Relaxation approximation and asymptotic stability of stratified solutions to the Incompressible Porous Media equation

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Joint work with Roberta Bianchini and Marius Paicu

The two-dimensional Incompressible Porous Media (IPM) system is the active scalar equation:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \\ \mathbf{u} = -\kappa \nabla P + \mathbf{g}\rho, \qquad \mathbf{g} = (0, -g)^T, \qquad \text{(Darcy law)} \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{IPM}$$

It models the dynamics of a fluid of density $\rho = \rho(t, x, y) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ through a porous medium according to the Darcy law.
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- Application: transport of a dissolved contaminant in porous media where the contaminant is convected with the subsurface water. For instance, one could be interested in the time taken by the pollutant to reach the water table below.
- Mathematical motivation: Less regular than 2D Euler.

The incompressibility condition together with Darcy's law implies that

$$\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}\partial_x\rho = (\mathcal{R}_1\mathcal{R}_2\rho, -\mathcal{R}_1^2\rho)$$

where $(\mathcal{R}_1, \mathcal{R}_2)$ is the two-dimensional homogeneous Riesz transform of order 0:

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- For this system, Córdoba, Gancedo and Orive (07') proved the local well-posedness in Hölder space C^{δ} with $0 < \delta < 1$ by the particle-trajectory method.
- What about global-in-time solutions close to equilibrium?

Litterature and stratification

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- Due to the form of the velocity $\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}\partial_x\rho$, all the steady states of (IPM) are stratified: constant in x.
- Among these steady states $\bar{\rho}_{eq} = g(y)$, there are only some for which one can hope to stabilise the system around. Here we focus on the linear and stable ones:

$$\bar{\rho}_{eq}(y) = \rho_0 - y$$

where $\rho_0 > 0$ is a constant averaged density.

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- Linearizing (IPM) around $\bar{\rho}_{eq}(y) = \rho_0 - y$, one obtains

$$\partial_t \widetilde{\rho} - \mathcal{R}_1^2 \widetilde{\rho} = (\mathcal{R}_2 \mathcal{R}_1 \widetilde{\rho}, -\mathcal{R}_1^2 \widetilde{\rho}) \cdot \nabla \widetilde{\rho}. \tag{IPM-diss}$$

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- This should lead to instability as it can be related to the Rayleigh–Bénard convection instability occurring even in the presence of diffusion.
- To sum-up: here, in a sense, we will rely on the fact that the stratification inherent in the model serves as a stabilising mechanism to derive global-in-time results.

- We refer to the work of Elgindi (17') about the justification of asymptotic stability of (IPM) in the whole space $\mathbb{R}^2$ for initial data in $H^{20}(\mathbb{R}^2)$.
- The analogous result in the periodic finite channel in $H^{10}(\mathbb{T}\times[-\pi,\pi])$ is due to Castro, Córdoba and Lear (19').
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Our contributions are:

- The asymptotic stability of (IPM) in $\dot{H}^{1^-}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$ with $s > 3$.
- A new relaxation approximation of (IPM) by the two-dimensional Boussinesq system with damped velocity.
- And, as a byproduct of the above two results, an existence result for the two-dimensional Boussinesq system with damped velocity.

Existence result for (IPM)

Theorem (Bianchini-CB-Paicu 2022)

Let $0 < \tau < 1$ *and* $s \geq 3 + \tau$. *For any initial datum* $\rho_{\rm in} \in \dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$, *there exists a constant value* $0 < \delta_0 \ll 1$ *such that, under the assumption*

$$\|\rho_{\rm in} - \bar{\rho}_{eq}\|_{\dot{H}^{1-\tau} \cap \dot{H}^s} \leq \delta_0,$$

there exists a unique global-in-time smooth solution $\widetilde{\rho}$ *to system* (IPM-diss) *satisfying the following inequality for all times* $t > 0$

$$\|\widetilde{\rho}\|_{L^\infty_T(\dot{H}^{1-\tau} \cap \dot{H}^s)} + \|\mathcal{R}_1 \widetilde{\rho}\|_{L^2_T(\dot{H}^{1-\tau} \cap \dot{H}^s)} + \|\nabla \mathcal{R}_1^2 \widetilde{\rho}\|_{L^1_T(L^\infty)} \lesssim \|\widetilde{\rho}_{\rm in}\|_{\dot{H}^{1-\tau} \cap \dot{H}^s},$$

where $\widetilde{\rho} = \rho - \bar{\rho}_{eq}$.

Idea of proof

Recall that the equation we are interested in reads:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = (\mathcal{R}_2 \mathcal{R}_1 \rho, -\mathcal{R}_1^2 \rho) \cdot \nabla \rho.$$

To justify the global-in-time existence of this equation, one way is to recover the following bound

$$\int_0^t \|(\nabla \mathcal{R}_1 \mathcal{R}_2 \rho, \nabla \mathcal{R}_1^2 \rho)\|_{L^\infty} < \infty.$$

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- But how can one retrieve such bound?

