The non-arithmetic cusped hyperbolic 3-orbifold of minimal volume

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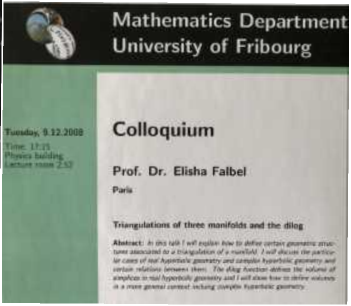
All the best, Elisha



Mathematics Department
University of Fribourg

Tuesday, 9.12.2008
Time: 17:15
Physics building
Lecture room 2.52

Colloquium

Prof. Dr. Elisha Falbel

Paris

Triangulations of three manifolds and the dilog

Abstract: *In this talk I will explain how to define certain geometric structures associated to a triangulation of a manifold. I will discuss the particular cases of real hyperbolic geometry and complex hyperbolic geometry and certain relations between them. The dilog function defines the volume of simplices in real hyperbolic geometry and I will show how to define volumes in a more general context including complex hyperbolic geometry.*

About small cusped hyperbolic orbifolds in low dimensions

Let $V^n = \mathbb{H}^n/\Gamma$ be a *cusped* hyperbolic n-orbifold, i.e. a quotient of (real) hyperbolic space $\mathbb{H}^n$ by a discrete group $\Gamma \subset \mathrm{Isom}\mathbb{H}^n$ with a non-compact fundamental polyhedron $P \subset \mathbb{H}^n$ of finite volume

If Γ has no torsion elements, then V^n is a smooth manifold

Aim. For $n \geq 2$, determine the orbifolds V^n of small(est) volume

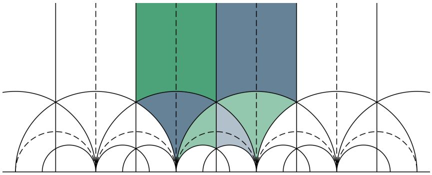
Remark. Much more difficult is the *compact* context

Siegel and the smallest cusped hyperbolic 2-orbifold

The two-dimensional case is well-known, both in the compact case and in the non-compact finite volume case (C. Siegel)

- The Coxeter group $[3,\infty] =$ •—•$\overset{\infty}{—}$• generated by the reflections in the sides of a triangle $(\frac{\pi}{2},\frac{\pi}{3},0)$ yields the cusped 2-orbifold of minimal volume

Notice. $[3,\infty]$ commensurable to the modular group $\mathrm{PSL}(2,\mathbb{Z})$



Hyperbolic Coxeter groups

- Let $P_C \subset \mathbb{H}^n$ be a convex polyhedron with finite volume and dihedral angles $\frac{\pi}{m_{ij}}$ with $m_{ij} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. The group Γ generated by the reflections $s_1, \ldots, s_k$ satisfying $(s_i s_j)^{m_{ij}} = 1$ in the facet hyperplanes of P_C is a *hyperbolic Coxeter group*
- The *Coxeter graph* is a weighted graph with vertices the generators $\{s_1, \ldots, s_k\}$ of the Coxeter group. Two generators s_i and s_j are joined by an edge with weight m_{ij} if they do not commute. If k is small, one often uses the description by the *Coxeter symbol*
- If Γ is a hyperbolic Coxeter group, then $\mathbb{H}^n/\Gamma$ is a hyperbolic *Coxeter orbifold*
- A hyperbolic manifold commensurable to a hyperbolic Coxeter orbifold is called a *Coxeter manifold*

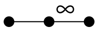
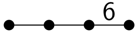
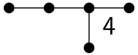
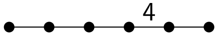
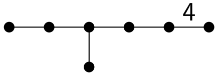
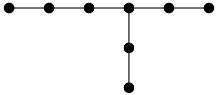
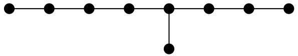
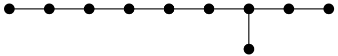
dim n	Coxeter graph	volume
2		$\frac{\pi}{6} \simeq 0.5236$
3		$\frac{1}{8}\text{Л}(\frac{\pi}{3}) \simeq 0.0423$
4		$\frac{\pi^2}{1{,}440} \simeq 6.85 \cdot 10^{-3}$
5		$\frac{7\zeta(3)}{46{,}080} \simeq 1.82 \cdot 10^{-4}$
6		$\frac{\pi^3}{777{,}600} \simeq 3.98 \cdot 10^{-6}$
7	$\mathbb{Z}_2 *$ 	$\frac{\sqrt{3}\,L(4{,}3)}{1{,}720{,}320} \simeq 9.46 \cdot 10^{-7}$
8		$\frac{\pi^4}{4{,}572{,}288{,}000} \simeq 2.13 \cdot 10^{-8}$
9		$\frac{\zeta(5)}{22{,}295{,}347{,}200} \simeq 4.65 \cdot 10^{-11}$

