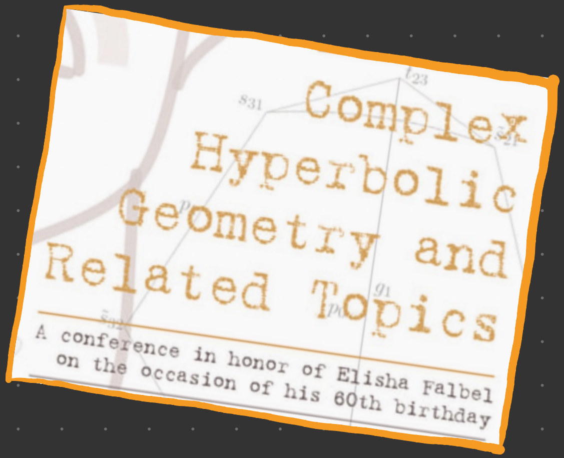
Twisted character varieties

Florent Schaffhauser
(Universidad de Los Andes, Bogotá)

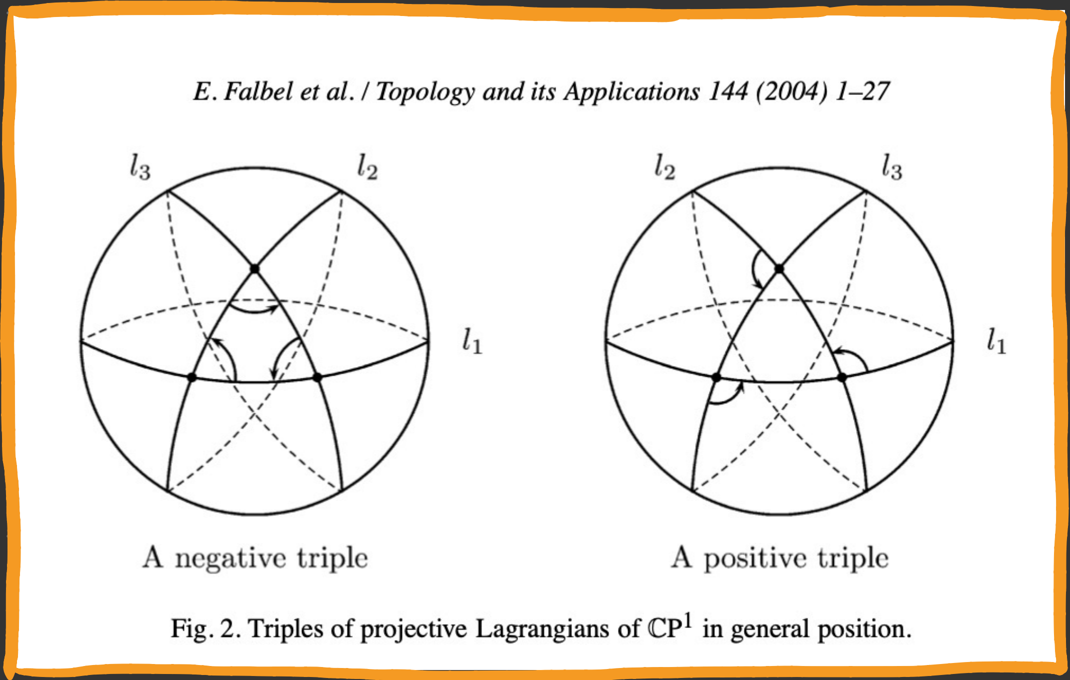


Complex Hyperbolic Geometry and Related Topics

A conference in honor of Elisha Falbel
on the occasion of his 60th birthday

Complex hyperbolic geometry
and related topics
Luminy, 4-8 July 2022

Triples of Lagrangians of $\mathbb{C}^2$



E. Falbel et al. / Topology and its Applications 144 (2004) 1–27

A negative triple

A positive triple

Fig. 2. Triples of projective Lagrangians of $\mathbb{C}P^1$ in general position.

