Statistical comparisons of spatio-temporal networks

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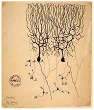
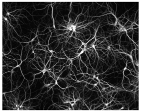
The brain is both a structural and functional network

Microscale

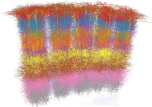


[Cajal, 1890]



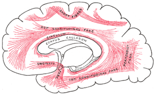
[Yuste, 2015]

Mesoscale



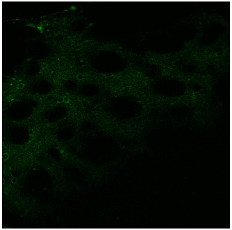
[Oberlaender *et al.*, 2013]

Macroscale



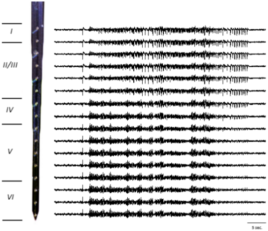
Neuroimagery

Optical imagery



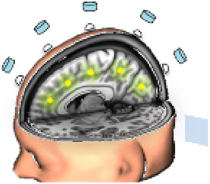
[Badreddine PhD]

Microelectrodes



Thanks to Florian Studer

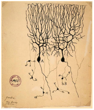
fMRI/EEG/MEG



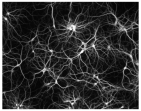
[De Vico Fallani *et. al*, 2016]

The brain is both a structural and functional network

Microscale

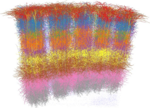


[Cajal, 1890]



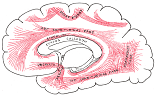
[Yuste, 2015]

Mesoscale

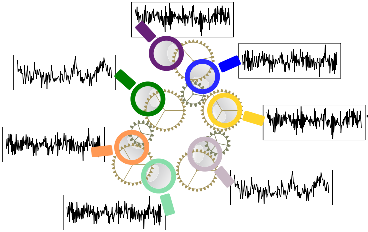


[Oberlaender *et al.*, 2013]

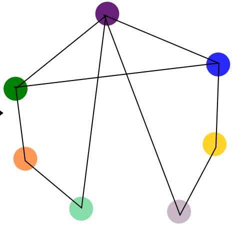
Macroscale



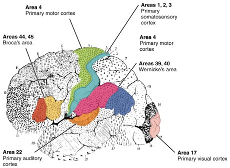
Mathematical modeling



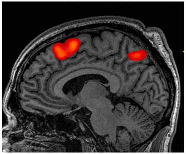
—— Network ——→



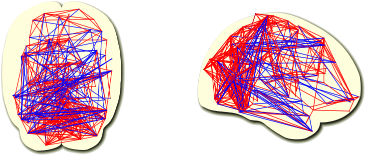
The brain is both a structural and functional network