Table: *The minimal volume cusped hyperbolic n-orbifolds; T. Hild [H]*

About the proof of the above volume minimality result

For n=2. Siegel exploiting fundamental area formula

For n=3 and n=4. Meyerhoff [M] and Hild-Kellerhals using local densities of periodic horosphere packings and crystallography

For general n, Hild [H] extended the methods in order to detect a minimal volume cusped hyperbolic n-orbifold as follows:

- Find a suitable candidate by looking at hyperbolic Coxeter groups of smallest rank $\geq n+1$ and with smallest covolume v_n in this class
- Prove that any cusped n-orbifold $\mathbb{H}^n/\Gamma$ with volume $\leq v_n$ has only 1 cusp $C = B_\infty/\Gamma_\infty \quad \rightsquigarrow$ lower covolume bound for Γ
- Identify the crystallographic group $\Gamma_\infty \subset \mathrm{Isom}(\mathbb{E}^{n-1})$
- Extend a fundamental polyhedron $P_\infty \subset \mathbb{E}^{n-1}$ to a fundamental polyhedron P for Γ

Arithmeticity and commensurability

Observe that all known **minimal volume** cusped hyperbolic n-orbifolds are **arithmetic**

- Two hyperbolic orbifolds $V_1 = \mathbb{H}^3/G_1$ and $V_2 = \mathbb{H}^3/G_2$ are **commensurable** if they have a common finite sheeted cover, that is, $\exists \gamma \in \operatorname{Isom}\mathbb{H}^3$ such that $G_1 \cap \gamma G_2 \gamma^{-1}$ has finite index in both G_1 and $\gamma G_2 \gamma^{-1}$
- The commensurability property for groups in $\operatorname{Isom}\mathbb{H}^3$ is an equivalence relation preserving discreteness, finite covolume, arithmeticity and having virtually fibred quotients
- By a result of Margulis, a lattice $G \subset \operatorname{Isom}\mathbb{H}^3$ is **non-arithmetic** if and only if its commensurator

  $$\operatorname{Comm}(G) = \{\, \gamma \in \operatorname{Isom}\mathbb{H}^n \mid G \text{ and } \gamma G \gamma^{-1} \text{ are commensurable} \,\}$$

  is a lattice (and contains G as a subgroup of finite index)

Arithmeticity criterion for hyperbolic Coxeter groups

For a **non-compact** hyperbolic Coxeter polyhedron $P \subset \mathbb{H}^n$ of finite volume associated to the Coxeter group $\Gamma \subset \operatorname{Isom}\mathbb{H}^n$, establish $2\,\mathrm{Gram}(P) =: (h_{ij})$ and build all the coefficient cycles (of length l) of the form

$$h_{i_1 i_2 \ldots i_l} := h_{i_1 i_2} h_{i_2 i_3} \cdot \ldots \cdot h_{i_{l-1} i_l} h_{i_l i_1}$$

The field $K(\Gamma) := \mathbb{Q}(\{h_{i_1 i_2 \ldots i_l}\})$ generated by all cycles of $2\,\mathrm{Gram}(P)$ is the smallest field of definition for Γ and equals the adjoint trace field of Γ, i.e. $K(\Gamma)$ is a commensurability invariant for Γ.