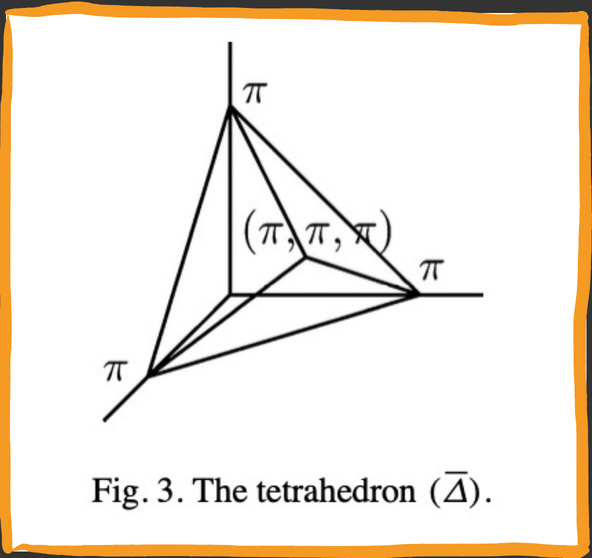
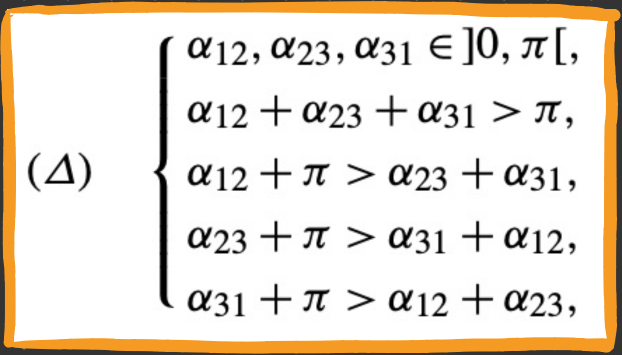


Fig. 3. The tetrahedron $(\bar{\Delta})$.

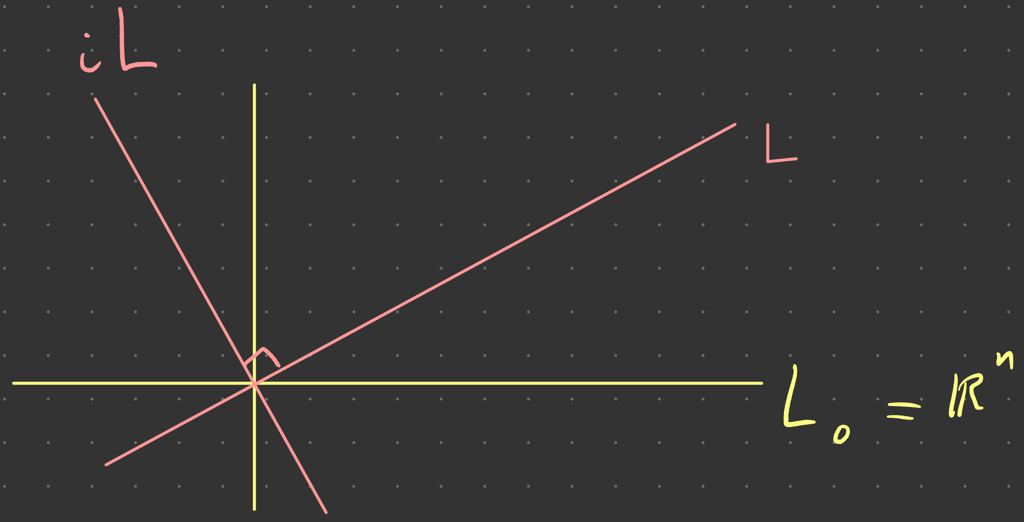


$$(\Delta) \quad \begin{cases} \alpha_{12}, \alpha_{23}, \alpha_{31} \in \,]0, \pi[, \\ \alpha_{12} + \alpha_{23} + \alpha_{31} > \pi, \\ \alpha_{12} + \pi > \alpha_{23} + \alpha_{31}, \\ \alpha_{23} + \pi > \alpha_{31} + \alpha_{12}, \\ \alpha_{31} + \pi > \alpha_{12} + \alpha_{23}, \end{cases}$$

Lagrangian triangle groups

[Falbel - Marco - Wentworth]

real subspace $L \subset \mathbb{C}^r$ such that $iL = L^\perp$



$$\sigma_L : \mathbb{C}^r = L \oplus iL \longrightarrow \mathbb{C}^r$$
$$x + iy \longmapsto x - iy$$

$\mathbb{C}$-antilinear involution

$u := \sigma_{L_1} \sigma_{L_2}$ is unitary.

$$(L_1, L_2, L_3) \longmapsto (\sigma_{L_1} \sigma_{L_2})(\sigma_{L_2} \sigma_{L_3})(\sigma_{L_3} \sigma_{L_1}) = 1$$

Coxeter groups

$$(L_1, L_2, L_3) \longmapsto \overbrace{(u_1, u_2, u_3) \mid u_1 u_2 u_3 = 1}^{\pi_1(\mathbb{CP}_1 \setminus \{s_1, s_2, s_3\}) \longrightarrow U(r)}$$

Assumption :

① fix the holonomy $u_j \in \underbrace{\mathcal{C}_j}_{\text{conjugacy class}}$

OR

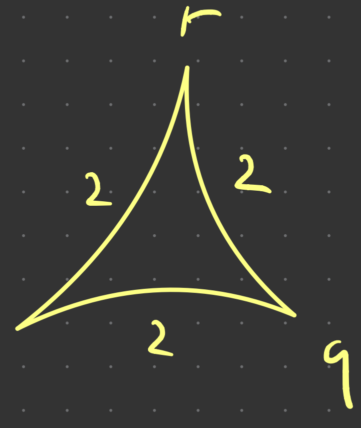
② fix an order : $u_j^{m_j} = 1$, $m_j \geq 2$

$$\langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r \rangle$$

Hyperbolic triangles

$$\langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r \rangle = \pi_1^{orb} X$$

$X =$



Representations ?