Phrenologic view

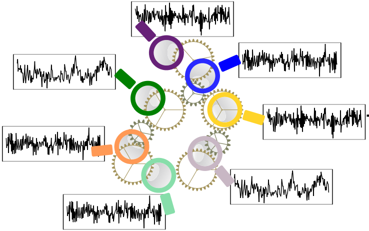


Neuroimagery

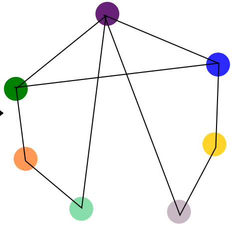


Connectivist view

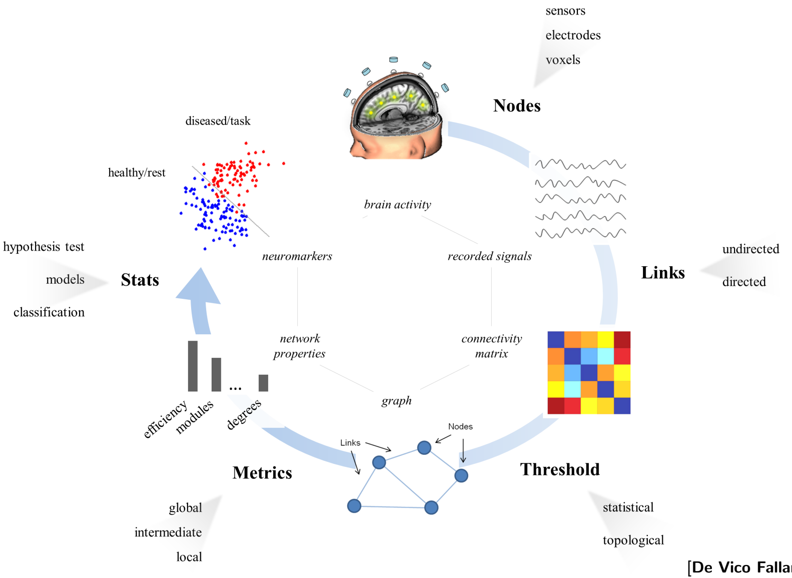
Mathematical modeling



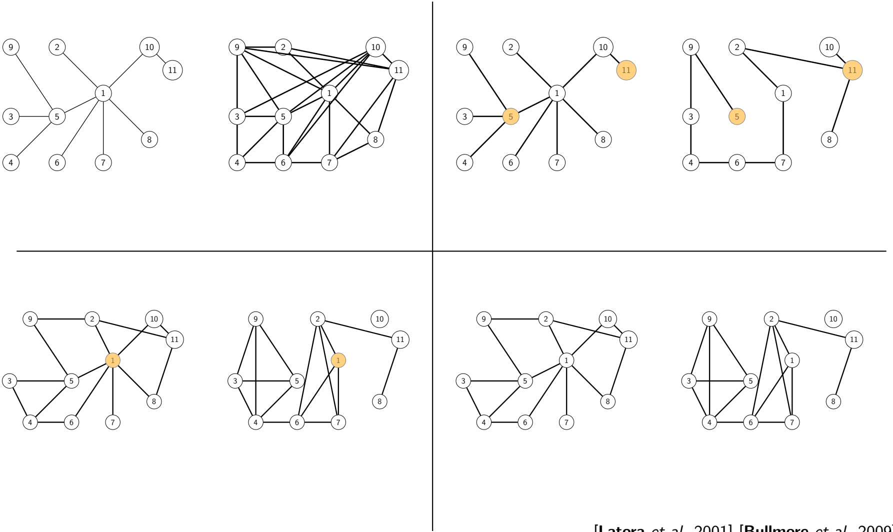
Network ⟶



Exploring the brain using networks analysis: pipeline

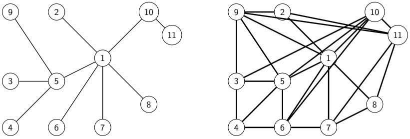


[De Vico Fallani *et al.*2014]



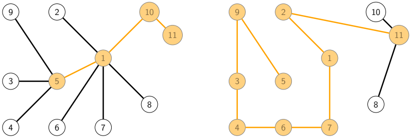
[**Latora** *et al.* 2001] [**Bullmore** *et al.* 2009]
[**Csárdi** *et al.* 2006]

Cost



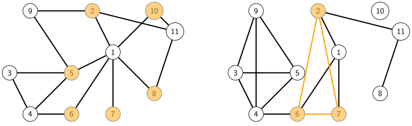
Low cost — High cost

Efficiency



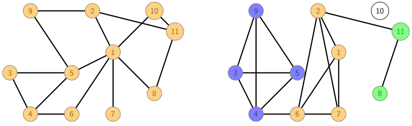
High efficiency — Low efficiency

Clustering



Low clust — High clust

Modularity

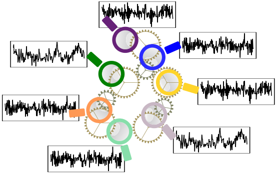


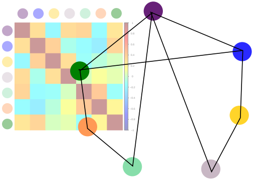
Low modularity — High modularity

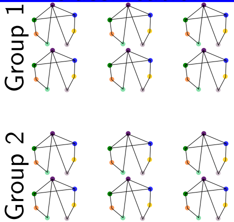
[**Latora** *et al.* 2001] [**Bullmore** *et al.* 2009]
[**Csárdi** *et al.* 2006]

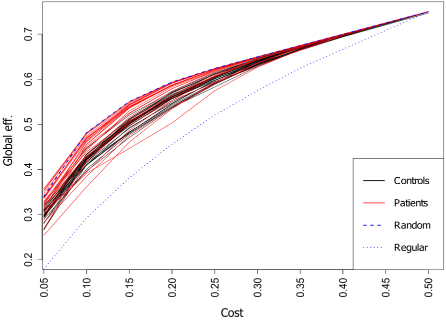
Comparisons of healthy volunteers and patients

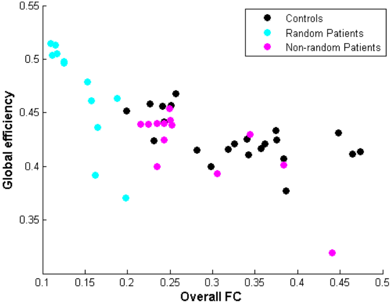
Observations	Graph inference	Comparaison, classification







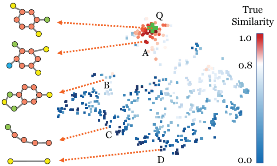




[**Malagurski** *et al.* 2019]

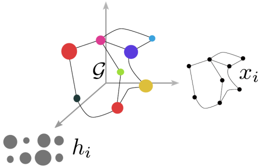
Graph comparisons: other methods

Graph matching
Edit distance



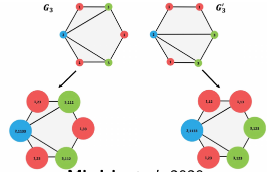
Bai *et al.* 2018

Fused Gromov Wasserstein distance



Vayer *et al.* 2020

Similarity of graphs



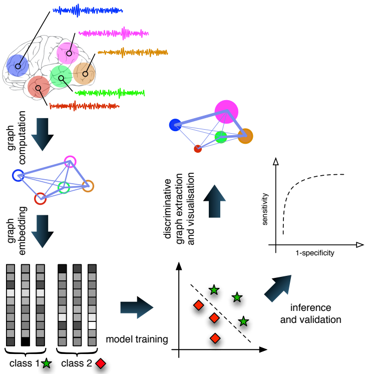
Mheich *et al.* 2020

Statistical tests

$$n^{1/2}(\phi(\widehat{L}_n) - \phi(\Lambda)) \longrightarrow N(0, \Sigma),$$

Ginestet *et al.* 2017

Graph comparisons: new methods needed



[**Richiardi** *et al.* 13]