Vinberg's criterion. *The Coxeter group $\Gamma \subset \operatorname{Isom}\mathbb{H}^n$ as given above (and its associated Coxeter orbifold $\mathbb{H}^n/\Gamma$) is arithmetic with field of definition $\mathbb{Q}$ if and only if all the cycles of $2\,\mathrm{Gram}(P)$ are rational integers.*

The six smallest cusped hyperbolic 3-orbifolds

In 1990, Adams [A1] determined the 6 smallest cusped hyperbolic 3-orbifolds, starting with $[3,3,6]$, by assuming an upper bound for their cusp volume and hence for their total volume

In the same conference volume *Topology '90*, W. Neumann and A. Reid characterised these orbifolds and showed that they are all arithmetic, most of them are commensurable to Coxeter orbifolds

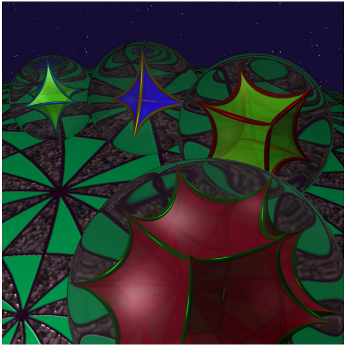
In subsequent work, Adams studied small N-cusped hyperbolic 3-orbifolds with $N \geq 2$ by similar but more elaborate methods

[$*$] Topology âĂŹ90 (Columbus, OH, 1990). Vol. 1. Ohio State Univ. Math. Res. Inst. Publ. de Gruyter, Berlin, 1992.

The non-arithmetic cusped orbifold of minimal volume

In joint work with Simon Drewitz [DK], we considered the *non-arithmetic case* of small volume cusped 3-orbifolds.

A natural candidate is the Coxeter group $[5, 3, 6]$ which relates to the symmetry group of an ideal dodecahedron with angle $\frac{\pi}{3}$.



Theorem (DK, 2021)

Among all non-arithmetic cusped hyperbolic 3-orbifolds, the 1-cusped quotient space V_* of $\mathbb{H}^3$ by the tetrahedral Coxeter group $[5,3,6]$ *has minimal volume. As such the orbifold V_* is unique, and its volume v_* is given explicitly by*

$$\frac{1}{2}\,Л(\frac{\pi}{3})+\frac{1}{4}\left\{Л(\frac{\pi}{6}+\frac{\pi}{5})+Л(\frac{\pi}{6}-\frac{\pi}{5})\right\}\approx 0.171502$$

Here, the Lobachevsky function $Л(x)$ is defined by

$$Л(x)=\tfrac{1}{2}\sum_{r=1}^{\infty}\frac{\sin(2rx)}{r^2}=-\int_0^x\log|2\,\sin t|\,dt\,,\,x\in\mathbb{R}\,.$$

The function $Л(x)$ is odd, π-periodic and satisfies the formula

$$\frac{1}{k}\,Л(kx)=\sum_{r=0}^{k-1}Л\left(x+\frac{r\pi}{k}\right)\,,\;k\in\mathbb{N}\,.$$

Some comments

- The reflection group $[5,3,6]$ yields $\mathrm{Sym}(D^{\infty}_{reg})$. By applying different face identifications to D^{∞}_{reg}, the orbifold V_* admits several non-isometric non-arithmetic cover manifolds (e.g. work of B. Everitt)
- The orbifold $V_* = \mathbb{H}^3/[5,3,6]$ does **not relate** to any Gromov–Piatetski-Shapiro construction since the Coxeter tetrahedron $[5,3,6]$ is not *splittable* in the sense of Fisher, Lafont, Miller and Stover (JEMS, 2021)
- As a by-product of the geometric methods used to prove our result, we obtain the following two-dimensional analogue:

Proposition

Among all non-arithmetic cusped hyperbolic 2-orbifolds, the 1-cusped quotient space V_* of $\mathbb{H}^2$ by the triangle Coxeter group $[5,\infty]$ has minimal area. As such the orbifold V_* is unique.

A comment about the smooth case

There are many (arithmetic and non-arithmetic) cusped hyperbolic 3-manifolds (knot and link complements and their Dehn fillings). In the small volume case, work of Gabai, Meyerhoff and Milley [GMM] based on MOM technology yields the following result in terms of the SnapPea census:

Theorem

Let M be a 1-cusped orientable hyperbolic 3-manifold of volume ≤ 2.848. Then, M is one of $m003$, $m004$, $m006$, $m007$, $m009$, $m010$, $m011$, $m015$, $m016$, or $m017$.

Note. Examining these 10 manifolds (see Maclachlan-Reid's book [MR, Section 13.6]), the examples in red are non-arithmetic and **not** commensurable with (a smooth cover of) V_* (invariant trace field!)

