Cavitation and Concentration in the Entropy Solutions of the Euler Equations & Related Nonlinear PDEs in Fluid Dynamics

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Euler Equations in Compressible Fluid Mechanics

$$\begin{cases} \rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0 \end{cases}$$

$U = (\rho, \rho\mathbf{v})^\top$ $\quad \rho$ – Density, $\quad \mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}^d$ – Velocity

$p = p(\rho) = \rho^2 e'(\rho)$ – Pressure with internal energy $e(\rho)$

For a polytropic perfect gas: $\quad p(\rho) = a_0 \rho^\gamma, \;\; e(\rho) = \frac{a_0}{\gamma-1}\rho^{\gamma-1}, \;\; \gamma > 1$

Paradigm: **Nonlinear Hyperbolic Conservation Laws**

$$\partial_t U + \nabla_{\mathbf{x}} \cdot \mathbf{F}(U) = 0$$

$U = (u_1, \cdots, u_m)^\top, \;\; \mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d, \;\; \nabla_{\mathbf{x}} = (\partial_{x_1}, \cdots, \partial_{x_d})$

$\mathbf{F} = (F_1, \cdots, F_d): \;\; \mathbb{R}^m \to (\mathbb{R}^m)^d$ is a nonlinear mapping.

Hyperbolicity in $\mathcal{D}$: For any $\omega \in S^{d-1}$, $\mathbf{u} \in \mathcal{D}$,

$$(\nabla_U \mathbf{F}(U) \cdot \omega)_{m\times m}\, \mathbf{r}_j(U,\omega) = \lambda_j(U,\omega)\, \mathbf{r}_j(U,\omega), \;\; 1 \le j \le m$$

$\lambda_j(U,\omega)$ **are real**

Challenges and Well-Posedness: Euler Equations

$$\partial_t U + \nabla_{\mathbf{x}} \cdot \mathbf{F}(U) = 0$$

Challenges: Singularities $\longrightarrow$ Discontinuous/Wild/Singular Solutions

- Shock Waves, Vortex Sheets, Vorticist Waves, Entropy Waves, ...
- Compactness and Oscillation $\Longleftrightarrow$ Uniqueness ??
- ***Cavitation/Decavitation** $\Longrightarrow$ Degeneracy, $\cdots$
- ***Concentration/Deconcentration** $\Longrightarrow$ ∞–Propagation Speed, $\cdots$
- $\ldots\ldots$

Entropy Solutions:

(i) $U(t, \mathbf{x}) \in BV, L^\infty, L^p, \mathcal{M}$;

(ii) For any convex entropy pair $(\eta, \mathbf{q})$, $\partial_t \eta(U) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(U) \le 0 \;\; \mathcal{D}'$

as long as $(\eta(U(t, \mathbf{x})), \mathbf{q}(U(t, \mathbf{x}))) \in \mathcal{D}'$, for $(\eta, \mathbf{q}) := (\eta, q_1, \ldots, q_d)$ that is a solution of $\nabla q_k(U) = \nabla \eta(U) \nabla \mathbf{F}_k(U), k = 1, \ldots, d$, and $\nabla^2 \eta(U) \ge 0$.

Posed Classes of Entropy Solutions in $BV, L^\infty, L^p, \mathcal{M}, \ldots$??

Nonlinear Hyperbolic Conservation Laws

Scalar Conservation Laws: L^∞ initial data

Maximum principle $\Rightarrow$ **Uniform bounded in L^∞** $\Rightarrow$ **Deconcentration**

1-**D Strictly Hyperbolic Systems**: BV initial data of small oscillation

BV-estimates (Glimm 1965): **Decavitation & Deconcentration**

- Glimm Scheme, Wave-Front Tracking Methods, $\cdots$
- Artificial Viscosity Methods, $\cdots$

See recent books: D. Serre: Cambridge University Press, 1999-2000
A. Bressan: Oxford University Press, 2000
C. M. Dafermos: Springer-Verlag, 2016 (4th Ed.)
......

Further Fundamental Issues:

- Large initial data without total variation ??
- Nonstrictly hyperbolic cases ??
- Multidimensional cases ??
- ......

1-D Isentropic Euler Equations: Cavitation/Concentration

$$\begin{cases} \rho_t + m_x = 0, & (m = \rho v) \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x = 0. \end{cases}$$

ρ – density, m – momentum, $v = \frac{m}{\rho}$ – velocity when $\rho > 0$

Eigenvalues: $\lambda_1(\rho, m) = v - \sqrt{p'(\rho)}$, $\lambda_2(\rho, m) = v + \sqrt{p'(\rho)}$

Cavitation $V \equiv \{\rho(t,x) = 0\}$: $(\lambda_2 - \lambda_1)(\rho(t,x), m(t,x)) = 0$ for $(t,x) \in V$

$\Longrightarrow$ strict hyperbolicity fails.

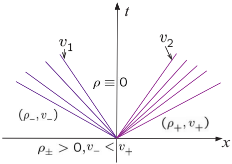
Concentration $S \equiv \{\rho(t,x) \sim \sum \alpha_j \delta_{S_j} + \rho_{\mathrm{nonatomic}}(t,x)\}$

$\Longrightarrow$ Infinite/ill-defined pressure, if it would occur

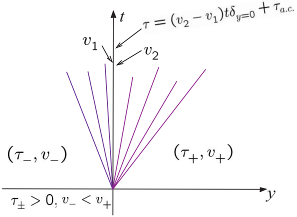
$\Longrightarrow$ ∞–propagation speed, if it would occur

Cavitation & Concentration: Pressure $p(\rho) = a_0\rho^\gamma, \gamma > 1$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$



$$(t,x) \to (t,y): \ y_t = \rho(t,x), \ y_x = -(\rho v)(t,x); \qquad \tau(t,y) = 1/\rho(t,x)$$



$$\partial_t \tau - \partial_y v = 0, \qquad \partial_t v + \partial_y p(1/\tau) = 0$$