Let us investigate the toy-model:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0. \tag{1}$$

In a Sobolev framework, performing standard energy estimates leads to, for any $s \in \mathbb{R}$,

$$\|\rho\|_{L^\infty_T(H^s)} + \|\mathcal{R}_1 \rho\|_{L^2_T(H^s)} \leq \|\rho_{in}\|_{H^s} \tag{2}$$

Issue: this only gives a L^2-in-time bound that is not enough to control the advection term (except if one assumes $s \geq 20$).

Anisotropic Besov spaces

To derive additional properties from $\partial_t\rho - \mathcal{R}_1^2\rho = 0$, we will use Littlewood-Paley decompositions adapted to $\mathcal{R}_1$ whose symbol is $\dfrac{\xi_1}{|\xi|}$.

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- $\dot{\Delta}_j$ the blocks associated to the direction $|\xi|$;
- $\dot{\Delta}_q^h$ the blocks associated to the direction ξ_1,

and we define the following *homogeneous anisotropic* Besov semi-norms:

$$\|f\|_{\dot{B}^{s_1,s_2}} \triangleq \left\| 2^{js_1} 2^{qs_2} \|\dot{\Delta}_j \dot{\Delta}_q^h f\|_{L^2(\mathbb{R}^d)} \right\|_{\ell^1(j\in\mathbb{Z}, k\in\mathbb{Z})}.$$

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• Recall that $\dot{\Delta}_j$ localises the support of the Fourier transform of a distribution in an annulus and $\dot{\Delta}_q^h$ localises it in a stripe. Therefore $\dot{\Delta}_j \dot{\Delta}_q^h$ localises in the intersection of an annulus and a stripe.

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• Main interest: Bernstein properties are available in both directions $|\xi|$ and ξ_1.

Anisotropic Besov approach

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• Such spaces have been used in the past by Chemin, Paicu, Zhang, Xin et al., for instance in the context of the anisotropic Navier-Stokes system and the MHD system.

- Applying both localisations, we get

$$\partial_t \dot{\Delta}_j \dot{\Delta}_q^h \rho - \mathcal{R}_1^2 \dot{\Delta}_j \dot{\Delta}_q^h \rho = 0.$$

Standard energy estimates yield

$$\frac{d}{dt} \|\dot{\Delta}_j \dot{\Delta}_q^h \rho\|_{L^2}^2 + \|\mathcal{R}_1 \dot{\Delta}_j \dot{\Delta}_q^h \rho\|_{L^2}^2 = 0$$

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$$\frac{d}{dt} \|\dot{\Delta}_j \dot{\Delta}_q^h \rho\|_{L^2}^2 + 2^{-2j} 2^{2q} \|\dot{\Delta}_j \dot{\Delta}_q^h \rho\|_{L^2}^2 = 0$$

A new bound

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Now, one can apply Gronwall-like inequality to "simplify the squares":

$$\|\dot{\Delta}_j \dot{\Delta}_q^h \rho(t)\|_{L^2} + 2^{-2j} 2^{2q} \int_0^t \|\dot{\Delta}_j \dot{\Delta}_q^h \rho\|_{L^2} \leq \|\dot{\Delta}_j \dot{\Delta}_q^h \rho_{in}\|_{L^2}$$

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Then, for any $s_1, s_2 \in \mathbb{R}$, multiplying by $2^{js_1} 2^{qs_2}$ and summing on $j, q \in \mathbb{Z}$:

$$\|\rho\|_{L_T^\infty(\dot{B}^{s_1,s_2})} + \|\rho\|_{L_T^1(\dot{B}^{s_1-2,s_2+2})} \lesssim \|\rho_{in}\|_{\dot{B}^{s_1,s_2}} \qquad (3)$$

Using the embedding $\dot{B}^{\frac{3}{2},\frac{1}{2}} \hookrightarrow \dot{W}^{1,\infty}$, one has:

$$\|\nabla\mathcal{R}_1^2\rho\|_{L^\infty} \lesssim \|\mathcal{R}_1^2\rho\|_{\dot{B}_{2,1}^{\frac{3}{2},\frac{1}{2}}} \lesssim \|\rho\|_{\dot{B}_{2,1}^{-\frac{1}{2},\frac{5}{2}}}.$$

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• And to deal with the nonlinearities, we develop new product laws, adapted to this anisotropic framework, and absorb them by the linear left-hand side side.

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Open question: Global well-posedness in the case $s_1 + s_2 = 2$ or $s > 2$?