About small cusped Coxeter 3-manifolds

In the special case of *Coxeter manifolds*, our theorem allows one to deduce the following result in terms of the non-arithmetic tetrahedral Coxeter group with cyclic graph $[(3^3, 6)]$.

Proposition (Drewitz)

The fundamental group of a non-arithmetic cusped hyperbolic Coxeter 3-manifold M_* of minimal volume is incommensurable with the Coxeter group $[5, 3, 6]$*; the volume of M_* is smaller than or equal to* $24 \cdot \mathrm{covol}([(3^3, 6)]) \approx 8.738570$*.*

Drewitz found 4 non-isometric multiply-cusped Coxeter 3-manifolds $M_1, \dots, M_4$ arising as 24-fold covers of $\mathbb{H}^3/[(3^3, 6)]$.

Conjecture

The four manifolds $M_1, \dots, M_4$ are of minimal volume among all non-arithmetic cusped hyperbolic Coxeter 3-manifolds.

Strategies for proving the Theorem...

Here are some indications for the proof (50 pages...) of

Theorem (DK, 2021)

Among all non-arithmetic cusped hyperbolic 3-orbifolds, the 1-cusped quotient space V_* of $\mathbb{H}^3$ by the tetrahedral Coxeter group $[5,3,6]$ *has minimal volume. As such the orbifold V_* is unique, and its volume v_* is given explicitly by*

$$\frac{1}{2}\,Л(\frac{\pi}{3})+\frac{1}{4}\left\{Л(\frac{\pi}{6}+\frac{\pi}{5})+Л(\frac{\pi}{6}-\frac{\pi}{5})\right\}\approx 0.171502.$$

Step 1. Show that a non-arithmetic cusped 3-orbifold V with volume $\leq \mathrm{vol}(V_*) \approx 0.171502$ has only 1 cusp by using Adams' results about the 4 smallest multiply-cusped orbifolds (all arithmetic) and their descriptions

...continued...

Step 2. Apply another work of Adams to show first that an (orientable) cusp C of V is *rigid*, i.e. Dehn filling cannot be performed. Hence, a cusp is of type $\{2,3,6\}$, $\{2,4,4\}$, or $\{3,3,3\}$. Secondly, exclude the case $\{3,3,3\}$.

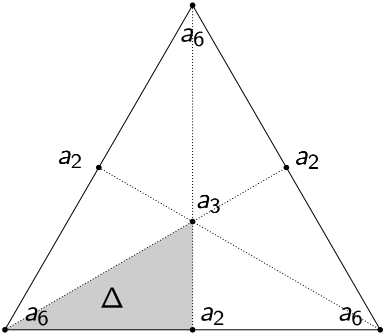
Step 3. Consider the cusp $C = B_\infty/\Gamma_\infty \subset V$ with B_∞ maximal whose horospherical boundary H_∞ is at height 1 from $\{x_3 = 0\}$. The Γ-images of B_∞ of diameter 1 are called *full-sized*. Furthermore, C can be identified by the chimney $P_\infty \times \mathbb{R}_{x_3 \geq 1}$ of volume

$$\mathrm{vol}(C) = \frac{\mathrm{vol}(P_\infty)}{2} \leq \mathrm{vol}(V) \quad \text{improved to} \quad \frac{\mathrm{vol}(C)}{d_3(\infty)} \leq \mathrm{vol}(V)$$

Estimate $\mathrm{vol}(C)$ for each type $\{2,3,6\}$ or $\{2,4,4\}$ and in terms of the distance $d \geq 1$ between full-sized horoballs

Step 4. Start with the case of **one** Γ_∞-equivalence class of full-sized horoballs.