Cavitation & Concentration: Pressure $p(\rho) = a_0\rho^\gamma, \gamma > 1$

$$\partial_t\rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$

Theorem (Global Existence of Entropy Solutions)

Let the Cauchy initial data satisfy

$$0 \le \rho_0(x) \le C_0, \qquad |m_0(x)| \le C_0\rho_0(x)$$

for some $C_0 > 0$. *Then there exists a global entropy solution* $(\rho, m)(t,x) = (\rho, \rho v)(t,x)$ *of the Cauchy problem such that*

$$0 \le \rho(t,x) \le C, \qquad |m(t,x)| \le C\rho(t,x),$$

where $C > 0$ *is a constant depending only on* $\gamma > 1$, $a_0 > 0$, *and* $C_0 > 0$.

DiPerna: $\gamma = \frac{N+2}{N}, N \ge 5$ odd, Ding-Luo & Chen: $\gamma \in (1, \frac{5}{3}]$,
Lions-Perthame-Tadmor: $\gamma \ge 3$, Lions-Perthame-Souganidis: $\gamma \in (\frac{5}{3}, 3)$,
Chen-LeFloch: General pressure laws

***Entropy Analysis for the measure-valued solution (Young measure) with compact support**

Isentropic Euler Equations: Vanishing Pressure Limit

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + \varepsilon\, p(\rho)) = 0.$$

Riemann Problem: $\rho_\pm > 0$

$$(\rho, v)|_{t=0} = \begin{cases} (\rho_-, v_-) & \text{for } x < 0, \\ (\rho_+, v_+) & \text{for } x > 0. \end{cases}$$

Consider the following two distinguished cases:

Two-rarefaction wave Riemann solution with $v_- < v_+$ **and** $\rho_\pm > 0$:

$$(\rho, v)(\frac{x}{t}) = \begin{cases} (\rho_-, v_-) & \text{for } x < v_- t, \\ \text{1-rarefaction wave} & \text{for } v_- t < x < v_*^\varepsilon t, \\ (\rho_*^\varepsilon, v_*^\varepsilon) & \text{for } v_- t < x < v_*^\varepsilon t, \\ \text{2-rarefaction wave} & \text{for } v_*^\varepsilon t < x < v_+ t, \\ (\rho_+, v_+) & \text{for } x > v_+ t. \end{cases} \tag{1}$$

Two-shock Riemann solution with $v_- > v_+$ **and** $\rho_\pm > 0$:

$$(\rho, v)(\frac{x}{t}) = \begin{cases} (\rho_-, v_-) & \text{for } x < \sigma_1^\varepsilon t, \\ (\rho_*^\varepsilon, v_*^\varepsilon) & \text{for } \sigma_1^\varepsilon t < x < \sigma_2^\varepsilon t, \\ (\rho_+, v_+) & \text{for } x > \sigma_2^\varepsilon t. \end{cases} \tag{2}$$

Vanishing Pressure Limit: Cavitation

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + \varepsilon p(\rho)) = 0$$

When $\varepsilon \to 0$, the two-rarefaction wave Riemann solution with $v_- < v_+$ and $\rho_\pm > 0$:

$$(\rho, v)(\frac{x}{t}) = \begin{cases} (\rho_-, v_-) & \text{for } x < v_- t, \\ \text{1-rarefaction wave} & \text{for } v_- t < x < v_*^\varepsilon t, \\ (\rho_*^\varepsilon, v_*^\varepsilon) & \text{for } v_- t < x < v_*^\varepsilon t, \\ \text{2-rarefaction wave} & \text{for } v_*^\varepsilon t < x < v_+ t, \\ (\rho_+, v_+) & \text{for } x > v_+ t \end{cases}$$

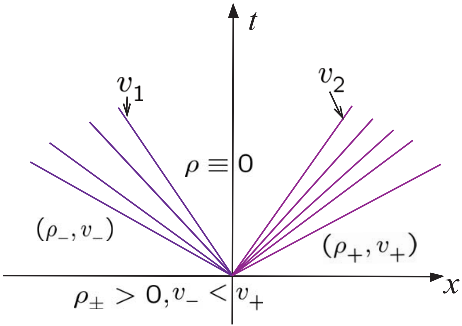
converges to a solution of the pressureless Euler equations containing a vacuum state that fills up the region formed by the two contact discontinuities $x = v_\pm t$:

$$(\rho, v)(\frac{x}{t}) = \begin{cases} (\rho_-, v_-) & \text{for } x < v_- t, \\ (0, \frac{x}{t}) & \text{for } v_- t < x < v_+, \\ (\rho_+, v_+) & \text{for } x > v_+ t. \end{cases}$$

G.-Q. Chen & H. Liu: SIAM J. Math. Anal. 34 (2003), 925–938

Formation Process of Cavitation as $\varepsilon \to 0$

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + \varepsilon p(\rho)) = 0. \end{cases}$$



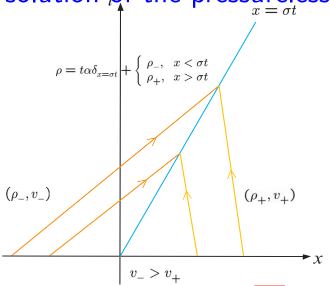
Vanishing Pressure Limit: Concentration

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + \varepsilon p(\rho)) = 0.$$

When $\varepsilon \to 0$, the two-shock Riemann solution with $v_- > v_+$ and $\rho_\pm > 0$:

$$(\rho, v)(\frac{x}{t}) = \begin{cases} (\rho_-, v_-) & \text{for } x < \sigma_1^\varepsilon t, \\ (\rho_*^\varepsilon, v_*^\varepsilon) & \text{for } \sigma_1^\varepsilon t < x < \sigma_2^\varepsilon t, \\ (\rho_+, v_+) & \text{for } x > \sigma_2^\varepsilon t \end{cases}$$

