Global well-posedness for the one-phase Muskat problem

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The Muskat problem

- Global well-posedness for the one-phase Muskat problem
  with H. Dong and H.Q. Nguyen (2022)

Porous medium equations

- Incompressibility condition

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- The Muskat problem (1934)

$$(\mu,\rho)(x,t) = \begin{cases} (\mu^1,\rho^1), & x \in D^1(t), \\ (\mu^2,\rho^2), & x \in D^2(t) = \mathbb{R}^2 \setminus \overline{D^1(t)}, \end{cases}$$

two incompressible and immiscible fluids.

Interface equation ($\mu^1 = 0 = \rho^1$)

The equations

$$\nabla \cdot u = 0, \qquad \mu^2 u = -\nabla p - \rho^2 (0, 1),$$

yields

$$\nabla^\perp \cdot u = 0, \qquad u = \nabla \phi, \qquad \mu^2 \phi = -p - \rho^2 y.$$

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For

$$D^2(t) = \{(x,y) \in \mathbb{R}^2, \quad y < f(x,t); \quad f : \mathbb{T} \times [0,T] \to \mathbb{R}\},$$

it is possible to get

$$\begin{cases} \Delta\phi = 0 \quad \text{in } D^2(t), \quad \nabla\phi \in L^2(D^2(t)), \\ \phi(x, f(x)) = -\frac{\rho^2}{\mu^2} f(x), \end{cases}$$

with

$$f_t(x) = u(x, f(x)) \cdot N(x) = \partial_N \phi(x, f(x)) = G(f)(-\frac{\rho^2}{\mu^2} f)(x).$$

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$$\partial_t f = -\frac{\rho^2}{\mu^2} G(f) f,$$

with the Dirichlet-Neumann operator $G(f)g$ given by

$$G(f)g(x) = \frac{1}{4\pi} p.v. \int_{\mathbb{T}} \frac{\sin(x-x') + \sinh(f(x)-f(x'))\partial_x f(x)}{\cosh(f(x)-f(x')) - \cos(x-x')} \theta(x') dx',$$

and $\theta : \mathbb{T} \to \mathbb{R}$ satisfies

$$\frac{1}{2}\theta(x) + \frac{1}{2\pi} p.v. \int_{\mathbb{T}} \frac{\sinh(f(x)-f(x')) - \sin(x-x')\partial_x f(x)}{\cosh(f(x)-f(x')) - \cos(x-x')} \theta(x') dx' = \partial_x g(x).$$

Linear Equation

At the linear level,

$$\begin{aligned}\partial_t f^L(x,t) &= -\frac{\rho^2}{\mu^2} p.v. \int \frac{\partial_x f^L(x',t)}{4\pi \tan((x-x')/2)} dx' \\ &= -\frac{\rho^2}{\mu^2} H(\partial_x f^L)(x,t) \\ &= -\frac{\rho^2}{\mu^2} \Lambda(f^L)(x,t), \qquad \Lambda = (-\Delta)^{1/2}.\end{aligned}$$

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Fourier transform:

$$\hat{f}^L(\xi,t) = \hat{f}_0(\xi,t) \exp\Big(-\frac{\rho^2}{\mu^2} |\xi| t\Big).$$

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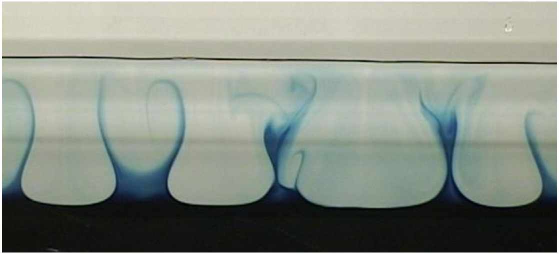
- $\frac{\rho^2}{\mu^2} > 0$ stable case: fluid below,
- $\frac{\rho^2}{\mu^2} < 0$ unstable case: fluid on top.