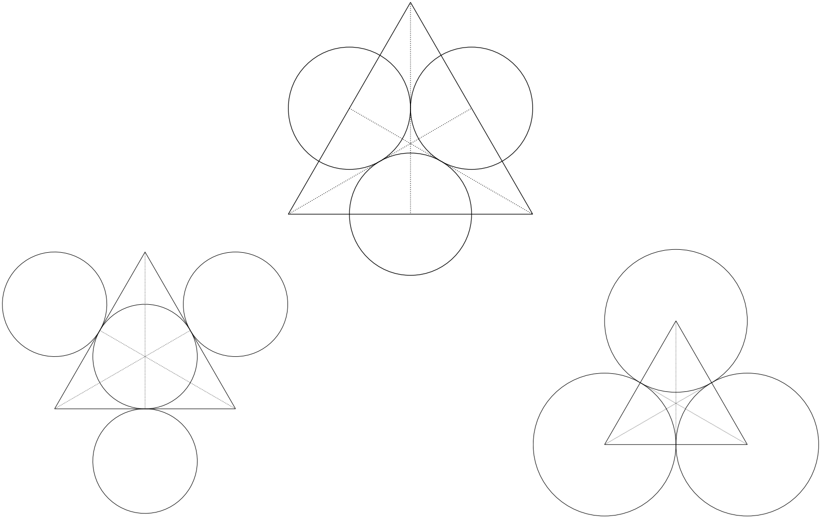
The cusp type $\{2, 3, 6\}$ with characteristic triangle $\Delta \subset D$



Full-sized horoballs could be centred in:

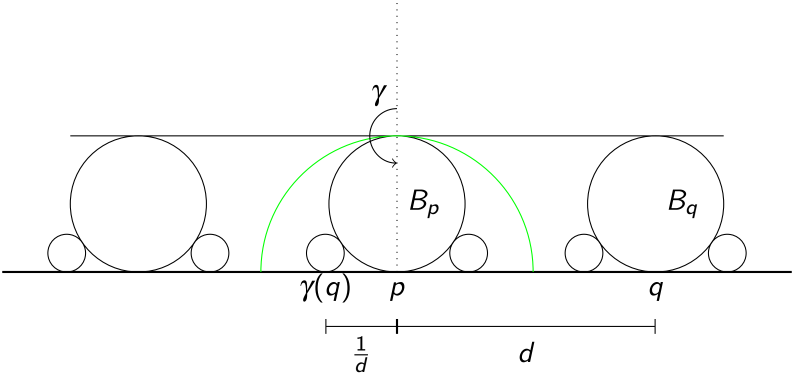
- singular point a_6...delicate !
- singular point a_3
- singular point a_2
- somewhere else....leading to a cusp volume $> v_*$

At least two full-sized horoballs touch each other ($d = 1$)



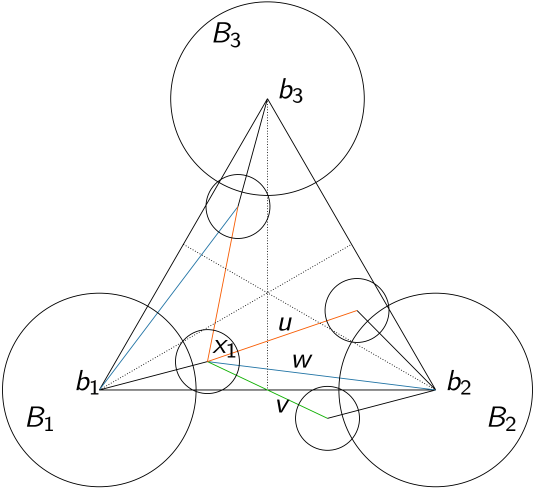
(Unique) realisations are commensurable to the arithmetic groups
$[3, 6, 3]$, $[4, 3, 6]$ and $[3, 3, 6]$

Full-sized horoballs do NOT touch each other ($d > 1$)



A $\frac{1}{d}$-ball is of (Euclidean) diameter $\frac{1}{d^2}$

A $\frac{1}{d}$-ball touches one full-sized horoball



$$\theta = \measuredangle x_1 b_2 \quad , \quad u^2 = d^2 + \frac{3}{d^2} - \sqrt{3}\sin\theta - 3\cos\theta$$

$$v^2 = d^2 + \frac{4}{d^2} - 4\cos\theta \quad , \quad w^2 = d^2 + \frac{1}{d^2} - 2\cos\theta$$