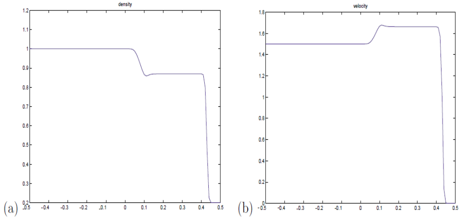
converges to a δ-shock solution of the pressureless Euler equations:



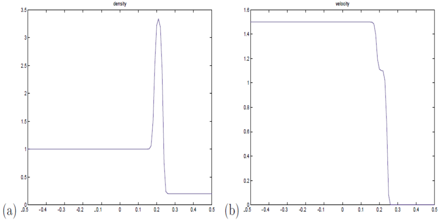
$$\alpha = \frac{1}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho v]) > 0, \qquad \sigma = \frac{\sqrt{\rho_+}v_+ + \sqrt{\rho_-}v_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} \in (v_+, v_-)$$

G.-Q. Chen & H. Liu: SIAM J. Math. Anal. 34 (2003), 925–938

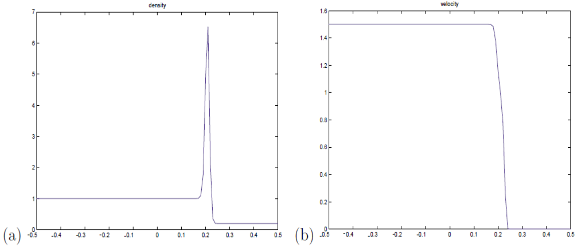
Formation Process of Concentration – δ-Shocks



Density and velocity for $\epsilon = 1.4$.



Density and velocity for $\epsilon = 0.07$.

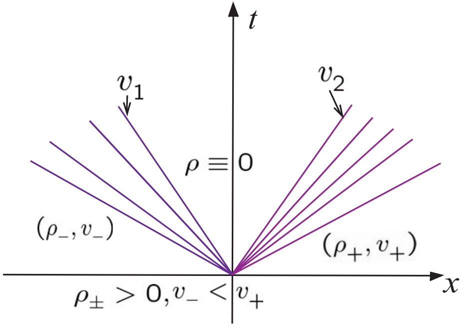


Density and velocity for $\epsilon = 0.0014$.

G.-Q. Chen & H. Liu: SIAM J. Math. Anal. 34 (2003), 925–938

Isothermal Limit: Process of Decavitation as $\gamma \to 1$

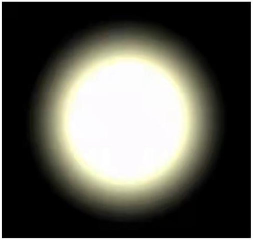
$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p_\gamma(\rho)) = 0, \end{cases} \qquad p_\gamma(\rho) = a_0 \rho^\gamma.$$

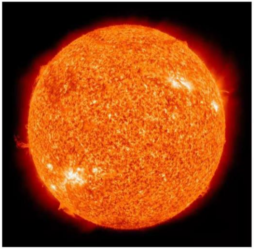


*G.-Q. Chen, F. Huang & T.-Y. Wang: Isothermal Limit of Entropy Solutions of the Euler Equations for Isentropic Gas Dynamics, arXiv:2202.02235, 2022.

Spherically Symmetric Solutions

- **The study of spherically symmetric solutions can date back to the 1950s and has been motivated by many important physical problems such as stellar dynamics including gaseous stars and supernova formation.**
- **Open Question: Could concentration (or cavitation) be formed at the origin, *i.e.*, the density becomes a Dirac measure (or zero) at the origin, especially when a focusing (defocusing) spherical shock is moving inward (outward) the origin?**





Multidimensional Isentropic Euler Equations

$$\begin{cases} \rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0. \end{cases}$$

$\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\quad \nabla_{\mathbf{x}}$ – Gradient w.r.t. $\mathbf{x} \in \mathbb{R}^d$
ρ – Density, $\quad \mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}^d$ – Velocity,
$p = p(\rho) = \rho^2 e'(\rho)$ – Pressure with internal energy $e(\rho)$

For a polytropic perfect gas: $\quad p(\rho) = a_0 \rho^\gamma, \;\; e(\rho) = \frac{a_0}{\gamma-1}\rho^{\gamma-1}, \;\; \gamma > 1$

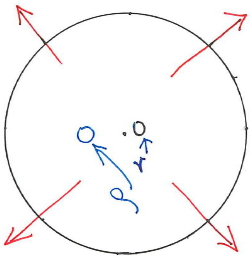
Spherically Symmetric Solutions:

$$\rho(t, \mathbf{x}) = \rho(t, r), \qquad \mathbf{v}(t, \mathbf{x}) = v(t, r)\frac{\mathbf{x}}{r}, \qquad r = |\mathbf{x}|.$$

Then the functions $(\rho, m) = (\rho, \rho v)$ **are governed by**

$$\begin{cases} \rho_t + m_r + \frac{d-1}{r} m = 0, \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_r + \frac{d-1}{r}\frac{m^2}{\rho} = 0. \end{cases}$$

Defocusing: Expanding Spherically Symmetric Solutions



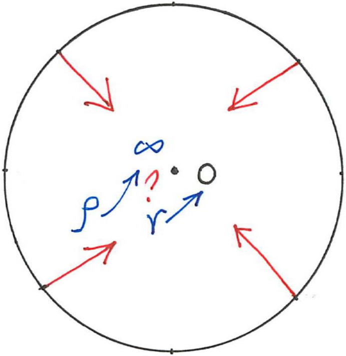
G.-Q. Chen: Proc. Royal Soc. Edinburgh, 127A (1997), 243–259.

$$0 \leq \int_0^{\rho_0(r)} \frac{\sqrt{p'(s)}}{s}\,\mathrm{d}s \leq v_0(r) \leq C < \infty$$

$\Longrightarrow$ **Formation of Cavitation near the origin**
via Finite Difference Scheme....