Objectives:

- use graph nodal statistics
- be invariant to permutation of nodes
- allow easy interpretation for medical researchers

Nodal statistics-based structural pattern on single graph

Nodal statistics-based structural pattern

Definition

We consider undirected unweigthed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and refer to a nodal statistics $s : \mathcal{V} \to s(\mathcal{V})$ any function of the adjacency matrix.
The **equivalence relation** $\sim_s$ **associated with a nodal statistics** s, on the nodes set $\mathcal{V}$ of a graph is:

$$v \sim_s u \iff s(u) = s(v).$$

The **equivalent relation associated with any collection of statistics** $\mathcal{S} = \{s_i\}_{i=1,..,n}$, is defined as:

$$a \sim_{\mathcal{S}} b \iff a \sim_{s_1} b,\, a \sim_{s_2} b,\, \ldots,\, a \sim_{s_n} b.$$

When the statistics is continuous, for some small ε

$$v \sim_s u \iff |s(u) - s(v)| \leqslant \varepsilon$$

Nodal statistics-based structural pattern on single graph

Structural Pattern

Definition

Le us define P the induced partition on $\mathcal{V}$ given $\mathcal{S}$,

$$P_{\mathcal{S}} = \frac{\mathcal{V}}{\sim_{\mathcal{S}}} = \{[a], \quad \forall a \in \mathcal{V}\}$$

defines the **structural pattern** of $\mathcal{G}$ associated with the statistics collection $\mathcal{S}$.

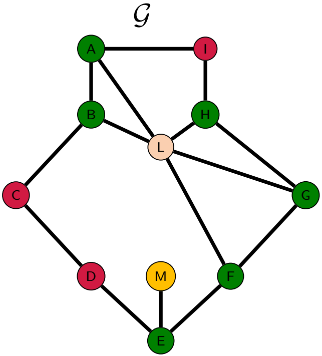
Node role

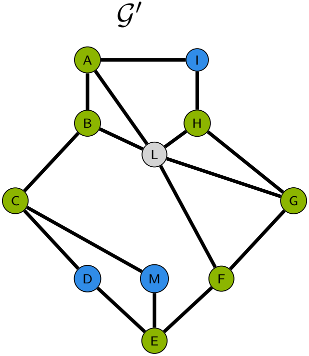
Definition

The class of equivalence $[a] = \{b \in \mathcal{V} | a \sim_s b \iff s(a) = s(b)\}$ corresponds to **node role**.

Illustrations with degree

Let $\mathcal{G}, \mathcal{G}'$ be two graphs having same vertices $\mathcal{V}$ and let $\mathcal{S}$ be the statistics degree whose associated partitions are $P_{\mathcal{S}}, P'_{\mathcal{S}}$ on $\mathcal{G}, \mathcal{G}'$ respectively.





$P_{\mathcal{S}} = [ABEFGH], [L], [CDI], [M]$ $\quad P'_{\mathcal{S}} = [ABCEFGH], [L], [DIM]$

Heterogeneity of nodal structural nodes

Definition

Let $\mathcal{G}$ be a graph having $\mathcal{V}$ vertices and let $\mathcal{S}$ be the statistics whose associated partitions are $P_{\mathcal{S}}$ on $\mathcal{G}$.
We define the power coefficient as $\widehat{\mathsf{PC}}$,

$$\widehat{\mathsf{PC}}_{\mathcal{G}}(\mathcal{S}) \quad = \quad 1 - \frac{\log \#\{\text{permutations preserving } P_{\mathcal{S}}\}}{\log \#\{\text{permutations of } \mathcal{V}\}}$$

Nodal structural roles

Properties of $\widehat{\text{PC}}$

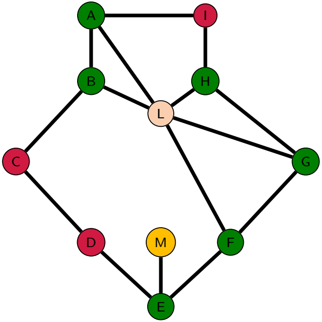
Proposition

- *The higher the $\widehat{PC}$, the more the collection of statistics $\mathcal{S}$ capture the heterogeneity of nodal structural roles in $\mathcal{G}$.*
  For a vertex-transitive graphs (i.e. all nodes are automorphically equivalent) $\widehat{PC}_{\mathcal{G}}(\mathcal{S}) = 0$ for all nodal statistics $\mathcal{S}$.
  If it exists a collection $\bar{\mathcal{S}}$ s.t. $\widehat{PC}_{\mathcal{G}}(\bar{\mathcal{S}}) = 1$ then the graph $\mathcal{G}$ does admit non-trivial automorphisms.
- *In the special case in which the permutations preserving $P_{\mathcal{S}}$ can be identified with the automorphisms of $\mathcal{G}$, PC can be interpreted as entropy of the network ensemble having $\mathcal{G}$ topology (**Bianconni** et al. 2007). In all other cases, PC encodes the amount of information given by $\mathcal{S}$ on the structure of $\mathcal{G}$ and it is a parsimony measure for $\mathcal{S}$.*

Illustrations with degree

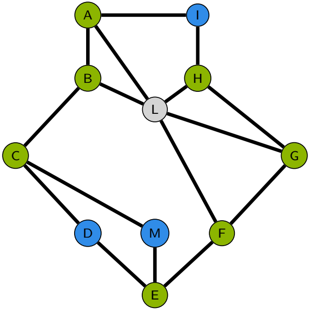
Let $\mathcal{G}, \mathcal{G}'$ be two graphs having same vertices $\mathcal{V}$ and let $\mathcal{S}$ be the statistics degree whose associated partitions are $P_{\mathcal{S}}, P'_{\mathcal{S}}$ on $\mathcal{G}, \mathcal{G}'$ respectively.