A representation of $\pi_1 X$ should induce

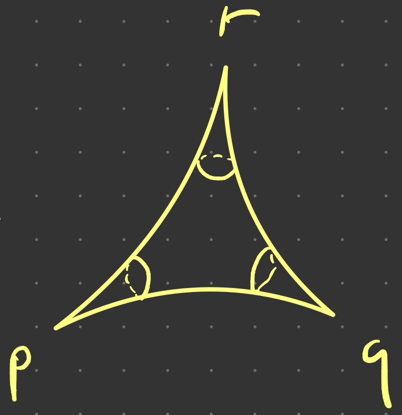
$$\begin{pmatrix} x \longmapsto \sigma_{L_1} \\ y \longmapsto \sigma_{L_2} \\ z \longmapsto \sigma_{L_3} \end{pmatrix}$$

$\sigma_{L_i} \in U(r) \rtimes \mathbb{Z}/2\mathbb{Z}$

$\sigma_{L_i} \cdot \sigma_{L_{i+1}} \in U(r)$

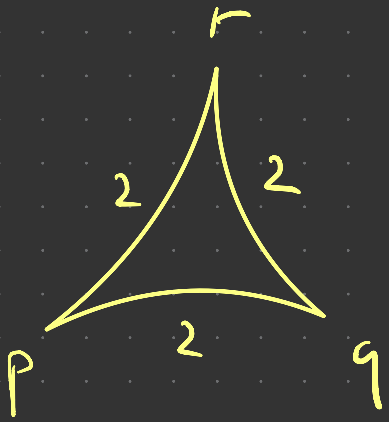
Index 2 subgroups

$X_{\mathbb{C}} =$



$\downarrow$ 2:1

$X =$



$$\pi_1 X_{\mathbb{C}} = \left\langle a, b, c \;\middle|\; abc = a^p = b^q = c^r \right\rangle$$

$$a \longmapsto xy, \quad b \longmapsto yz, \quad c \longmapsto zx$$

$$\begin{tikzcd}
1 \arrow[r] & \pi_1 X_{\mathbb{C}} \arrow[r] \arrow[d, "\mathfrak{X}|_{\pi_1 X_{\mathbb{C}}}"] & \pi_1 X \arrow[r, "\varphi"] \arrow[d, "\mathfrak{X}"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & U(r) \arrow[r] & U(r) \rtimes \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

"Twisted" character varieties

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}\left(\pi_1 X ; U(r) \rtimes \mathbb{Z}/2\mathbb{Z}\right) \big/ U(r) \arrow[r, "\text{FW '06}"] & \mathrm{Hom}\left(\pi_1 X_{\mathbb{C}} ; U(r)\right) \big/ U(r) \end{tikzcd}$$

$$\underbrace{\mathfrak{X}}_{\simeq H^1_{\varphi}(\pi_1 X ; U(r))} \longmapsto \underbrace{\mathfrak{X}\big|_{\pi_1 X_{\mathbb{C}}}}_{= H^1(\pi_1 X_{\mathbb{C}} ; U(r))}$$

$$\varphi : \pi_1 X \to \mathbb{Z}/2\mathbb{Z} \curvearrowright U(r) \text{ via } u \longmapsto \bar{u}.$$

$$\gamma \mapsto \varphi_\gamma$$

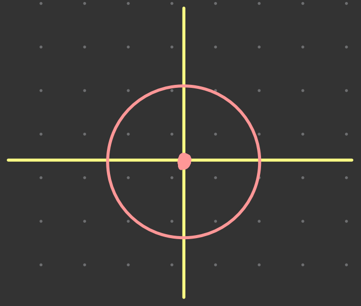
$$e : \pi_1 X \longrightarrow U(r)$$

$$\gamma_1 \gamma_2 \longmapsto e(\gamma_1)\, \varphi_{\gamma_1}(e(\gamma_2))$$

$$\boxed{\begin{array}{l} \hat{e} : \pi_1 X \longrightarrow U(r) \rtimes \mathbb{Z}/2\mathbb{Z} \\ \qquad \gamma \mapsto (e(\gamma), \varphi_\gamma) \\ \text{is a group morphism.} \end{array}}$$