Γ_∞ is a reflection group, D has a mirror symmetry

- Since the simplicial horoball density $d_3(\infty) \approx 0.853276$,

$$\mathrm{vol}(V) \geq \frac{\mathrm{vol}(C)}{d_3(\infty)} = \frac{\sqrt{3}\, d^2}{48\, d_3(\infty)} > \mathrm{vol}(V_*) = v_* \approx 0.171502$$

  implies the rough bound $1 < d \leq 2.013813$

- Next, one checks easily that

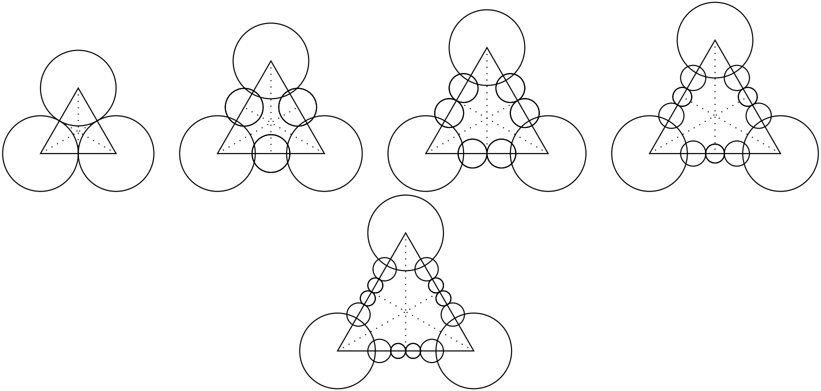
$$w \geq 1\,, \text{ and this fact leads to } \quad d \geq 1.515464$$

- It suffices to assume that D has a mirror symmetry. Indeed, if Γ_∞ is **orientation-preserving**, the bound $d \geq 1.515464$ yields

$$\mathrm{vol}(V) \geq \frac{\mathrm{vol}(C)}{d_3(\infty)} > v_*$$

A hierarchy of $\frac{1}{d_k}$-balls ($d = d_1$)

1. Each $(\frac{1}{d})$-ball touches two full-sized balls
2. Each $(\frac{1}{d})$-ball touches one full-sized and one $(\frac{1}{d})$-ball
3. Each $(\frac{1}{d})$-ball touches one full-sized ball and *no* other $(\frac{1}{d})$-ball.



$$\rightsquigarrow \qquad d = 2\cos\left(\frac{\pi}{2k_{max}+2}\right) \quad ; \quad d = 2\cos\left(\frac{\pi}{2k_{max}+3}\right)$$

1st partial solution for $\frac{1}{d_k}$-balls aligned on edges of D

Put the two results from above together and write
$d =: 2\cos\alpha_\kappa \leq 2$ with $\kappa = k_{max}$.

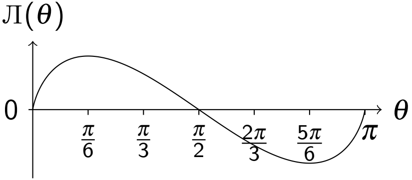
Proposition

In the case of the $\{2,3,6\}$-cusp where Γ_∞ is a reflection group, and the $\frac{1}{d_k}$-balls are aligned on the edges of D, the minimal possible orbifold volume is obtained from the (truncated) Coxeter orthoscheme $R(\alpha_\kappa, \frac{\pi}{3})$ where $d = 2\cos\alpha_\kappa$

By Schläfli's volume differential formula, the volume of $R(\alpha_\kappa, \frac{\pi}{3})$ is strictly increasing as a function of d....and there is a closed formula for $R(\alpha_\kappa, \frac{\pi}{3})$:

Volume of $R(\alpha, \beta)$ in terms of Lobachevsky's function

$$\text{Л}(\theta) = -\int_0^\theta \ln|2\sin t|\, dt$$



For a *(truncated) orthoscheme* with (at least) one ideal vertex $R(\alpha, \beta) = [\alpha, \beta, \frac{\pi}{2} - \beta]$, $0 \leq \alpha, \beta < \frac{\pi}{2}$, the volume is

$$\text{vol}(R(\alpha, \beta)) = \frac{1}{4}\left\{\text{Л}(\frac{\pi}{2} + \alpha - \beta) - \text{Л}(\frac{\pi}{2} + \alpha + \beta) + 2\,\text{Л}(\beta)\right\}$$

Consequence.

