Local theory for thick spray equations

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Description of a spray

Dispersed phase of particles (ex: droplets, dust specks) within a gas

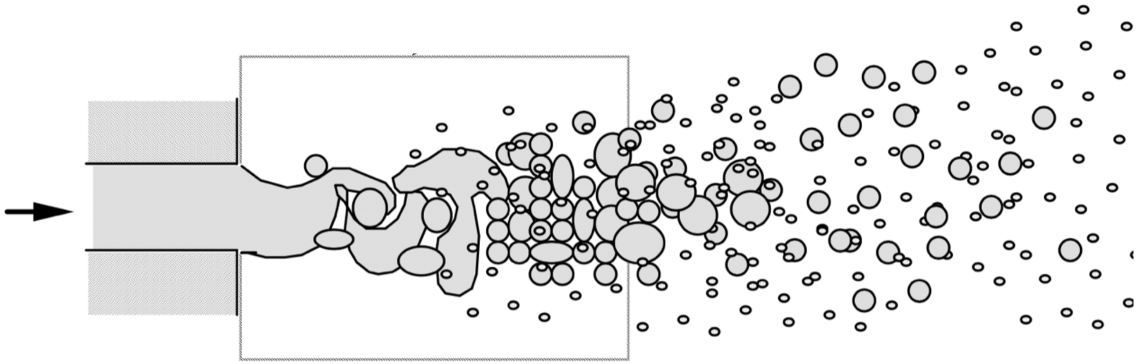
- Gas:
  - macroscopic description $\to$ fluid mechanics equations
  - $u(t,x) \in \mathbb{R}^d$, $\varrho(t,x) \in \mathbb{R}^+$: velocity and density of the fluid
- Particles:
  - mesoscopic description $\to$ kinetic equation ($\sim$ Boltzmann, Vlasov...)
  - $f(t,x,v) \in \mathbb{R}^+$: phase-space distribution function for the particles
  - kinetic moments:

$$\rho_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \quad j_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)v\,\mathrm{d}v$$

Critical quantity: gas volume fraction $\alpha(t,x) := 1 - \rho_f(t,x) \in [0,1]$

Thick spray equations: "α not close to 1"

Computer Modeling of Sprays - Reitz - 1996



$$\begin{cases} \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) - \Delta_x u + \alpha\nabla_x p(\varrho) = \displaystyle\int_{\mathbb{R}^d} (v-u) f \,\mathrm{d}v, \\ \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\big(f(u-v) - f\nabla_x p(\varrho)\big) = \mathcal{Q}(f,f) \end{cases} \quad \text{(TS)}$$

- Compressible barotropic Navier-Stokes system + Vlasov-Boltzmann eq.
- **Coupling** through: **drag term** $(u-v)$ + retroaction, **pressure gradient** $\nabla_x p(\varrho)$, and **volume fraction** $\alpha = 1 - \rho_f$

Some naive estimates (on $\mathbb{T}_x^d \times \mathbb{R}_v^d$)...

- Transport kinetic equation on f (neglect collisions):

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v(f(u-v)) - \nabla_x p(\varrho) \cdot \nabla_v f = 0$$

$$\Longrightarrow \|f(t)\|_{\mathcal{H}^\ell_{x,v}} \lesssim \mathrm{C}\Big(\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{\ell+1}_x)}\Big)$$

- Transport equation on ϱ:

$$\partial_t \varrho + u \cdot \nabla_x \varrho + \frac{1}{1-\rho_f} \mathrm{div}_x \left[F + u\right] \varrho = 0, \quad F = j_f - \rho_f u$$

$$\Longrightarrow \|\varrho(t)\|_{\mathrm{H}^\ell_x} \lesssim \mathrm{C}\left(\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{\ell+1}_{x,v})}\right)$$

- **Loss of 2 derivatives for** ϱ due to coupling: formally prevents a direct existence result $\Longrightarrow$ "open problem" for thick spray equations.

- **Key fact**: ϱ depends on f only through its **moments in velocity**:

$$\partial_t \varrho + u \cdot \nabla_x \varrho + \frac{\varrho}{1-\rho_f} \mathrm{div}_x (j_f - \rho_f u) = \text{l.o.t.}$$

$\rightarrow$ use a smoothing averaging effect for $j_{\partial_x^m f}$ and $\rho_{\partial_x^m f}$

- Reminiscent of **singular Vlasov equations**, with loss of derivatives
  - Requires some condition on the data: prevent instabilities
  - Local W-P in Sobolev for Vlasov-Benney by [Han-Kwan, Rousset, 2016]

$\exists$ a **Penrose stability condition** (**P**) adapted to (**TS**) ($\sim$ ellipticity condition): small data, or Maxwellians f^{in} or small smooth perturbation...

Theorem (E. & Han-Kwan). *Take smooth enough initial data such that* $(f^{\text{in}}, \varrho^{\text{in}})$ *satisfies* (**P**). $\exists m \in \mathbb{N}$, $\exists T > 0$ *and a unique solution to* (**TS**)*:*

$$f \in \mathscr{C}\left([0,T]; \mathcal{H}^{m-1}_{x,v}\right), \quad \varrho \in \mathrm{L}^2(0,T; \mathrm{H}^m_x), \quad u \in \mathscr{C}\left([0,T]; \mathrm{H}^m_x\right) \cap \mathrm{L}^2(0,T; \mathrm{H}^{m+1}_x),$$