$\mathfrak{X}$ from above

Differential equations with symmetry

$D =$ 

$$A(z) = \begin{pmatrix} z^2 & -z \\ z & z^2 \end{pmatrix} = {}^t A(-z)$$

$$U'(z) = A(z)\,U(z)$$

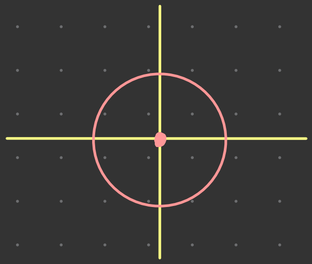
⇝ germs of solutions = locally constant sheaf of 2-dim. $\mathbb{C}$-vector spaces

[in fact, constant, since D is simply connected]

⇝ bases of solutions = locally constant sheaf of $GL(2;\mathbb{C})$-spaces [torsors]

if $M(z)$ is such a basis, so is $N(z) := {}^t M(z)^{-1}$

Recovering symmetry via monodromy

$D =$ 

$$A(z) = \begin{pmatrix} z^2 & -z \\ z & z^2 \end{pmatrix}$$

$$U'(z) = A(z)\,U(z)$$

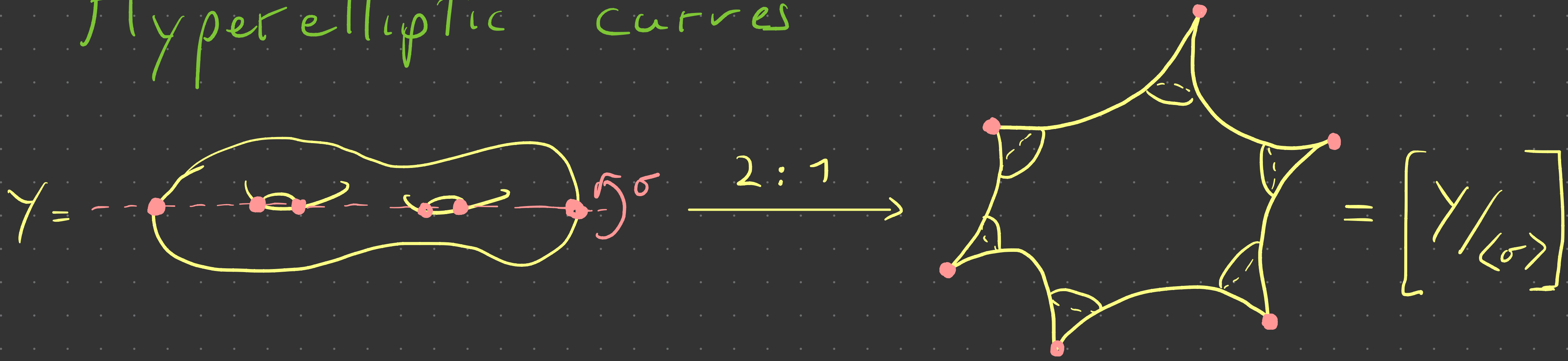
$\Big\}$ Trivial monodromy

$\mathbb{Z}/2\mathbb{Z} \curvearrowright GL(2, \mathbb{C})$ via $g \longmapsto {}^t g^{-1}$

symmetry of the equation ${}^t A(-z) = A(z)$ $\rightsquigarrow$ germs of solutions $=$ $\mathbb{Z}/2\mathbb{Z}$-equivariant sheaf

generalized monodromy (on $[D/(\mathbb{Z}/2\mathbb{Z})]$) $\rightsquigarrow$ $\mathbb{Z}/2\mathbb{Z}$-action on bases of solutions $\Big\}$ $M(z) \mapsto {}^t M(z)^{-1}$

Hyperelliptic curves



"anti-invariant rank r local system"

$$\begin{tikzcd} \mathcal{V} \arrow[d] \\ Y \end{tikzcd}$$

such that $(\sigma^*\mathcal{V})^\vee \simeq \mathcal{V}$

How to define / characterize them via monodromy?

[global version of the previous example]

Extended monodromy

$$\begin{tikzcd}
1 \arrow[r] & \pi_1 Y \arrow[r] \arrow[d] & \pi_1[Y/\langle\sigma\rangle] \arrow[r, "\varphi"] \arrow[d] & \langle\sigma\rangle \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & GL(r;\mathbb{C}) \arrow[r] & GL(r;\mathbb{C}) \rtimes \langle\sigma\rangle \arrow[r] & \langle\sigma\rangle \arrow[r] & 1
\end{tikzcd}$$

$$GL(r;\mathbb{C}) \circlearrowright \mathrm{Hom}_{\langle\sigma\rangle}\left(\pi_1[Y/\langle\sigma\rangle]; GL(r;\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}\right)$$

$$\rightsquigarrow H^1_{\varphi}\left(\pi_1[Y/\langle\sigma\rangle]; GL(r;\mathbb{C})\right)$$

with $\langle\sigma\rangle \curvearrowright GL(r;\mathbb{C})$

via $g \longmapsto {}^t g^{-1}$

Twisted character varieties

Lie group

group morphism $\left\{ \begin{array}{l} \varphi : \Pi \to \mathrm{Aut}(G) \\ \gamma \mapsto \varphi_\gamma \end{array} \right.$

$$H^1_\varphi(\Pi ; G) = \frac{\left\{ \rho : \Pi \to G \;\middle|\; \overbrace{\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\varphi_{\gamma_1}(\rho(\gamma_2))}^{\text{"twisted" representation}} \right\}}{\rho \sim (g\cdot\rho) : \gamma \mapsto g\,\rho(\gamma)\,\varphi_\gamma(g^{-1})}$$