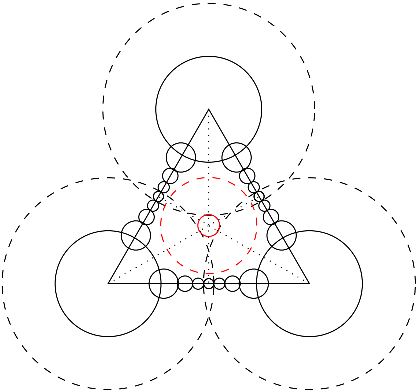
$$\text{vol}(R(\alpha_k, \frac{\pi}{3})) \geq \text{vol}(R(\frac{\pi}{7}, \frac{\pi}{3})) > 0.317811 > v_*$$

2nd partial solution for $\frac{1}{d_k}$-balls aligned on edges of D

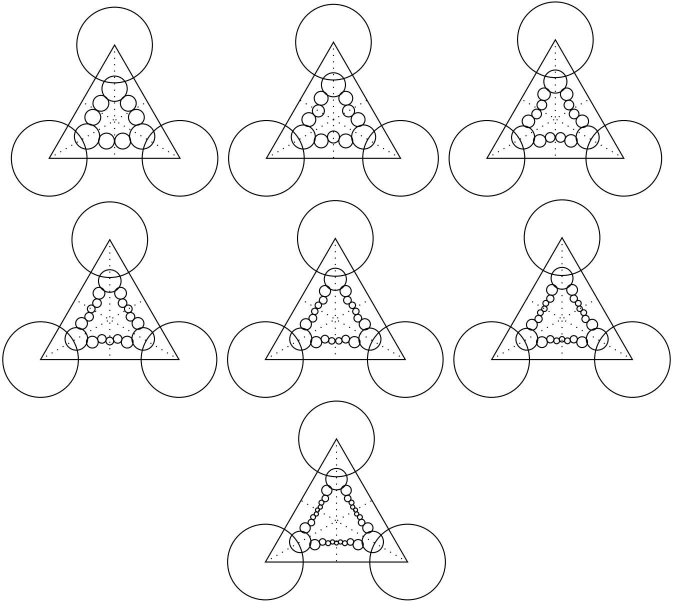
Proposition

Under the same assumptions as above and with $d > 2$, the minimal possible volume is obtained by an orbifold related to a truncated orthoscheme $R(\beta, \frac{\pi}{6})$ where $0 < \beta < \frac{\pi}{15}$.

Consequence. $\mathrm{vol}(R(\beta, \frac{\pi}{6})) \geq \mathrm{vol}(R(\frac{\pi}{15}, \frac{\pi}{6})) > 0.416491 > v_*$

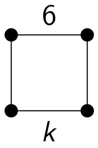


$\frac{1}{d}$-balls centred on the angle bisectors of D



Solution in the diagonal case

The solution is given by a (truncated) Coxeter simplex group with Coxeter graph



where k is an integer with $k \geq 3$. For the volume of V, one gets

- for $3 \leq k \leq 6$, $\quad \mathrm{vol}(V) \geq \mathrm{vol}([3^3, 6]) > 2\, v_*$
- for $k > 7$, the edge length e of D is $d + \sqrt{3} \geq 2\sqrt{3}$ and hence

$$\mathrm{vol}(C) = \frac{e^2\sqrt{3}}{48} \geq \frac{\sqrt{3}}{4} > 0.43 > 2\, v_*$$

Further steps to analyse

- For the cusp type $\{2,3,6\}$, full-sized horoballs centred at singular points a_3
- For the cusp type $\{2,3,6\}$, full-sized horoballs centred at singular points a_2
- Perform a similar analyse for the cusp type $\{2,4,4\}$
- For both cusp types, consider the case of several Γ_∞-equivalence classes of full-sized horoballs

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[DK] S. Drewitz, R. Kellerhals, *The non-arithmetic cusped hyperbolic 3-orbifold of minimal volume*, preprint 50 pages, arXiv:2106.12279.

References

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[A2] C. Adams, *Volumes of hyperbolic 3-orbifolds with multiple cusps*, Indiana Univ. Math. J. 41 (1992), 149–172.

[GMM] D. Gabai, R. Meyerhoff, P. Milley, *Minimum volume cusped hyperbolic three-manifolds*, J. Amer. Math. Soc. 22 (2009), 1157-1215.

[H] T. Hild, *The cusped hyperbolic orbifolds of minimal volume in dimensions less than ten*, J. Algebra 313 (2007), 208–222.

[K] R. Kellerhals, *On the volume of hyperbolic polyhedra*, Math. Ann. 285 (1989), 541–569.

[M] R. Meyerhoff, *The cusped hyperbolic 3-orbifold of minimum volume*, Bull. Amer. Math. Soc. (N.S.) 13 (1985), 154–156.

Thank you for your attention