• For initial data **below** H^2, Kiselev and Yao ('22) proved the time-growth of the norm $\|\rho - \bar{\rho}\|_{H^{s'}}$ for any $s' \geq 1$ and any stratified smooth steady state $\bar{\rho}$ arbitrarily close to the solution, in the bounded strip $\mathbb{T} \times [-\pi, \pi]$.

Relaxation approximation of (IPM)

The two-dimensional Boussinesq system reads

$$\begin{cases} \partial_t \eta + \mathbf{u} \cdot \nabla \eta = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \eta \mathbf{g}, \qquad \mathbf{g} = (0, -g), \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{E}$$

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Considering a damping in the equation of the vorticity and linearizing around the same linear steady states as before, it is shown by Bianchini and Natalini that (E) can be recast into

$$\begin{cases} \partial_t b - \mathcal{R}_1 \Omega = (\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot (\nabla b), \\ \partial_t \Omega - \mathcal{R}_1 b + \dfrac{\Omega}{\varepsilon} - = \Lambda^{-1}[(\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot (\nabla \Lambda \Omega)], \end{cases} \tag{2D-B}$$

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For this system:

- Wan (19') proved the global well-posedness in H^s with $s \geq 5$.
- Bianchini and Natalini (21') derived time-decay estimates in the same setting.
- Quid of $\varepsilon \to 0$?

Formal link between (IPM) and (2D-B)

Let us have a closer look at the linear structure:

$$\begin{cases} \partial_t b - \mathcal{R}_1 \Omega = 0, \\ \partial_t \Omega - \mathcal{R}_1 b + \dfrac{\Omega}{\varepsilon} = 0. \end{cases} \tag{4}$$

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Taking inspiration from the theory of partially dissipative systems, in the following "diffusive" scaling:

$$(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)(\tau, x) \triangleq (b, \frac{\Omega}{\varepsilon})(t, x) \quad \text{with} \quad \tau = \varepsilon t, \tag{5}$$

the system (2D-B), in the scaled unknowns $(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)$, reads:

$$\begin{cases} \partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1 \widetilde{\Omega}^\varepsilon = 0, \\ \varepsilon^2 \partial_t \widetilde{\Omega}^\varepsilon - \mathcal{R}_1 \widetilde{b}^\varepsilon + \widetilde{\Omega}^\varepsilon = 0. \end{cases} \tag{6}$$

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Formally, as $\varepsilon \to 0$, the second equation gives the Darcy's law $\widetilde{\Omega}^\varepsilon = \mathcal{R}_1 \widetilde{b}^\varepsilon$ and inserting it in the first one gives the linear part of the incompressible porous media equation:

$$\partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1^2 \widetilde{b}^\varepsilon = 0.$$

Theorem (Bianchini-CB-Paicu ('22))

Let $(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)$ be the unique solution of (2D-B) associated to $(b_{\rm in}, \Omega_{in}) \in H^{3^+}$. Then, for any $0 < s' < s$ and $0 < \tau < \tau' < 1$, as $\varepsilon \to 0$,

$$\widetilde{b}^\varepsilon \to \rho \text{ strongly in } C([0,T], \dot{H}^{1-\tau'}_{loc} \cap \dot{H}^{s-s'}_{loc}),$$

where ρ is the unique solution of (IPM) associated to the initial data b_{in}.

Moreover, we recover the Darcy law in the following sense:

$$\|\widetilde{\Omega}^\varepsilon - \mathcal{R}_1 \widetilde{b}^\varepsilon\|_{L^1_T(B^{\frac32,\frac12} \cap B^{\frac12,\frac12})} \leq \varepsilon \mathcal{M}(0).$$

Relaxation Theorem

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Moreover, we recover the Darcy law in the following sense:

$$\|\widetilde{\Omega}^\varepsilon - \mathcal{R}_1 \widetilde{b}^\varepsilon\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}} \cap B^{\frac{1}{2},\frac{1}{2}})} \leq \varepsilon \mathcal{M}(0).$$

Proof: follows from uniform estimates established for the system (2D-B). Again, we extract crucial a priori bounds for the solution thanks to the use of anisotropic Besov spaces.

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Future researches:

- Explicit convergence rate?
- What about more general operators?
- Which nonlinearities we can handle for general operators.

Thank you for your attention.

Lemma (Embedding in Sobolev space)

Let $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$ *such that* $\tau_1 < s_1 + s_2 < \tau_2$ *and* $s_2 > 0$. *If* $a \in \dot{H}^{\tau_1}(\mathbb{R}^2) \cap \dot{H}^{\tau_2}(\mathbb{R}^2)$ *and* $a \in B^{s_1,s_2}$, *then*

$$\|a\|_{B^{s_1,s_2}} \lesssim \|a\|_{B^{s_1+s_2}} \lesssim \|a\|_{\dot{H}^{\tau_1}} + \|a\|_{\dot{H}^{\tau_2}}.$$