<u>Main example:</u>

$\Pi < PGL(2;\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}})$

discrete, cocompact

$$\Pi \simeq \pi_1\Big(\underbrace{\left[\mathbb{H}^2_{\mathbb{R}}/\Pi\right]}_{X_\Pi}\Big)$$

$\rightsquigarrow$ local systems on X_Π?

Twisted local systems

$$\varphi : \pi \to \mathrm{Aut}(G)$$
$$\gamma \mapsto \varphi_\gamma$$

<u>Goal:</u>

$$\begin{tikzcd} \left\{\begin{array}{c}\varphi\text{-twisted}\\ \text{representations}\\ \rho : \pi \to G\end{array}\right\} \arrow[r] & \left\{\begin{array}{c}\text{sheaf of parallel}\\ \text{sections of a}\\ \text{flat bundle}\\ \text{on } X_\pi\end{array}\right\} \arrow[l, dashed, bend right=60, "\text{monodromy}"'] \end{tikzcd}$$

$\rightsquigarrow$ the flat bundle on X_π is given by $\mathcal{E}(\rho) := \left(\mathbb{H}^2_{\mathbb{R}} \times G\right) / \pi$ where

$$\gamma \cdot (\xi, h) = (\gamma \cdot \xi, \rho(\gamma)\, \varphi_\gamma(h))$$

⚠ 2 When φ is non-trivial, $\mathcal{E}(\rho)$ is not a principal G-bundle in general, but a "twisted" version of it.

Principal homogeneous spaces (a.k.a. torsors)

$$\varphi : \Pi \to \mathrm{Aut}(G)$$

group bundle on X_Π $\Bigg($ $\mathcal{G}_\varphi := \left(\mathbb{H}^2_{\mathbb{R}} \times G\right)\big/_{\Pi}$ where Π acts via $\gamma\cdot(\xi, g) := (\gamma\cdot\xi, \varphi_\gamma(g))$

⇝ $\mathcal{G}_\varphi$ acts on $\mathcal{E}(\rho)$: $\mathcal{E}(\rho) \times_{X_\Pi} \mathcal{G}_\varphi \to \mathcal{E}(\rho)$

⇝ the associated map

$$\begin{aligned} \mathcal{E}(\rho) \times_{X_\Pi} \mathcal{G}_\varphi &\longrightarrow \mathcal{E}(\rho) \times_{X_\Pi} \mathcal{E}(\rho) \\ (v, g) &\longmapsto (v, v\cdot g) \end{aligned}$$

is an isomorphism

⇝ $\forall x \in X_\Pi$, $\mathcal{E}(\rho)_x \simeq \left(\mathcal{G}_\varphi\right)_x$ non-canonically

Betti space

"Twisted character varieties parameterize twisted local systems."

$$\begin{tikzcd} H^1_\varphi(\pi_1 X; G) \arrow[r] & \{\varphi\text{- twisted } G\text{-local systems}\} \arrow[l, bend right, dashed, "\text{monodromy}"'] \\ (\rho: \pi_1 X \to G) \arrow[r, mapsto] & E(\rho) = (\tilde{X} \times G) / \pi_1 X \end{tikzcd}$$

Simpson's question (1991):

What is the corresponding Dolbeault space?

group bundles

New assumption: X is endowed with a complex analytic structure

Example: $X = \left[\mathbb{H}^2_{\mathbb{R}} / \Gamma \right]$, $\underset{\text{discrete}}{\Gamma < PSL(2;\mathbb{R})} = \mathrm{Isom}^+(\mathbb{H}^2_{\mathbb{R}})$

$$\begin{tikzcd} \mathcal{G} \arrow[d] \\ X \end{tikzcd}$$

$\mathcal{G}$ a complex analytic group bundle:

$\mathcal{G} \times_X \mathcal{G} \longrightarrow \mathcal{G}$

\+ associativity

\+ neutral section

$1_{\mathcal{G}} : X \longrightarrow \mathcal{G}$

Example:

- $X \times G$, with G a complex Lie group
- $\mathrm{Ad}(P)$, where P is a principal G-bundle

Principal homogeneous spaces

= a bundle $\begin{matrix}\mathcal{E}\\ \downarrow\\ X\end{matrix}$ endowed with an action $\mathcal{E}\times_X \mathcal{G} \longrightarrow \mathcal{E}$ such that the associated morphism $\mathcal{E}\times_X \mathcal{G} \longrightarrow \mathcal{E}\times_X \mathcal{E}$ is an isomorphism.