$\mathcal{G}$



$\widehat{\mathrm{PC}}_{\mathcal{G}}(\mathcal{S}) = 0.52$

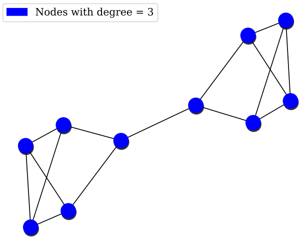
$\mathcal{G}'$

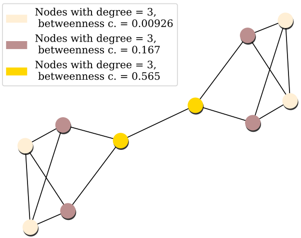


$\widehat{\mathrm{PC}}_{\mathcal{G}'}(\mathcal{S}) = 0.41$

However degree may be not sufficient

Depending on the graph structure, one statistic may be not sufficient to capture important features of graph.





Comparisons of statistics induced by graph structures

Orthogonal statistics for heterogeneity evaluation of a collection elements

Two nodal statistics are said to be perfectly orthogonal if their union-associated equivalent relation induces the trivial partition: all nodes belong to a single element set.

Definition

Let $\mathcal{G}$ be a graph having $\mathcal{V}$ vertices and let $\mathcal{S}$ be the statistics whose associated partitions are $P_{\mathcal{S}}$ on $\mathcal{G}$.

$$O_{\mathcal{G}}(\mathcal{S}) = \frac{|\{v \in \mathcal{V} \text{ s.t. } \#[v]_{\sim_{\mathcal{S}}} \neq 1\}|}{|\mathcal{V}|}$$

$O_{\mathcal{G}}(\mathcal{S})$ is the ratio between the number of nodes in non-trivial classes and the total number of vertices and corresponds to an orthogonality score. By definition, $\mathcal{S}$ is perfectly orthogonal if and only if $O_{\mathcal{G}}(\mathcal{S}) = 0$.

Structural patterns for graph collections characterization

Correspondence structural pattern score

Definition

Let $\mathcal{G}, \mathcal{G}'$ be two graphs having same vertices $\mathcal{V}$ and let $\mathcal{S}$ be a statistics collection whose associated partitions are $P_{\mathcal{S}}, P'_{\mathcal{S}}$ on $\mathcal{G}, \mathcal{G}'$ respectively. Given bijective mapping from $P_{\mathcal{S}}, P'_{\mathcal{S}}$ to an initial segment of the natural numbers as enumerations, let $c(v_i), c'(v_i)$ be the enumeration of the classes of v_i, the correspondence structural pattern score between $\mathcal{G}, \mathcal{G}'$ is defined as:

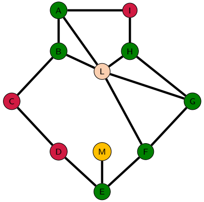
$$C(\mathcal{G}, \mathcal{G}') = \max_{\pi \in \Pi} \frac{1}{|\mathcal{V}|} \sum_{i=1}^{|\mathcal{V}|} \mathcal{X}(\pi(c(v_i)) = c'(v_i))$$

where Π is the set of all coupling between the elements in $P_{\mathcal{S}}$ and the elements in $P'_{\mathcal{S}}$ and $\mathcal{X}$ is the indicator function.

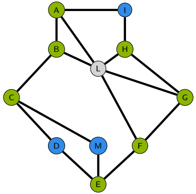
Illustrations

Let $\mathcal{G}, \mathcal{G}'$ be two graphs having same vertices $\mathcal{V}$.

$\mathcal{G}$



$\mathcal{G}'$

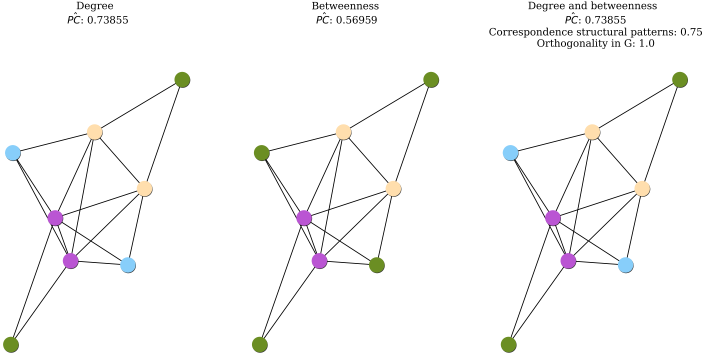


$\mathcal{G}'$ \ $\mathcal{G}$	L	C D I	A B E F G H	M
L	1			
D I M		2		1
A B C E F G H		1	6	

