, *and* $q \in [1,+\infty]$, *with possibly* $s + 2\theta = n/p$ *when* $q = 1$,

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Corollary

Let $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (-1+1/p, 2+1/p)$, (s,p,q) satisfies the same conditions. Then, boundedly

$$\mathbb{P} : \dot{\mathrm{B}}^{s}_{p,q,\mathcal{H}}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{B}}^{s,\sigma}_{p,q,\mathcal{H}}(\mathbb{R}^n_+).$$

$\mathrm{L}^q\dot{\mathrm{B}}^s_{p,q}$-Maximal Regularity : the linear Hodge-Stokes system (for k-differential forms)

$$\begin{cases} \partial_t u - \Delta u + \mathrm{d}\pi = f, \text{ on } (0,+\infty)\times\mathbb{R}^n_+, \\ \delta u = 0, \text{ on } (0,+\infty)\times\mathbb{R}^n_+, \\ \nu \lrcorner (u, \mathrm{d}u)_{|_{\partial\mathbb{R}^n_+}} = 0, \text{ on } (0,+\infty)\times\partial\mathbb{R}^n_+, \\ u(0) = u_0, \text{ on } \mathbb{R}^n_+. \end{cases}$$

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The Hall-MHD System in arbitrary dimension : the equations

The usual Hall-MHD system for $u, b : \mathbb{R}^3 \longrightarrow \mathbb{C}^3$, $h_0 \geqslant 0$

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi = \mathrm{curl}\, b \times b - u \times \mathrm{curl}\, u, \text{ on } (0,+\infty) \times \mathbb{R}^3, \\ \partial_t b + \mathrm{curl}\,\mathrm{curl}\, b = \mathrm{curl}([u - h_0 \,\mathrm{curl}\, b] \times b), \text{ on } (0,+\infty) \times \mathbb{R}^3, \\ \mathrm{div}\, u, \mathrm{div}\, b = 0, \text{ on } (0,+\infty) \times \mathbb{R}^3, \\ (u,b)(0) = (u_0, b_0), \text{ on } \mathbb{R}^3. \end{cases}$$

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Here u is a 1-form, b is a 2-form.

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Here u is a 1-form, b is a 2-form. Then on $\mathbb{R}^n_+$ with Hodge boundary conditions :

$$\begin{cases} \partial_t u - \Delta u + \mathrm{d}\pi = -\delta b \lrcorner b + u \lrcorner \mathrm{d}u, \text{ on } (0,+\infty)\times\mathbb{R}^n_+, \\ \partial_t b + \mathrm{d}\mathrm{d}^* b = \mathrm{d}([u - h_0\delta b]\lrcorner b), \text{ on } (0,+\infty)\times\mathbb{R}^n_+, \\ \delta u = 0, \text{ on } (0,+\infty)\times\mathbb{R}^n_+, \\ \mathrm{d}b = 0, \text{ on } (0,+\infty)\times\mathbb{R}^n_+, \\ \nu \lrcorner (u, \mathrm{d}u, b, \mathrm{d}b)_{|_{\partial\mathbb{R}^n_+}} = 0, \text{ on } (0,+\infty)\times\partial\mathbb{R}^n_+, \\ (u,b)(0) = (u_0,b_0), \text{ on } \mathbb{R}^n_+. \end{cases}$$

The Hall-MHD System in arbitrary dimension : W-P

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Theorem

Let $p \in (n-1, 2n)$. For $(u_0, b_0) \in \dot{\mathrm{B}}^{n/p-1,\sigma}_{p,1,\mathfrak{t}} \times (\dot{\mathrm{B}}^{n/p-1,\gamma}_{p,1} \cap \dot{\mathrm{B}}^{n/p,\gamma}_{p,1,\mathfrak{t}})$ sufficiently small in norm,
$\exists!$ global mild solution $(u,b) \in \mathrm{C}^0_b(\mathbb{R}_+, \dot{\mathrm{B}}^{n/p-1,\sigma}_{p,1,\mathfrak{t}} \times (\dot{\mathrm{B}}^{n/p-1,\gamma}_{p,1} \cap \dot{\mathrm{B}}^{n/p,\gamma}_{p,1,\mathfrak{t}})) \cap \mathrm{L}^1(\mathbb{R}_+, \dot{\mathrm{B}}^{n/p+1}_{p,1} \times (\dot{\mathrm{B}}^{n/p+1}_{p,1} \cap \dot{\mathrm{B}}^{n/p+2}_{p,1}))$ that remains small enough.

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Idea : Previous linear theory and arguments similar to **[Danchin,Tan;arXiv:1911.03246, Sections 1 & 3]**.