* M. Slemrod: PRSE, 1996: Spherical Self-Similar Piston Problem
* F. Huang, T.-H. Li & D. Yuan 2019, $\cdots\cdots$

Focusing: Imploding Spherically Symmetric Solutions



Guderley 1942, Courant-Friedrichs 1945, ...

Merle-Raphaël-Ronianski-Szeftel 2022: Singularity of Self-Similar Solutions

Rauch 1986: No BV or L^∞ Bounds

Longstanding Problem: Does the concentration occur generically?

$\Longleftrightarrow$ Does the density develop into a measure at the origin generically?

Spherically Symmetric Solutions for the Euler Equations via Navier-Stokes Viscosity Limits

Theorem (Chen-Wang: ARMA 2022, Chen-Schrecker: ARMA 2018
Chen-Perepelitsa: CMP 2015)

Let the initial functions (ρ_0, m_0) satisfy the relative finite-energy conditions with $\bar{\rho} := \lim_{r\to\infty} \rho_0(r) \geq 0$.

$\Longrightarrow$ **There exists a sequence of Navier-Stokes-type approximate solutions $(\rho^\varepsilon, m^\varepsilon), m^\varepsilon = \rho^\varepsilon v^\varepsilon$, for $\varepsilon > 0$ such that, when $\varepsilon \to 0$, there exists a subsequence of $(\rho^\varepsilon, m^\varepsilon)$ that converges strongly almost everywhere to a finite-energy spherically symmetric entropy solution (ρ, m) with**

$$\rho(t, \mathbf{x}) = \rho(t, |\mathbf{x}|), \quad (\rho\mathbf{v})(t, \mathbf{x}) = m(t, |\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \qquad \text{for all } \gamma > 1.$$

***There DO exist entropy solutions (as zero viscosity limits) even $\bar{\rho} > 0$ with ∞–propagation speed, but without concentration at the origin!!**

Entropy Analysis I

$$\partial_t U + \partial_r F(U) = G(U, r), \qquad U \in \mathbb{R}^2$$

Entropy-Entropy Flux Pair (η, q) if they satisfy the 2×2 hyperbolic system:

$$\nabla q(U) = \nabla \eta(U) \nabla F(U).$$

For smooth solution U, $\partial_t \eta(U) + \partial_r q(U) = \nabla \eta(U) G(U, r)$.

If the system is endowed with globally defined Riemann invariants $w_i(U), 1 \le i \le 2$, satisfying $\nabla w_i(U) \cdot \nabla F(U) = \lambda_i(U) \nabla w_i(U)$ so that

$$q_{w_i} = \lambda_i \eta_{w_i}, \qquad i = 1, 2.$$

That is, the entropy function η is determined by

$$\eta_{w_1 w_2} + \frac{\lambda_{2 w_1}}{\lambda_2 - \lambda_1} \eta_{w_2} - \frac{\lambda_{1 w_2}}{\lambda_2 - \lambda_1} \eta_{w_1} = 0.$$

For the Euler system, η is determined by the **Euler-Poisson-Darboux equation**:

$$\eta_{w_1 w_2} + \frac{\alpha}{w_2 - w_1}(\eta_{w_2} - \eta_{w_1}) = 0, \qquad \alpha = \frac{3 - \gamma}{2(\gamma - 1)}.$$

Entropy Analysis - II

$$\begin{cases} \rho_t + m_r = -\frac{d-1}{r} m, & (m = \rho v) \\ m_t + (\frac{m^2}{\rho} + p(\rho))_r = -\frac{d-1}{r}\frac{m^2}{\rho}. \end{cases}$$

Strict Hyperbolicity – fails: $\lambda_2 - \lambda_1 = 2\sqrt{p'(\rho)} \to 0$ when $\rho \to 0$ (vacuum)

Entropy Pair (η, q): $\nabla q(U) = \nabla \eta(U) \nabla F(U)$ for $U = (\rho, m)^\top$

Convex Entropy: $\nabla^2 \eta(U) > 0$ **Weak Entropy**: $\eta(\rho, \rho v)|_{\rho=0} = 0$

Weak entropy pairs are represented as

$$\eta^\psi(\rho, \rho v) = \int_{\mathbb{R}} \chi(s)\psi(s)\,\mathrm{d}s, \quad q^\psi(\rho, \rho v) = \int_{\mathbb{R}} (\theta s + (1-\theta)v)\chi(s)\psi(s)\,\mathrm{d}s$$

by C^2-functions $\psi(s)$, where $\chi(s)$ is the weak entropy kernel:

$$\chi(s) := [\rho^{2\theta} - (v-s)^2]_+^\alpha, \qquad \theta = \frac{\gamma - 1}{2}, \alpha = \frac{3-\gamma}{2(\gamma-1)}$$

Physical Convex Entropy: **Mechanical energy-energy flux pair** (η_*, q_*)**:**

$$\eta_*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho), \qquad q_*(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + m(e(\rho) + \frac{p}{\rho})$$

Entropy Analysis - III: L^p–Compactness Framework

Theorem (L^p-Compensated Compactness Framework)

Let a function sequence $(\rho^\varepsilon, m^\varepsilon)(t,r)$ defined on a compact domain $\Omega \Subset \mathbb{R}_+ \times \mathbb{R}_+$ satisfy

- *There exists a constant $C > 0$, independent of $\varepsilon > 0$, such that*

$$\|\rho^\varepsilon\|_{L^{\max\{\gamma+1,\gamma+\theta\}}(\Omega)} + \Big\|\frac{(m^\varepsilon)^3}{(\rho^\varepsilon)^2}\Big\|_{L^1(\Omega)} \le C \qquad \text{for } \theta = \frac{\gamma-1}{2}.$$