$$C(\mathcal{G}, \mathcal{G}') = \frac{1+2+6}{11} = 0.81$$

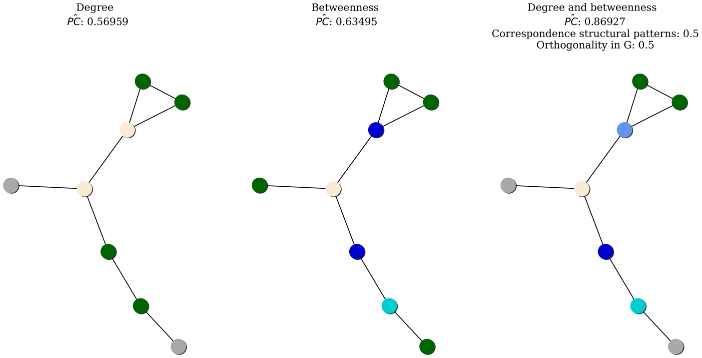
Illustrations

One statistics is enough informative



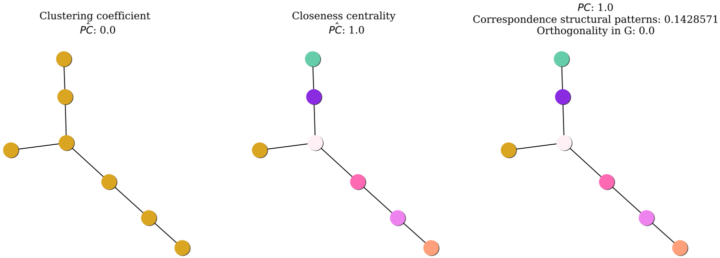
Illustrations

Two nodal statistics are more informative



Illustrations

Two perfectly orthogonal nodal statistics



Property of orthogonality

Proposition

- *a nodal statistics whose induced partition is composed of classes having each one a unique element, is perfectly orthogonal with every nodal statistics*
- *if collection of statistics is perfectly orthogonal, all other collection having as subset that collection is perfectly orthogonal as well*
- *if a perfectly orthogonal statistics set exists on a graph, then the graph does not admit non-trivial automorphisms*

Property of correspondance structural pattern

Proposition

- *If for every class in $P_{\mathcal{S}}$ there exists one class of $P'_{\mathcal{S}}$ having all and only its elements, then $P_{\mathcal{S}} = P'_{\mathcal{S}}$ and $C(\mathcal{G}, \mathcal{G}') = 1$. The opposite is also true.*
- *all graphs defined on the same node set, having same degree sequence, have a correspondence of structural patterns associated with the degree statistics equals to 1*
- *the minimum values of structural pattern score is given by $\frac{1}{|\mathcal{V}|}$. (At least one class of $P_{\mathcal{S}}$ shares one element with one of the classes in $P'_{\mathcal{S}}$).*
- *if on the same graph, the structural patterns score of different nodal statistics reaches the minimal value, then the nodal statistics are perfectly orthogonal*

Structural patterns for graph collections characterization

Nodal-percentage of participation

Definition

Given a graphs collection $G = \{\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k) \text{ s.t. } \mathcal{V}_k = \mathcal{V}\}$,and a statistics collection $\mathcal{S}$ we count the percentage of participation of each node of $\mathcal{V}$ in non-trivial classes:

$$\forall v \in \mathcal{V} \quad \mathrm{PP}_G^{\mathcal{S}}(v) = \mathrm{PP}_G(v) = \frac{|\{\mathcal{G}_k \in G \text{ s.t. } \#[v]_{\sim_{\mathcal{S}}}^{\mathcal{G}_k} \neq 1\}|}{|G|} \tag{1}$$

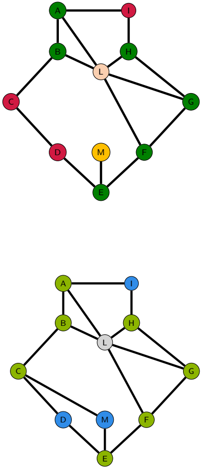
with $[v]_{\sim_{\mathcal{S}}}^{\mathcal{G}_k}$ the class of v in $\mathcal{G}_k$ in the partition induced by $\mathcal{S}$.

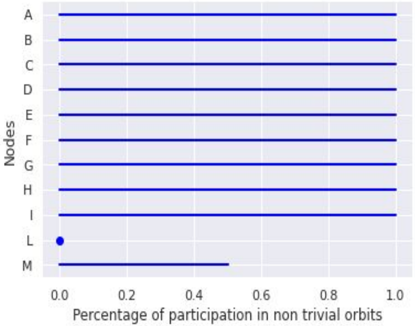
Illustrations

Let $\mathcal{G}, \mathcal{G}'$ be two graphs having same vertices $\mathcal{V}$.