Muskat (1934);   Saffman, Taylor (1958)

Hele-Shaw cell:



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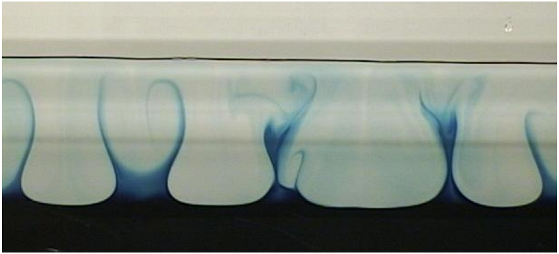

- Classical results:

Duchon, Robert-83; Dombre, Pumir, Siggia-92; Constantin, Pugh-93; Chen-93; Hou, Lowengrub, Shelley-94; Yi-96; Escher, Simonett-97; Otto-97&99

Recent results: (60 authors)

Kim-03; Siegel, Caflish, Howison-04; Ambrose-04&07&14; Guo, Hallstrom, Spirn-07; Córdoba, **G.**-07,09&10; Córdoba, **G.**, Orive-08; Córdoba^2, **G.**-09,11&13; Escher, A.V. Matioc, B.V. Matioc-10; Castro, Córdoba, Fefferman, **G.**, López-Fernández-11&12; Escher, B.V. Matioc-11; Córdoba, Faraco, **G.**-11; Ye, Tanveer-11&12; Székelyhidi-12; Constantin, Córdoba, **G.**, Strain-13; Knüpfer, Masmoudi-13&15; Castro, Córdoba, Fefferman, **G.**-13&16; Beck, Sosoe, Wong-13; **G.**, Strain-14; Granero-Belinchón-14; Berselli, Córdoba, Granero-Belinchón-14; Gómez-Serrano, Granero-Belinchon-14; Cheng, Granero-Belinchon, Shkoller-15; Córdoba, Gómez-Serrano, Zlatos-15&16; Constantin, Córdoba, **G.**, Piazza, Strain-16; Constantin, **G.**, Shvydkoy,Vicol-17; Deng, Lei, Lin-17; Córdoba, Pernas-Castaño-17; Patel, Strain-17; Clemens, Székelyhidi-18; Cameron-19; Castro, Faraco, Mengual-19; Chang-Lara, Guillen, Schwab-19; **G.**, García-Juárez, Patel, Strain-19; Matioc-19; Alazard, Lazar-20; Alazard, Meunier, Smets-20; H.Q. Nguyen, Pausader-20; Jacobs, Kim, Alpár-Mészarós-20; H.Q. Nguyen-20; **G.**, Granero-Belinchón, Scrobogna-20

- Last two years results:

Abels, Matioc-21; Cameron-21; A.V. Matioc, B.V. Matioc-21 Castro, Córdoba, Faraco-21; Córdoba, Lazar-21; **G.**, García-Juárez, Patel, Strain-21; Alazard, Q.H. Nguyen-21[3]; Flynn, H.Q. Nguyen-21; **G.**, Lazar-21; Mengual-21; H.Q. Nguyen-21; Q.H. Nguyen-21; Alonso-Orán, Granero-Belinchón-21; García-Juárez, Gómez-Serrano, H.Q. Nguyen, Pausader-21; Agrawal, Patel, Wu-22; Bocchi; **G.**-22, Geng, Granero-Belinchón-22; Alazard, Magliocca, Meunier-22; A.V. Matioc, B.V. Matioc-22; H.Q. Nguyen-22, Haziot, Pausader-22, H.Q. Nguyen, Tice-22, Shi-22

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- Two-fluids equation:

$$\partial_t f(x) = \frac{\rho^2 - \rho^1}{4\pi\mu} p.v. \int_{\mathbb{T}} \frac{\sin(x-x')(\partial_x f(x) - \partial_x f(x'))}{\cosh(f(x)-f(x'))-\cos(x-x')} dx',$$

One-fluid vs Two-fluids

	∃! local sub.	∃ global small critical	squirt	splash	turning	∃! local critical	global ∃ large data
1-F	✓	✓	×	✓	×	?	?
2-F	✓	✓	×	×	✓	✓	×

sub.≡subcritical

Global well-posedness for large data

Theorem (H. Dong, G., H.Q. Nguyen-21)

For all $f_0 \in W^{1,\infty}(\mathbb{T})$*, there exists*

$$f \in C(\mathbb{T} \times [0,\infty)) \cap L^\infty([0,\infty); W^{1,\infty}(\mathbb{T})), \quad \partial_t f \in L^\infty([0,\infty); L^2(\mathbb{T}))$$

such that $f|_{t=0} = f_0$*,* f *satisfies One-Fluid-Muskat in* $L^\infty_t L^2_x$*, and*

$$\|f(t)\|_{W^{1,\infty}(\mathbb{T})} \leq \|f_0\|_{W^{1,\infty}(\mathbb{T})} \quad a.e.\ t > 0.$$

Moreover, f *is unique in its class (viscosity solution).*

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- This is the first construction of unique global strong solutions for the Muskat problem with initial data of arbitrary size in a critical space.