Example: $\mathcal{G} = X \times G$

principal homogeneous $\mathcal{G}$-space = principal G-bundle

Higgs bundles

$\mathcal{G}_\varphi = \left[(\mathbb{H}^2_{\mathbb{R}} \times G)/\Pi \right]$ with G a complex analytic Lie group and $\varphi : \Pi \to \mathrm{Aut}(G)$, $\gamma \mapsto \varphi_\gamma$ a group morphism

$\mathrm{Lie}(\mathcal{G}_\varphi) = \left[(\mathbb{H}^2_{\mathbb{R}} \times \mathrm{Lie}(G))/\Pi \right]$ $\quad \gamma \in \Pi \curvearrowright \mathrm{Lie}(G)$ via $T_{1_G}\varphi_\gamma$

Definition A $\mathcal{G}_\varphi$ - Higgs bundle is a pair $(\mathcal{E}, \theta)$ where $\mathcal{E}$ is a $\mathcal{G}_\varphi$-torsor and θ is a morphism of vector bundles

$$\theta : TX \longrightarrow \mathrm{ad}(\mathcal{E}) := \left(\mathcal{E} \times_X \mathrm{Lie}(\mathcal{G}_\varphi)\right)/\mathcal{G}_\varphi .$$

$\theta \in H^0(X; \Omega^1_X \otimes \mathrm{ad}(\mathcal{E}))$: Higgs Field

Rank 1 Higgs bundles

$G = \mathbb{C}^*$ $\qquad \mathcal{G} = X \times \mathbb{C}^*$

$\mathcal{E} \leftrightarrow \mathcal{L}$, a holomorphic line bundle

$\mathrm{ad}(\mathcal{E}) \leftrightarrow \mathrm{End}(\mathcal{L}) \simeq \mathcal{L}^\vee \otimes \mathcal{L} \simeq X \times \mathbb{C}$

$$\begin{tikzcd} \left(\Theta : TX \arrow[rr] \arrow[dr] & & X \times \mathbb{C}\right) \arrow[dl] \arrow[r, leftrightarrow] & \text{a holomorphic 1-form} \\ & X & & \end{tikzcd}$$

Dolbeault space: $(\mathcal{L}, \Theta)$ such that $\deg \mathcal{L} = 0$

$$\rightsquigarrow \underbrace{\mathrm{Jac}(X) \times H^0(X, \Omega^1_X)}_{T^\vee \mathrm{Jac}(X)}$$

Nonabelian Hodge correspondence (Simpson)

$$\left\{\begin{matrix}\text{semistable degree } 0\\ \text{rank } r \text{ Higgs bundles}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{rank } r\\ \text{flat bundles}\end{matrix}\right\}$$

$$\longleftrightarrow \quad H^1(\pi_1 X; GL(r;\mathbb{C}))$$

Rank 1 case ($\dim_{\mathbb{C}} X = 1$)

$$\begin{tikzcd} \mathrm{Jac}(X) \times H^0(X;\Omega^1_X) \arrow[r, squiggly, shift left, "\text{linearization}"] & \underbrace{H^1(X;\mathcal{O}_X) \oplus H^0(X;\Omega^1_X)} \arrow[l, squiggly, shift left, "\text{exponential sequence}"] \end{tikzcd}$$

Dolbeault's thm: $H^{0,1}(X) \oplus H^{1,0}(X)$

Hodge decomposition: $H^1(X;\underline{\mathbb{C}}_X)$

$$T^\vee \mathrm{Jac}(X) \underset{\text{homeo}}{\simeq} H^1(X;\underline{\mathbb{C}}^*_X) \underset{\text{homeo}}{\simeq} \mathrm{Hom}(\pi_1 X;\mathbb{C}^*)$$

A twisted example (rank 1)

$Y \to [Y/\langle\sigma\rangle]$ $\quad \sigma^2 = \mathrm{id}_Y$ (e.g. hyperelliptic curve)