$\mathcal{G}$

$\mathcal{G}'$





- Nodes A-I share their roles with other nodes in both graphs
- node L has a specific role in the two graphs
- role of node M depends on graphs instance

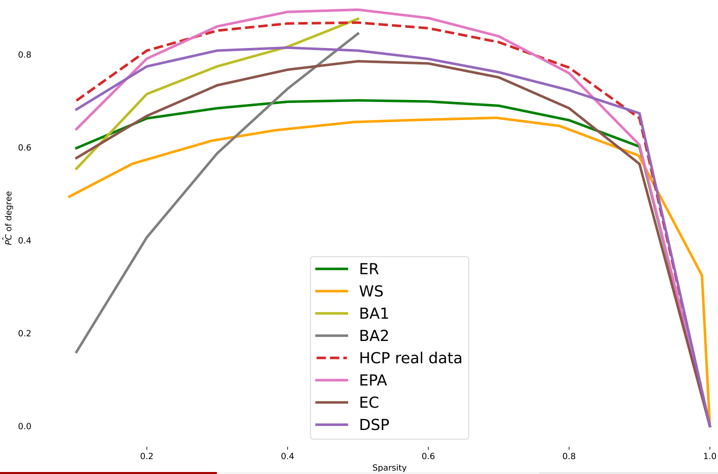
Results: $\widehat{PC}$ for different sparsity graphs models

Using different graph generative models and one real datasets:

- Erdős-Rényi model (ER)
- Watts-Strogatz model (WS)
- Barabási-Albert model (BA)
- Degree sequence preserving model (DSP)
- Economical preferential attachment model (EPA)
- Economical clustering model (EC)
- resting-state fMRI data human connectome project (HCP)

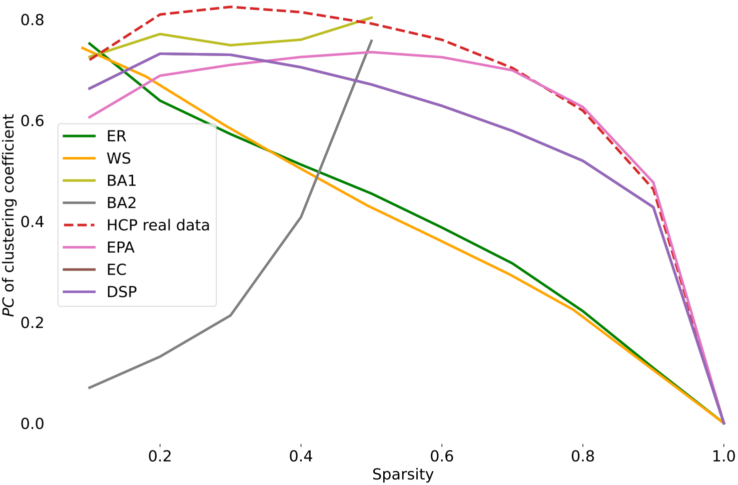
Results: $\widehat{PC}$ for different sparsity graph models

$\widehat{PC}$ is able to determine a prefered sparsity for degree

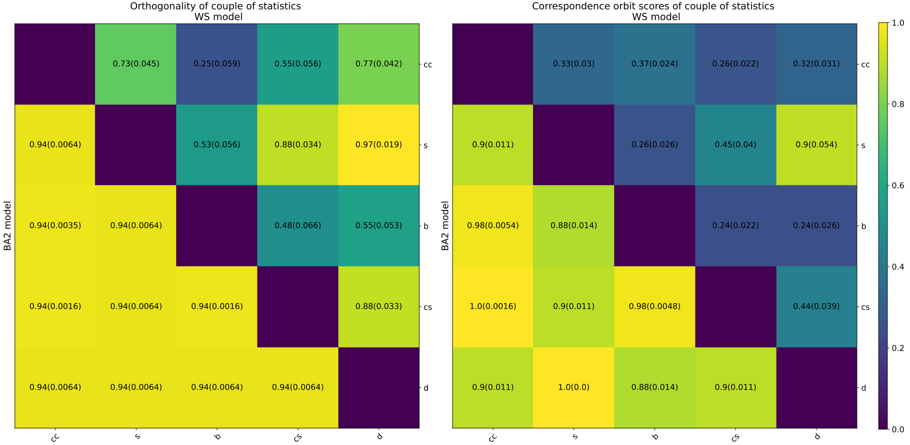


Results: $\widehat{PC}$ for different sparsity graph models

$\widehat{PC}$ behaviour is dependent on statistics



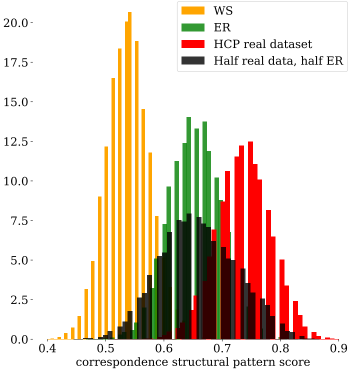
Results of orthogonality on simulated WS and BA models

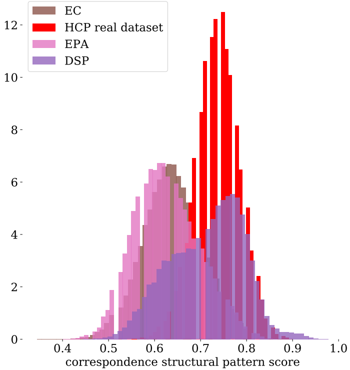


cc= clustering coefficient, s=second order centrality, cs=closeness centrality, b=betweenness centrality, d=degree