High-regularity vs Low-regularity

- High-regularity:
  - Easy to get maximum principle
  - Easy to satisfy the contour equation
  - Easy to get uniqueness
  - Difficult to propagate regularity

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Existence

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- Property: $f_1, f_2 \in W^{1,\infty}(\mathbb{T})$, $C^{1,1}(x_0)$, $f_1 \leq f_2$ and $f_1(x_0) = f_2(x_0)$ then

$$-G(f_1)f_1(x_0) \leq -G(f_2)f_2(x_0).$$

- Property: $f_1, f_2 \in C_T C_x^2$ two solutions with $f_1(0) \leq f_2(0)$ then

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- Property: with $f(x+y,0) \leq f(x,0) + L|y|$ we get

$$\|f\|_{W^{1,\infty}}(t) \leq \|f_0\|_{W^{1,\infty}}.$$

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- Regularized system:

$$\partial_t f^\varepsilon = -\frac{\rho^2}{\mu^2} G(f^\varepsilon)(f^\varepsilon) + \varepsilon \Delta f^\varepsilon, \qquad f^\varepsilon(x,0) = (K_\varepsilon * f_0)(x).$$

We are able to probe global-in-time existence for the system above.

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- New inequalities:

$$\|G(f)g\|_{L^2} \leq C(1 + \|\partial_x f\|_{L^\infty})^2 \|\partial_x g\|_{L^2},$$

$$\|\theta\|_{L^2} \leq C(1 + \|\partial_x f\|_{L^\infty})^{\frac{5}{2}} \|\partial_x f\|_{L^2}.$$

- Taking limits at the nonlinear parts:

$$N_1 = \frac{1}{2\pi}\partial_x \int_{\mathbb{T}} \arctan\Big(\frac{\tanh(\frac{f^\varepsilon(x)-f^\varepsilon(x')}{2})}{\tan(\frac{x-x'}{2})}\Big)\theta^\epsilon(x')dx',$$

and

$$N_2 = \frac{1}{4\pi}\partial_x p.v. \int_{\mathbb{T}} \log\big(\cosh(f^\varepsilon(x) - f^\varepsilon(x')) - \cos(x - x')\big)\theta^\varepsilon(x')dx'.$$

Uniqueness

• Def: $f : \mathbb{T} \times [0, T]$ is a viscosity subsolution (super) if
(i) f is upper semicontinuous (lower semicontinuous) on $\mathbb{T} \times [0, T]$
(ii) $\forall\ \psi$ with $\partial_t \psi \in C(\mathbb{T} \times (0, T))$ and $\psi \in C((0, T); C^{1,1}(\mathbb{T}))$, if $f - \psi$ attains a global maximum (minimum) over $\mathbb{T} \times [t_0 - r, t_0]$ at $(x_0, t_0) \in \mathbb{T} \times (0, T)$ for some $r > 0$, then

$$\partial_t \psi(x_0, t_0) \leq -\frac{\rho^2}{\mu^2}\big(G(\psi)\psi\big)(x_0, t_0) \quad (\geq).$$

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• Consistency: f be a subs. (supers.), $f \in W^{1,\infty}(\mathbb{T} \times (0,T))$ and $C^{1,1}(x_0, t_0)$. Then

$$\partial_t f(x_0, t_0) \leq -\frac{\rho^2}{\mu^2}\big(G(f)f\big)(x_0, t_0) \quad (\geq).$$

• Highly needed: For $f \in W^{1,\infty}(\mathbb{T})$ and $C^{1,1}(x_0)$, its harmonic extension ϕ satisfies:

$$
\begin{aligned}
&|\phi(x, y) - f(x_0) - (x - x_0, y - f(x_0)) \cdot \nabla\phi(x_0, f(x_0))| \\
&\qquad\qquad\qquad \leq M\Big|(x - x_0)^2 + (y - f(x_0))^2\Big|^{\frac{1+\alpha}{2}}.
\end{aligned}
$$

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$$|\phi(x,y) - f(x_0) - (x - x_0, y - f(x_0)) \cdot \nabla\phi(x_0, f(x_0))| \\ \leq M\Big|(x - x_0)^2 + (y - f(x_0))^2\Big|^{\frac{1+\alpha}{2}}.$$

• Comparison: For f subs. and g supers., if $f(0) \leq g(0)$, then

$$f(t) \leq g(t), \quad \forall (x,t) \in \mathbb{T} \times [0, T].$$

Sup and inf-convolution.

What about 3D?

- Implicit kernels
- More difficult to solve the regularized system
- More difficult to obtain the harmonic extension property

Thank you!