$\langle\sigma\rangle \curvearrowright \mathbb{C}^*$ via $z \mapsto z^{-1}$

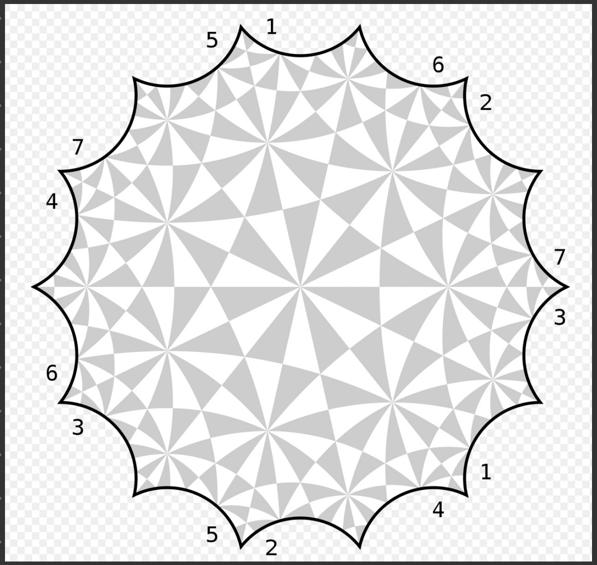
$$H^1_{\mathrm{Betti}}([Y/\langle\sigma\rangle]; \mathbb{C}^*) \simeq \mathrm{Hom}_{\langle\sigma\rangle}\left(\pi_1[Y/\langle\sigma\rangle]; \mathbb{C}^* \rtimes \langle\sigma\rangle\right) \big/ \mathbb{C}^*$$

$$\mathcal{G}_\sigma = \left[(Y \times \mathbb{C}^*)/\langle\sigma\rangle\right]$$

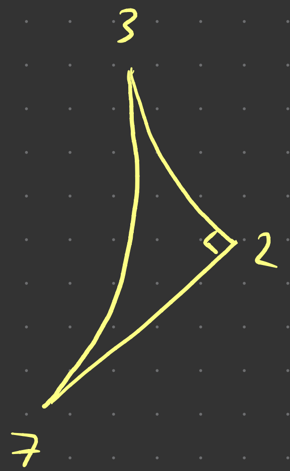
$$\underbrace{H^1_{\mathrm{Dol}}([Y/\langle\sigma\rangle]; \mathbb{C}^*)}_{T^\vee \mathrm{Bun}(\mathcal{G}_\sigma)_0} \simeq \underbrace{\mathrm{Prym}(Y \to [Y/\langle\sigma\rangle])}_{\{\mathcal{L} \in \mathrm{Jac}(Y) \mid (\sigma^*\mathcal{L})^\vee \simeq \mathcal{L}\}_0} \times \underbrace{\mathrm{Fix}_{\langle\sigma\rangle} H^0(Y; \Omega^1_Y)}_{-\sigma^*\omega = \omega}$$

An example

$\mathcal{K} :=$



$\xrightarrow{336:1}$



$= \underbrace{\mathcal{K} / \mathrm{Aut}^{\pm}(\mathcal{K})}_{\mathcal{T}}$

The Klein quartic

(genus 3 Riemann surface
with 168 automorphisms)

A hyperbolic
(2,3,7) triangle

$$1 \to \pi_1 \mathcal{K} \to \pi_1 \mathcal{T} \to \underbrace{\mathrm{Aut}^{\pm}(\mathcal{K})}_{\Gamma \quad + (\Gamma \to \mathbb{Z}/2\mathbb{Z})} \to 1$$

Hitchin components (j/w Alessandrini–Lee)

JEMS 2022

We look at Hitchin components:

- representations into groups such as $PGL(n;\mathbb{R})$
  $PSp^{\pm}(2n;\mathbb{R})$
  $PO(p,p+1)$
  $\vdots$
  G_2
- Hitchin representations are deformations of $\underbrace{\text{Fuchsian representations}}$

$$\begin{tikzcd}
 & PGL(2;\mathbb{R}) \arrow[d, "\kappa"] \\
\pi_1 X \arrow[ur, dashed] \arrow[r] & PGL(n;\mathbb{R})
\end{tikzcd}$$

by definition

- Hitchin representations form a connected component of $\mathrm{Hom}(\pi_1 X; PGL(n;\mathbb{R}))/PGL(n;\mathbb{R})$.

Parameterization by symmetric differentials

Orbifold version of Hitchin's approach:

$$\begin{tikzcd}
\mathrm{Hom}^{irr}_{\mathbb{Z}/2\mathbb{Z}}\left(\pi_1\big(\underbrace{\mathcal{K}/\Gamma}_{\mathcal{C}}\big); PGL(n;\mathbb{C})\times\mathbb{Z}/2\mathbb{Z}\right)\big/PGL(n;\mathbb{C}) \arrow[r, leftrightarrow, "NAHC"] & \left\{\begin{array}{l}\text{stable}\\ \Gamma\text{- equivariant}\\ PGL(n;\mathbb{C})\text{- Higgs}\\ \text{bundles of}\\ \text{degree } 0\end{array}\right\} \arrow[d, "\text{Hitchin fibration}"] \\
\mathrm{Hit}(\pi_1\mathcal{C}; PGL(n;\mathbb{R})) \arrow[u, hook] \arrow[r, phantom, "\underset{\text{homeo}}{\simeq}"] & \underbrace{\bigoplus_{j=2}^{n} \mathrm{Fix}_\Gamma H^0(\mathcal{K}; \Omega_{\mathcal{K}}^{\otimes j})}_{H^0(\mathcal{C};\, \Omega_{\mathcal{C}}^{\otimes j})} \arrow[u, dashed, bend left, "\text{Hitchin section}"]
\end{tikzcd}$$