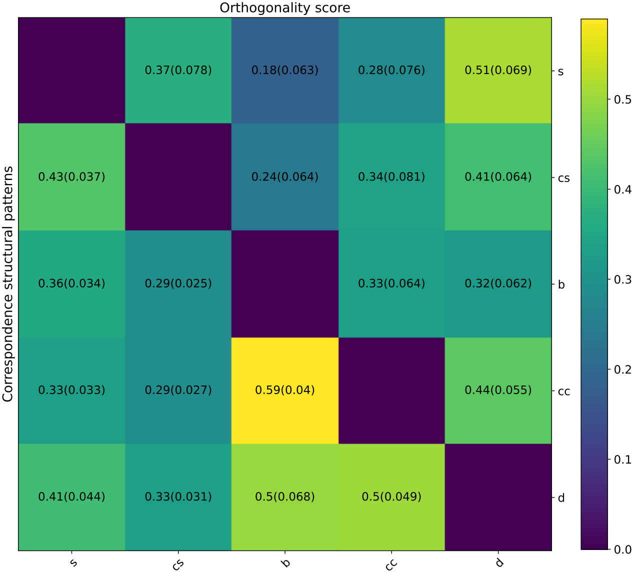
Results of correspondance score on graph models

Real datasets have the highest correspondance score

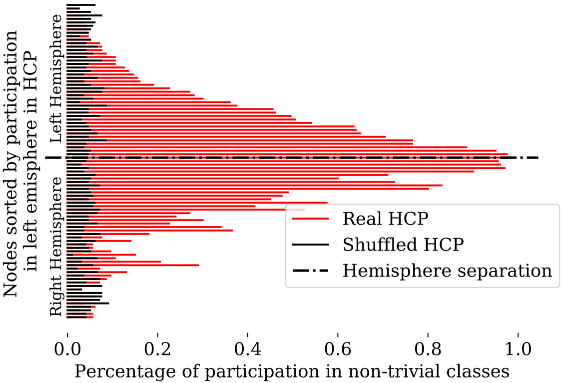




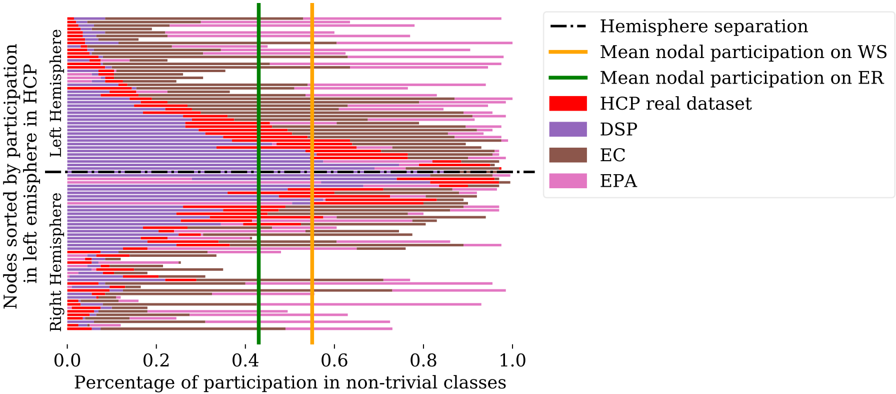
Results of correspondance and orthogonality score on brain data



Results of percentage of participation to non-trivial classes on brain data



Results of percentage of participation to non-trivial classes on brain data



Results of fMRI resting state datasets on healthy volunteers and patients

Freely avalaible datasets. Do not hesitate to use them!

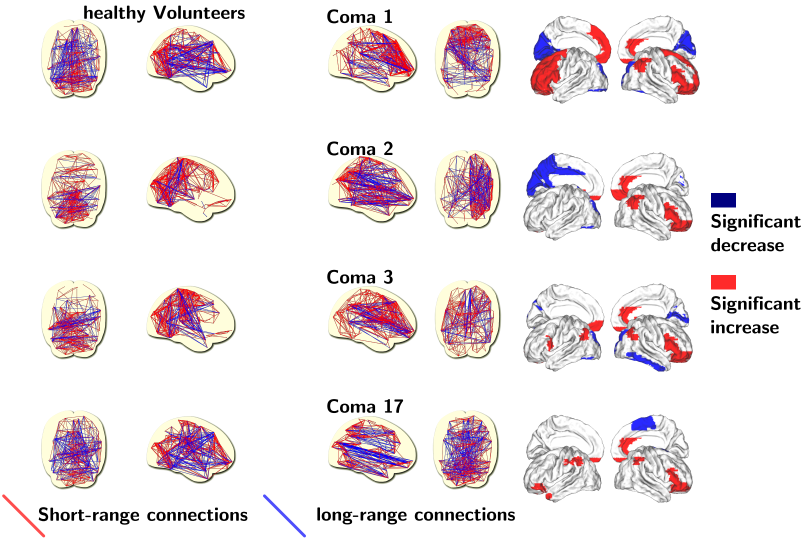
Provider	Subjects	Scanning Parameters				Age range
	Total	B0	TR	#Vol	Frequency Band	years
ChuStr *Achard et al. 2012*	HC(20)-CO(17)	1.5T	3000 ms	405	0.042-0.084 Hz	25-45, 21-82
HCP *Termenon et al. 2016*	HC(100×2)	3T	720 ms	1200	0.043-0.087 Hz	20-43
iShare *Tsuchida et al. 2017*	HC(1814)	3T	850 ms	1046	0.037-0.074 Hz	18-35
Gin-Chuga *Ramirez et al. 2019*	HC(11)-PD(13)	3T	1000 ms	500	0.031-0.063 Hz	46-70, 51-70