- *For any weak entropy pair generated by compactly supported test function $\psi \in C_c^2(\mathbb{R})$ such that the corresponding sequence of entropy dissipation measures*

$$\partial_t \eta^\psi(\rho^\varepsilon, m^\varepsilon) + \partial_r q^\psi(\rho^\varepsilon, m^\varepsilon) \qquad \text{is compact in } H^{-1}(\Omega).$$

Then there exist both a subsequence (still denoted) $(\rho^\varepsilon, m^\varepsilon)(t,r)$ and a measurable vector function $(\rho, m)(t,r)$ such that

$$(\rho^\varepsilon, m^\varepsilon)(t,r) \to (\rho, m)(t,r) \qquad \text{a.e. as } \varepsilon \to 0.$$

L^p–Framework for General $\gamma > 1$: Chen-Perepelitsa, CPAM 2010

* DiPerna, Ding-Luo-Chen, Lions-Perthame-Souganidis-Tadmor, Chen-LeFloch, LeFloch-Westdickenberg, $\cdots$

Multidimensional Euler-Poisson Equations

$$\begin{cases} \rho_t + \nabla\cdot\mathcal{M} = 0, \\ \mathcal{M}_t + \nabla\cdot\big(\frac{\mathcal{M}\otimes\mathcal{M}}{\rho}\big) + \nabla p + \rho\nabla\Phi = 0, \\ \Delta\Phi = \kappa\rho, \qquad\qquad \mathbf{x} = (x_1,\dots,x_d)\in\mathbb{R}^d. \end{cases}$$

ρ – Density, $\mathbf{v} = (v_1,\dots,v_d)\in\mathbb{R}^d$ – Velocity, $\nabla_{\mathbf{x}}$ – Gradient w.r.t. $\mathbf{x}\in\mathbb{R}^d$
Φ – Gravitational potential of gaseous stars if $\kappa = 4\pi g > 0$ when $d = 3$
& plasma electric field potential if $\kappa < 0$

Spherically Symmetric Solutions:

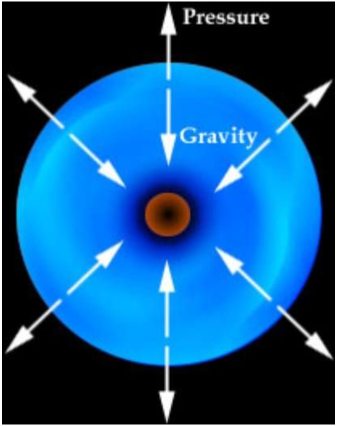
$$\rho(t,\mathbf{x}) = \rho(t,r), \quad \mathbf{v}(t,\mathbf{x}) = v(t,r)\frac{\mathbf{x}}{r}, \quad \Phi(t,\mathbf{x}) = \Phi(t,r), \qquad\qquad r = |\mathbf{x}|.$$

Then the functions $(\rho, m) = (\rho, \rho v)$ are governed by

$$\begin{cases} \rho_t + m_r = -\frac{d-1}{r}m, \\ m_t + \big(\frac{m^2}{\rho} + p(\rho)\big)_r = -\rho\Phi_r - \frac{d-1}{r}\frac{m^2}{\rho}, \\ \Phi_{rr} + \frac{d-1}{r}\Phi_r = \kappa\rho. \end{cases}$$

Mathematical Model of Self-Gravitating Newtonian Gaseous Stars

A gaseous star is modeled as a compactly supported gaseous fluid surrounded by vacuum subject to self-gravitation.



Euler-Poisson Equations with $\kappa > 0$
Self-Gravitational Gaseous Stars: Smooth Solutions

- Chandrasekhar 1938:
  - $\gamma > \frac{2d}{d+2}$ (*e.g.* $\gamma > \frac{6}{5}$ for $d = 3$) is necessary to ensure the global existence of finite-energy solutions with finite mass, which corresponding to the one for the Lane-Emden solutions.
  - There no exist steady white dwarf star with total mass larger than the Chandrasekhar limit M_{ch} when $\gamma \in (\frac{6}{5}, \frac{4}{3}]$ for $d = 3$.
- Goldreich-Webber 1980 (see also Deng-Xiang-Yang 2003, Fu-Lin 1998, Makino 1992): There exist homologous self-similar collapsing solutions when $\gamma = \frac{4}{3}$ for $d = 3$.
- **Guo-Hadzic-Jang** (ARMA 2021): $\exists$ ∞**–D family of collapsing solutions.**
  $\gamma \in (1, \frac{4}{3})$ (mass supercritical) & Mach number $\gg 1 \Longrightarrow$ **Concentration**

  Lei-Gu 2016, Luo-Xin-Zeng 2014, Makino 1986, $\cdots\cdots$.

Weak Solutions outside a solid ball $|\mathbf{x}| \geq 1$: Makino 1997, Xiao 2016, $\cdots$

Open Problem: **?** $\exists$ **Global Weak Entropy Solutions including the Origin**??
Even under Self-Gravitation?

