The effect of turbulence on reaction-diffusion equations

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Reaction–diffusion equations

For $\ell \geq 1$ consider the following problem for $v = (v_i)_{i=1}^{\ell} : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}^\ell$

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Example

For reversible chemical–reactions

$$p_1 V_1 + \cdots + p_\ell V_\ell \underset{k_-}{\overset{k_+}{\rightleftharpoons}} q_1 V_1 + \cdots + q_\ell V_\ell$$

one has

$$f_i(v) = (q_i - p_i)\Big[k_- \prod_{1 \leq i \leq \ell} v_j^{p_j} - k_+ \prod_{1 \leq i \leq \ell} v_j^{q_j}\Big] \qquad \text{(law of mass action)}.$$

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2. **(Positivity):** $f_i(v_1, \dots, v_{i-1}, 0, v_{i+1}, \dots, v_\ell) \geq 0$.
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$$\sup_{t \in [0,\tau)} \int_{\mathbb{T}^d} v(x,t)\, dx \leq c_0 \int_{\mathbb{T}^d} v_0 \, dx \qquad \text{(a-priori } L^1\text{-bounds).}$$

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Main problem: Growth of the nonlinearity/only L^1-estimate available.

Transport noise as model for small scale

Reaction-diffusion with transport

$$\partial_t v_i = \nu_i \Delta v_i + (u \cdot \nabla) v_i + f_i(v), \quad \text{on } \mathbb{T}^d, \tag{1}$$

where u is the velocity of an underlined fluid flow.

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By (1) and large/small scale decomposition:

$$\partial_t v_i - \nu_i \Delta v_i = (u_L \cdot \nabla) v + f_i(v) + \sum_{k\geq 1} \underbrace{\theta_k (\sigma_k \cdot \nabla) v}_{\text{Stochastic transport}} \circ \dot{\beta}_t^k.$$

For a special sequence of divergence free field $(\sigma_k)_{k\geq 1}$ consider

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Delayed blow–up (A.)

Under the above assumption, if $v(\cdot, 0) \in L^\infty(\mathbb{T}^d; [0, \infty)^\ell)$, then for each

$$\varepsilon \in (0, 1) \quad \text{and} \quad T \in (0, \infty)$$

there exists θ_k's such that the unique maximal solution

$$v : [0, \tau) \times \Omega \times \mathbb{T}^d \to [0, \infty)^\ell \quad \text{to (SR) verifies} \quad \mathbb{P}(\tau \geq T) > 1 - \varepsilon.$$

Agresti ('22). Delayed blow-up and enhanced diffusion by transport noise for systems of reaction-diffusion equations. *ArXiv.*

Some comments

$$\partial_t v_i - \nu_i \Delta v_i = f_i(\cdot, v) + \sum_{k\geq 1} \theta_k (\sigma_k \cdot \nabla) v_i \circ \dot{\beta}_t^k \quad \text{on } \mathbb{T}^d.$$

- A turbulent flow can prolong the life of solutions up to $T \gg 1$ (delayed blow-up).

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- The noise does not improve energy estimates or L^q-estimates:

$$\int_{\mathbb{T}^d} |v|^{q-2} [(\sigma_k \cdot \nabla) v_i] v_i \, dx = 0 \quad \text{as} \quad \mathrm{div}\sigma_k = 0.$$

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- In practice $\sigma_k = e^{ik\cdot x} a_k$ where $a_k \in \mathbb{R}^d$. The choice of θ_k's is not unique: For all $N \geq 1$ one can choose θ satisfying

$$\theta_k = 0 \qquad \text{for all } |k| \leq N.$$

Hence the noise acts only on high Fourier modes (high mode transportation).

The key result: “weak” enhanced diffusion

$$\partial_t v_i - \nu_i \Delta v_i = f_i(\cdot, v) + \sum_{k\geq 1} \theta_k (\sigma_k \cdot \nabla) v_i \circ \dot{\beta}_t^k \quad \text{on } \mathbb{T}^d. \tag{SR}$$

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Let $v(\cdot,0) \in L^\infty(\mathbb{T}^d;[0,\infty)^\ell)$ and fix $\varepsilon \in (0,1)$, $T \in (0,\infty)$,

$$\nu > 0, \quad q > \frac{d(h-1)}{2} \quad \text{and} \quad r \in (1,\infty).$$

Then there exist $\theta \in \ell^2$ such that the unique solution (v,τ) to (SR) verifies

$$\mathbb{P}(\tau \geq T, \|v - v_{\mathrm{det}}\|_{L^r(0,T;L^q)} \leq \varepsilon) > 1 - \varepsilon.$$

where v_{det} is the unique solution on $[0, T]$ to

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- (NECESSITY OF L^q-THEORY) Almost all relevant physical situations require $q > 2$.
- (SUBCRITICALITY) The space $L_x^{d(h-1)/2}$ is (locally) scaling invariant for (SR). Hence $L_t^r(L_x^q)$ is "close" to the "natural trace" space $L_t^\infty(L_x^{d(h-1)/2})$ for (SR).