Dimension

$$\dim \mathrm{Hit}(\pi_1 \overset{(2,3,7)}{\mathcal{C}}; PGL(n;\mathbb{R}))$$

$$= \frac{(n-1)(n-2)}{2} - \sum_{i=2}^{n-1} \left(\sum_{j=1}^{n-1} \left\lceil \frac{j+1}{i} - 1 \right\rceil \right) e_i$$

$$e_i = \begin{cases} 1 & \text{if } i = 2,3,7 \\ 0 & \text{otherwise} \end{cases}$$

n	dim	
2	0	(Thurston)
3	0	(Choi-Goldman)
4,5	0	$\rightsquigarrow$ rigidity of Hitchin (2,3,7) groups
$6 \leq n \leq 11$	1	

1-parameter family in $\mathrm{Hit}(\pi_1 \mathcal{K}; PGL(7;\mathbb{R}))$

$$e \longmapsto e|_{\pi_1 \mathcal{K}}$$

Vanishing differentials

$$\mathrm{Hit}(\pi_1 \mathcal{C}; PGL(6;\mathbb{R})) \simeq \overbrace{H^0(\mathcal{C}; \Omega_{\mathcal{C}}^{\otimes 6})}^{\nexists\ \Gamma\text{-invariant } d\text{-differentials on } \mathcal{K} \text{ for } 2 \leq d \leq 5}$$

$$\simeq \mathrm{Hit}(\pi_1 \mathcal{C}; PSp^{\pm}(6;\mathbb{R}))$$

Consequence: If a Hitchin representation

$$\rho : \pi_1 \mathcal{K} \to PGL(6;\mathbb{R})$$

extends to

$$\hat{\rho} : \pi_1 \mathcal{C} \to PGL(6;\mathbb{R})$$

then $\mathrm{Im}\,\rho \subset PSp^{\pm}(6;\mathbb{R})$.

In particular, ρ is not Zariski-dense.

Deformations of $PSL(2;\mathbb{R})$-structures

M: closed 3-manifold with $\overbrace{\mathcal{D}_{PSL(2;\mathbb{R})}(M)}^{\text{deformation space}} \neq \emptyset$.

Example: $M = UX$ unit tangent bundle of a closed orientable surface

Facts:

- M as above is a Seifert manifold, a d-folded cover of $U\mathcal{X}$, where $\mathcal{X}$ is a closed orientable orbi-surface.
- There is a canonical map
$$\mathcal{D}_{PSL(2;\mathbb{R})}(M) \longrightarrow \mathcal{D}_{\mathbb{R}P_3}(M).$$

Rigidity

Theorem (ALS, 2022)

The image of the canonical map

$$\mathcal{D}_{PSL(2;\mathbb{R})}(M) \longrightarrow \mathcal{D}_{\mathbb{RP}_3}(M)$$

is a connected component of $\mathcal{D}_{\mathbb{RP}_3}(M)$, homeomorphic to

$$\underbrace{\mathrm{Hit}(\pi_1 \mathcal{X}; PGL(4;\mathbb{R}))}_{= \text{ a point if } \mathcal{X} = \mathcal{T}(2,3,7).}.$$

Remarks

- When $M = UX$ (X = closed orientable surface) the theorem above is due to Guichard & Wienhard (2008).

- More generally, we have:

$$\mathcal{D}_{PSL(2;\mathbb{R})}(M) \longrightarrow \mathcal{D}^{\circ}_{\mathbb{RP}_3^{\omega}}(M) \longrightarrow \mathcal{D}^{\circ}_{\mathbb{RP}_3}(M)$$

$$\mathrm{Hit}(\pi_1 \mathcal{Y}; PGL(2;\mathbb{R})) \hookrightarrow \mathrm{Hit}(\pi_1 \mathcal{Y}; PSp^{\pm}(4;\mathbb{R})) \hookrightarrow \mathrm{Hit}(\pi_1 \mathcal{Y}; PGL(4;\mathbb{R}))$$

$$-3\chi^{top}(Y) + 2k + \ell \qquad -10\chi^{top}(Y) + 8k + 4\ell - 2k_2 - 2k_3 - \ell_2 - \ell_3 \qquad -15\chi^{top}(Y) + 12k + 6\ell - 4k_2 - 2k_3 - 2\ell_2 - \ell_3$$

$\mathcal{Y} = (2,3,7)$: the canonical projective structure of M is rigid.

$(3,3,4)$: M is contact-rigid but not rigid.

$(2,4,5)$: not rigid but $\mathcal{D}^{\circ}_{\mathbb{RP}_3}(M) = \mathcal{D}^{\circ}_{\mathbb{RP}_3^{\omega}}(M)$.