Finite Initial Total-Energy and Total-Mass

Initial Condition:

$$(\rho, \mathcal{M})|_{t=0} = (\rho_0(\mathbf{x}), \mathcal{M}_0(\mathbf{x})) = (\rho_0(|\mathbf{x}|), m_0(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}) \longrightarrow (0, \mathbf{0}) \ \text{ as } |\mathbf{x}| \to \infty$$

Asymptotic Condition:

$$\Phi(t, \mathbf{x}) = \Phi(t, |\mathbf{x}|) \longrightarrow 0 \qquad \text{as } |\mathbf{x}| \to \infty$$

Finite initial total-energy:

$$E_0 := \int_{\mathbb{R}^d} \Big(\frac{1}{2}\Big|\frac{\mathcal{M}_0}{\sqrt{\rho_0}}\Big|^2 + \rho_0 e(\rho_0)\Big)(\mathbf{x})\,\mathrm{d}\mathbf{x} < \infty \qquad \text{for } \kappa > 0.$$

Finite initial total-mass:

$$M := \int_{\mathbb{R}^d} \rho_0(\mathbf{x})\,\mathrm{d}\mathbf{x} = \omega_d \int_0^\infty \rho_0(r)\, r^{d-1}\mathrm{d}r < \infty,$$

where $e(\rho) := \frac{a_0}{\gamma-1}\rho^{\gamma-1}$ represents the internal energy, and $\omega_d := \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ denotes the surface area of the unit sphere in $\mathbb{R}^d$.

Spherically Symmetric Solutions for the Euler-Poisson Equations via inviscid Navier-Stokes-Poisson-type Limits

Theorem (Chen-He-Wang-Yuan: CPAM 2022)

Let $(\rho_0, m_0)(|\mathbf{x}|)$ satisfy the finite-energy and finite-mass conditions.
$\Longrightarrow$ There exist Navier-Stokes-Poisson-type viscosity solutions $(\rho^\varepsilon, m^\varepsilon, \Phi^\varepsilon)$ for $\varepsilon > 0$ such that, when $\varepsilon \to 0$, there exists a subsequence of $(\rho^\varepsilon, m^\varepsilon, \Phi^\varepsilon)$ that converges strongly a.e. to a finite-energy spherically symmetric entropy solution $(\rho, m, \Phi)(t, r)$ with

$$\rho(t,\mathbf{x}) = \rho(t,|\mathbf{x}|), \quad \mathcal{M}(t,\mathbf{x}) = m(t,|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}, \quad \Phi(t,\mathbf{x}) = \Phi(t,|\mathbf{x}|)$$

for $\kappa > 0$ when $\gamma > \frac{2(d-1)}{d}$

or $\gamma \in (\frac{2d}{d+2}, \frac{2(d-1)}{d}]$ with the critical mass $M_c(\gamma)$

*There exist entropy solutions (as inviscid Navier-Stokes limits) with ∞–propagation speed, but without concentration, at the origin even under self-gravitation!!

Main Strategies

- Design an appropriate free boundary problem with
  - appropriate approximate initial data
  - stress-free boundary condition

  to construct the approximate solutions (involving the initial location $b > 0$ of the free boundary – a large parameter, besides the small parameter $\varepsilon > 0$) for CNSPEs.
- Obtain the trace estimates in the energy estimates & adopt the Bresch-Desjardins entropy to make uniform estimates of the approximate solutions, independent of $\varepsilon > 0$ and $b > 0$.
- Prove that the Navier-Stokes-Poisson viscosity solutions satisfy the L^p–compensated compactness framework after first taking $b \to \infty$, which then ensures the strong convergence of the viscosity solutions as $\varepsilon \to 0$.
- Verify that the strong limit functions are finite-energy global solutions of the compressible Euler-Poisson equations with large initial data of spherical symmetry.

Navier-Stokes-Poisson Approximate Solutions

Consider the following approximate free boundary problem for CNSPEs:

$$\begin{cases} \rho_t + (\rho v)_r + \frac{d-1}{r}\rho v = 0, \\ (\rho v)_t + (\rho v^2 + p)_r + \frac{d-1}{r}\rho v^2 + \frac{\kappa\rho}{r^{d-1}}\int_a^r \rho(t,y)\, y^{d-1}\mathrm{d}y \\ \qquad\qquad = \varepsilon\big(\rho(v_r + \frac{d-1}{r}v)\big)_r - \varepsilon\frac{d-1}{r}v\rho_r, \end{cases}$$

for $(t,r) \in \Omega_T := \{(t,r)\ :\ a \le r \le b(t),\, 0 \le t \le T\}$ (moving domain), with $a = b^{-1}$ for $b \gg 1$ and $\{r = b(t) : 0 < t \le T\}$ as a free boundary:

$$b'(t) = v(t, b(t)) \text{ for } t > 0, \qquad b(0) = b,$$

- On the free boundary $r = b(t)$, the stress-free boundary condition:

$$\big(p(\rho) - \epsilon\rho(v_r + \frac{d-1}{r}v)\big)(t, b(t)) = 0 \qquad \text{for } t > 0.$$

- On the fixed boundary $r = a = b^{-1}$, the Dirichlet boundary condition:

$$v|_{r=a} = 0 \qquad \text{for } t > 0.$$

- The initial condition: $(\rho, \rho v)|_{t=0} = (\rho_0^{\epsilon,b}, \rho_0^{\epsilon,b}v_0^{\epsilon,b})(r)$ for $r \in [a,b]$. $(\rho_0^{\epsilon,b}, v_0^{\epsilon,b})(r)$ are smooth/compatible and $0 < C_{\epsilon,b}^{-1} \le \rho_0^{\epsilon,b}(r) \le C_{\epsilon,b} < \infty$.

*Duan-Li, JDE 2015: $\kappa > 0$ with $\gamma \in (\frac{6}{5}, \frac{4}{3}] \Longrightarrow$ General as needed for $d \ge 2$.

