# Random Algebraic Geometry

Antonio Lerario

Real Algebraic Geometry
Géometrie algébrique réelle
CIRM, October 2022

In the last years there has been an increasing interest into the statistical behaviour of algebraic sets over non-algebraically closed fields: when the notion of "generic" is no longer available, one seeks for a "random" study of the objects of interest. In this course, divided into four lectures, I will present the major ideas in the subject (lecture notes will be made available):

1. Generic and random. In the first lecture I will discuss how to switch from the notion of generic, from classical algebraic geometry, to the notion of random. Of course, this depends on the choice of the probability distribution on the "moduli space" of the objects of interest. I will discuss what are the reasonable choices in the case $\mathbb{K} = \mathbb{C}$ (where still these questions make sense, and "random" and "generic" are synonymous) and in the case $\mathbb{K} = \mathbb{R}$ (where spherical harmonics play a crucial role).

2. Degree and volume. In the second lecture I will try to explain to what extent the right notion of degree, in the probabilistic context, is the notion of volume. I will introduce the classical kinematic formula, over $\mathbb{R}$ and over $\mathbb{C}$. In the complex case I will connect to the Bernstein-Khovanskii-Kouchnirenko Theorem and, in the real case, explain how these ideas can be used to construct a probabilistic version of Schubert Calculus.

3. The square–root law and the topology of random hypersurfaces. In the third lecture I will focus on the case $\mathbb{K} = \mathbb{R}$ and explain in which sense random real algebraic geometry behaves as the "square root" of complex algebraic geometry. I will discuss a probabilistic version of Hilbert's Sixteenth Problem, following the work of Gayet & Welschinger (introducing a local random version of Nash and Tognoli's Theorem and of Morse theory for the study of Betti numbers of random hypersurfaces) and of Diatta & Lerario (showing that "most" hypersurfaces of degree $d$ are isotopic to hypersurfaces of degree $\sqrt{d \log d}$).

4. The nonarchimedean world. In the last lecture I will discuss how to export some of the ideas from the previous lectures to the case $\mathbb{K} = \mathbb{Q}_p$, leaving with some open questions.