Main steps in the proof:

1. (Scaling limit). Construct $(\theta_k^{(n)})_{k\geq 1}$'s such that solution $v^{(n)}$ of

$$\partial_t v_i^{(n)} - \nu_i \Delta v_i^{(n)} = \phi(R^{-1}\|v^{(n)}\|_{L^r(0,T;L^q)}) f_i(\cdot, v^{(n)}) + \sum_{k\geq 1} \theta_k^{(n)} (\sigma_k \cdot \nabla) v_i^{(n)} \circ \dot{\beta}_t^k, \quad (\mathrm{RD}_n)$$

where $\phi|_{[0,1]} = 1$, converges to the solution of the PDEs with enhanced diffusion

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New ideas:

- Use L^p-theory instead of L^2 (maximal L^p-regularity and critical spaces for SPDEs).

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- Use L^p-theory instead of L^2 (maximal L^p-regularity and critical spaces for SPDEs).
- Measure the convergence $v^{(n)} \to v_{\mathrm{det}}$ in $L^r(0,T;L^q)$ with $r < \infty$ and $q > \frac{d(h-1)}{2}$.

Main steps in the proof:

1. (Scaling limit). Construct $(\theta_k^{(n)})_{k\geq 1}$'s such that solution $v^{(n)}$ of

$$\partial_t v_i^{(n)} - \nu_i \Delta v_i^{(n)} = \phi(R^{-1}\|v^{(n)}\|_{L^r(0,T;L^q)}) f_i(\cdot, v^{(n)}) + \sum_{k\geq 1}\theta_k^{(n)}(\sigma_k\cdot\nabla) v_i^{(n)} \circ \dot{\beta}_t^k, \quad (\mathrm{RD}_n)$$

   where $\phi|_{[0,1]} = 1$, converges to the solution of the PDEs with enhanced diffusion

$$\partial_t v_{\mathrm{det,i}} - (\nu + \nu_i)\Delta v_{\mathrm{det,i}} = f_i(\cdot, v_{\mathrm{det}}). \quad (\mathrm{R}_\nu)$$

2. Choose $\nu > 0$ so that (R_ν) is globally well-posed.
3. Write $v^{(n)} = (v^{(n)} - v_{\mathrm{det}}) + v_{\mathrm{det}}$ and use that $v^{(n)} - v_{\mathrm{det}}$ are close for n large.

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- Taking $n\to\infty$ in (RD_n) can be interpreted as "zooming out" from small scales.
- In the homogenization language:

$(\mathrm{R}_\nu) \iff$ Effective equation.

Enhanced diffusion $\iff$ Large scale regularity.

A–priori estimates via Moser type iterations

Applying Itô formula and $|f_i(y)| \lesssim 1 + |y|^h \lesssim 1 + \sum_{1\leq i\leq \ell} |y_i|^h$, one has:

$$\sup_{r\in(0,t]} \|v_i(r)\|_{L^q}^q + \nu_i \int_0^t \int_{\mathbb{T}^d} |v_i|^{q-2}|\nabla v_i|^2 \,dxds$$
$$\lesssim \|v_{0,i}\|_{L^q}^q + \sum_{1\leq i\leq \ell} \int_0^t \int_{\mathbb{T}^d} (1 + |v_i|^{q+h-1}) \,dxds.$$

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Main step towards the a-priori estimates:

- **Sub-criticality**: (i.e. $q>\frac{d(h-1)}{2}$) implies the existence of $\beta\in(0,1)$ and $r<\infty$

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- **Enhanced dissipation**: For some deterministic $R > 0$,

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- **Conclusion**: $\displaystyle\int_0^T (1 + |v_i|^{q+h-1}) \, dxds \le C_R + \frac{\nu_i}{2\ell} \int_0^T \int_{\mathbb{T}^d} |v_i|^{q-2}|\nabla v_i|^2 \, dxds.$

Open problems and work in progress

- (WORK IN PROGRESS) The case $T = \infty$ for chemical reaction–diffusion equations.

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   - A. *Delayed blow-up and enhanced diffusion by transport noise for systems of reaction-diffusion equations*, ArXiv.
2. L^p-theory for stochastic reaction–diffusion equations:
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Thank you for your attention!