Basic Energy Estimates for the Approximate Solutions: $\kappa > 0$

The approximate solution $(\rho, v)(t,r) := (\rho^{\epsilon,b}, v^{\epsilon,b})(t,r)$ satisfies the following energy identity:

$$\int_a^{b(t)} \big(\frac{1}{2}\rho v^2 + \rho e(\rho)\big)(t,r)\, r^{d-1}\mathrm{d}r - \frac{\kappa}{2}\int_a^{\infty} \frac{1}{r^{d-1}}\Big(\int_a^r \rho(t,y)\, y^{d-1}\mathrm{d}y\Big)^2 \mathrm{d}r$$
$$+ \epsilon \int_0^t \int_a^{b(s)} \big(\rho v_r^2 + (d-1)\rho \frac{v^2}{r^2}\big)(t,r)\, r^{d-1}\mathrm{d}r\mathrm{d}s$$
$$+ (n-1)\epsilon \int_0^t (\rho v^2)(s, b(s)) b(s)^{d-2}\, \mathrm{d}s$$
$$= \int_a^b \Big(\big(\frac{1}{2}\rho_0 v_0^2 + \rho_0 e(\rho_0)\big)(r) - \frac{\kappa}{2}\frac{1}{r^{2(d-1)}}\Big(\int_a^r \rho_0(t,y) y^{d-1}\mathrm{d}y\Big)^2\Big)\, r^{d-1}\mathrm{d}r,$$

where $\rho(t,r)$ is understood to be 0 for $r \in [0,a] \cup (b,\infty)$ in the 2^{nd} term of the right-hand side (RHS) and the 2^{nd} term of the left-hand side (LHS).

There are the **two cases**: $\gamma > \frac{2(d-1)}{d}$; $\gamma \in (\frac{2d}{d+2}, \frac{2(d-1)}{d}]$.

BD-Type Entropy Estimate

Given any fixed $T > 0$, then, for all $t \in [0, T]$,

$$
\begin{aligned}
&\epsilon^2 \int_a^{b(t)} \frac{|\rho(t,r)_r|^2}{\rho(t,r)}\, r^{d-1} \mathrm{d}r + \epsilon \int_0^t \int_a^{b(s)} |(\rho^{\frac{\gamma}{2}})_r|^2\, r^{d-1} \mathrm{d}r\mathrm{d}s \\
&\quad + p(\rho(t,b(t)))\, b^d(t) + \frac{1}{\epsilon} \int_0^t p(\rho(s,b(s)))p'(\rho(s,b(s)))\, b^d(s)\, \mathrm{d}s \\
&\le C(E_0, M, T).
\end{aligned}
$$

To obtain the derivative estimate of the density, we use the entropy identified by D. Bresch and B. Desjardins (2007).

To close the bound, we need to control the boundary term $p(\rho_0(b))b^d$ for the approximate initial data.

To resolve this issue, we construct the approximate initial data $(\rho_0^{\epsilon,b}, u_0^{\epsilon,b})$ so that $p(\rho_0^{\epsilon,b}(b))b^d$ are uniformly bounded.

Expanding of Domain Ω_T with Free Boundary

Given $T > 0$ and $\epsilon \in (0, \epsilon_0]$, there exists a positive constant $B(M, E_0, T, \epsilon) > 0$ such that, if $b \geq B(M, E_0, T, \epsilon)$,

$$b(t) \geq \frac{b}{2} \qquad \text{for } t \in [0, T]. \qquad\qquad (**)$$

* For the free boundary problem, a follow-up point is whether the free boundary domain Ω_T will expand to the whole space as $b \to \infty$; otherwise, it would not be a good approximation to the original Cauchy problem.

* We solve this difficulty by proving (**), provided $b \gg 1$.

***Uniform higher integrability**: For any $K \Subset [a, b(t)]$ and any $t \in [0, T]$,

$$\|\rho^{b,\varepsilon}\|_{L^{\max\{\gamma+1,\gamma+\theta\}}([0,T]\times K)} + \|\rho^{b,\varepsilon}(v^{b,\varepsilon})^3\|_{L^1([0,T]\times K)} \leq C(K, M, E_0, T).$$

Existence of Global Weak Solutions of CNSPEs

- Based on these uniform estimates just presented, we take the limit, $b \to \infty$, to obtain the global weak viscosity solutions of CNSPEs.
- Let (η, q) be a weak entropy pair for any smooth compact supported function $\psi(s)$ on $\mathbb{R}$. Then, for $\epsilon \in (0, \epsilon_0]$, the Navier-Stokes-Poisson viscosity solutions $(\rho^\epsilon, m^\epsilon)$ satisfy that

$$\partial_t \eta(\rho^\epsilon, m^\epsilon) + \partial_r q(\rho^\epsilon, m^\epsilon) \qquad \text{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+).$$

- Given any $T \in (0, \infty)$, the following uniform bounds hold for all $t \in [0, T]$:

$$\int_0^\infty \rho^\epsilon(t,r)\, r^{d-1}\mathrm{d}r = \int_0^\infty \rho_0^\epsilon(r)\, r^{d-1}\mathrm{d}r = M,$$

$$\int_0^\infty \eta^*(\rho^\epsilon, m^\epsilon)(t,r)\, r^{d-1}\mathrm{d}r + \epsilon \int_{\mathbb{R}^2_+} \frac{(m^\epsilon)^2(t,r)}{\rho^\epsilon(t,r)}\, r^{d-3}\mathrm{d}r\mathrm{d}t + \|\Phi^\epsilon(t)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}$$
$$+ \int_0^\infty \Big(\int_0^r \rho^\epsilon(t,y)\, y^{d-1}\mathrm{d}z\Big)\rho^\epsilon(t,r)\, r\mathrm{d}r + \|\nabla\Phi^\epsilon(t)\|_{L^2(\mathbb{R}^d)} \le C(M, E_0),$$

$$\epsilon^2 \int_0^\infty \big|(\sqrt{\rho^\epsilon(t,r)})_r\big|^2 r^{d-1}\mathrm{d}r + \epsilon \int_0^T \int_0^\infty \big|((\rho^\epsilon)^{\frac{\gamma}{2}})_r\big|^2 r^{d-1}\mathrm{d}r\mathrm{d}t \le C(M, E_0, T),$$

$$\|\rho^\varepsilon\|_{L^{\max\{\gamma+1,\gamma+\theta\}}([0,T]\times K)} + \Big\|\frac{(m^\epsilon)^3}{(\rho^\varepsilon)^2}\Big\|_{L^1([0,T]\times K)} \le C(K, M, E_0, T) \quad \text{for all } K \Subset (0,\infty).$$

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for $\kappa > 0$ when $\gamma > \frac{2(d-1)}{d}$

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*There exist entropy solutions (as inviscid Navier-Stokes limits) with ∞–propagation speed, but without concentration, at the origin even under self-gravitation!!

M-D Euler-Poisson Equations for White Dwarf Stars

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p + \rho \nabla \Phi = 0, \\ \Delta \Phi = \kappa \rho. \end{cases}$$

ρ – Density, $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$ – Velocity

Φ – Self-consistent electric field potential, $\kappa > 0$.

$p = p(\rho) = \rho^2 e'(\rho)$ – General pressure with internal energy $e(\rho)$

For a white dwarf star (Chandrasekhar 1938),

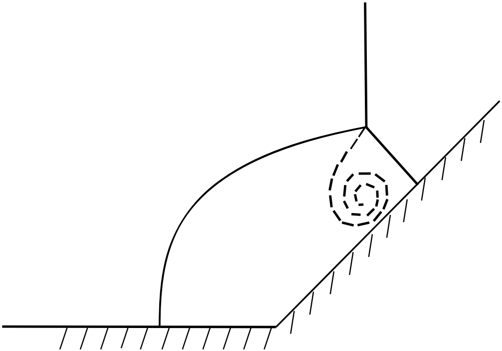
$$p(\rho) = A \int_0^{B\rho^{\frac{1}{3}}} \frac{\sigma^4}{\sqrt{D + \sigma^2}} \, \mathrm{d}\sigma \qquad \text{for } \rho > 0,$$

where A, B and D are positive constants.

$\Longrightarrow \qquad p(\rho) \cong \rho^{\frac{5}{3}}$ as $\rho \to 0$, $\qquad p(\rho) \cong \rho^{\frac{4}{3}}$ as $\rho \to \infty$.

*G.-Q. Chen, F. Huang, T.-H. Li, W. Wang, and Y. Wang:
Global Existence of Spherically Symmetric Solutions of the Compressible Euler-Poisson Equations for White Dwarf Stars, Preprint 2022.

Shock Reflection-Diffraction: Mach Reflection



? Does cavitation/concentration form at the center of vorticity wave?

? Right space for vorticity ω?

? Chord-arc $z(s) = z_0 + \int_0^s e^{ib(s)} ds$, $b \in BMO$?

***Chen-Feldman 2018 (Research Monograph): The Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures, 832 pages, Annals of Mathematics Studies, 197, Princeton University Press, 2018**

Challenges and Well-Posedness: Euler Equations

$$\partial_t U + \nabla_{\mathbf{x}} \cdot \mathbf{F}(U) = 0$$

Challenges: Singularities $\longrightarrow$ Discontinuous/Wild/Singular Solutions

- Shock Waves, Vortex Sheets, Vorticity Waves, Entropy Waves, ...
- Compactness and Oscillation $\Longleftrightarrow$ Uniqueness ??
- ***Cavitation and Decavitation** $\Longrightarrow$ Degeneracy, $\cdots$
- ***Concentration/Deconcentration** $\Longrightarrow$ ∞–Propagation Speed, $\cdots$
- $\ldots\ldots$

Entropy Solutions:

(i) $U(t, \mathbf{x}) \in BV, L^\infty, L^p, \mathcal{M}$;

(ii) For any convex entropy pair $(\eta, \mathbf{q})$, $\partial_t \eta(U) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(U) \leq 0 \ \ \mathcal{D}'$

as long as $(\eta(U(t,\mathbf{x})), \mathbf{q}(U(t,\mathbf{x}))) \in \mathcal{D}'$, for $(\eta, \mathbf{q}) := (\eta, q_1, \ldots, q_d)$ that is a solution of $\nabla q_k(U) = \nabla \eta(U) \nabla \mathbf{F}_k(U), k = 1, \ldots, d$, and $\nabla^2 \eta(U) \geq 0$.

Posed Classes of Entropy Solutions in $BV, L^\infty, L^p, \mathcal{M}, \ldots$??

Entropy Methods for the Analysis of Entropy Solutions of Multidimensional Conservation Laws?

A general mathematical framework may be derived from the theory of divergence-measure fields via the entropy methods, which are based on the **Entropy Solutions**:

(i) $U(t, \mathbf{x}) \in \mathcal{M},\ L^\infty,\ L^p,$ plus additional features when available;

(ii) For any convex entropy pair $(\eta, \mathbf{q})$,

$$\partial_t \eta(U) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(U) \le 0 \qquad \mathcal{D}'$$

as long as $(\eta(U(t, \mathbf{x})), \mathbf{q}(U(t, \mathbf{x}))) \in \mathcal{D}'$.

$$\Longrightarrow \quad \mathrm{div}_{(t,\mathbf{x})}(\eta(U(t, \mathbf{x})), \mathbf{q}(U(t, \mathbf{x}))) \in \mathcal{M}$$

$\Longrightarrow$ $(\eta(U(t, \mathbf{x})), \mathbf{q}(U(t, \mathbf{x}))) \in \mathcal{DM}(\mathbb{R}_+ \times \mathbb{R}^d)$ (divergence-measure field)

$\Longrightarrow$ Integration by parts, normal traces, ...

$\Longrightarrow$ Properties of entropy solutions, ...,

via **Entropy Methods** and **Theory of Divergence-Measure Fields**

*Chen-Frid: ARMA 147 (1999), 308–357; CMP 236 (2003), 251–280

*Chen-Torres-Ziemer, Frid, Chen-Comi-Torres, $\cdots\cdots$

*Chen-Torres: Notices Amer. Math. Soc. 171(2) (2021), 1282–1290