Focus on comatose patients

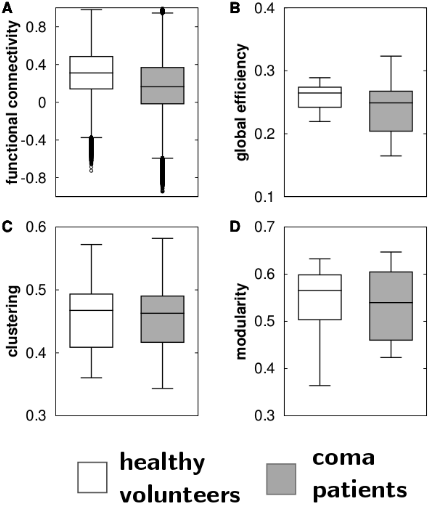
- **90 and 417 anatomical regions:** space average of the fMRI time series over all voxels in 90 (AAL) and 417 regions
- **SPM preprocessing:** correction for geometrical displacements
- **Resting state:** lying quietly with eyes closed during 20 minutes
- **Group comparison:**
  20 young healthy volunteers, 17 patients in coma

[**Achard** *et al.* 2012]

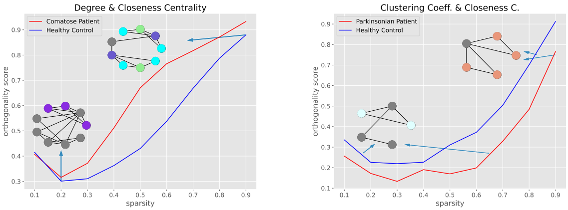
Examples of connectivity graphs



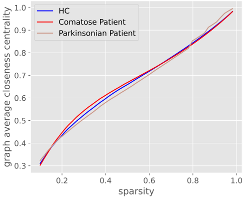
No significant difference on global measure of functional connectivity



Results of orthogonality scores on real datasets

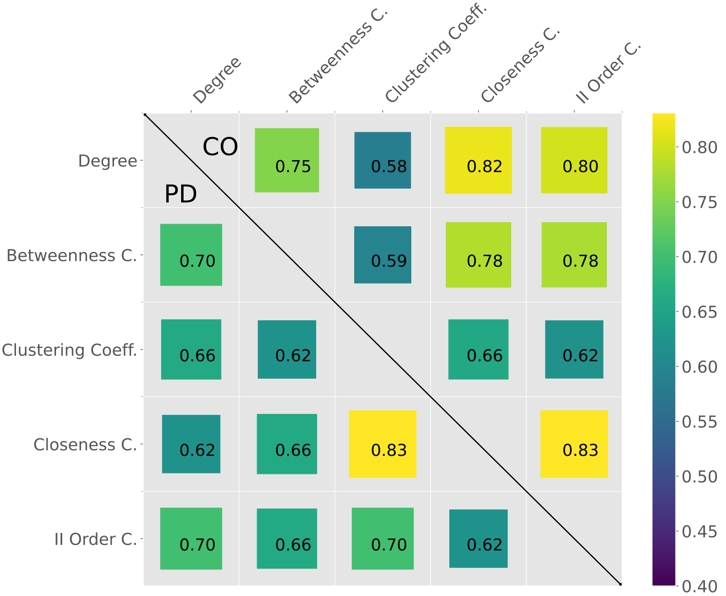


(a) Orthogonality centroids

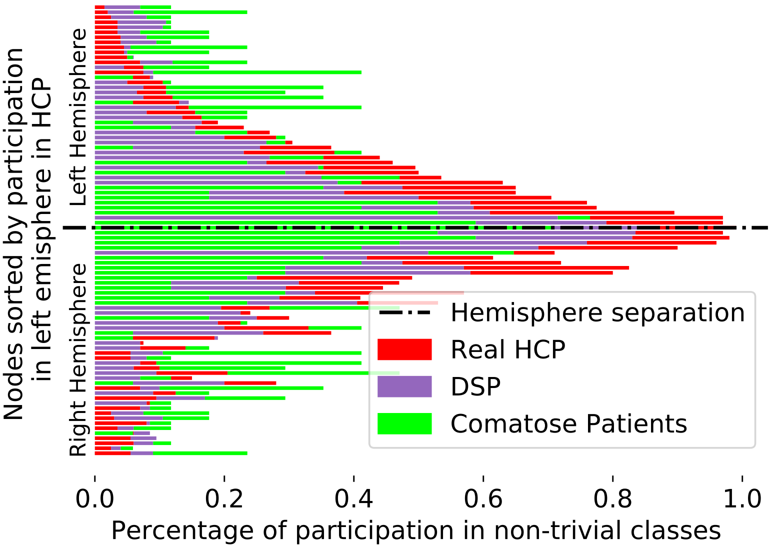


(b) Mean graph average

Illustrations of classification on orthogonality curves



Illustrations at the nodal level



Conclusion and future work

Conclusion

- New framework for comparison of spatio-temporal models
- Description at nodal level
- Comparisons of graph statistics

Future work

- Derive properties for graph statistics
- Build a statistical test
- Apply to different data

arxiv.org/abs/2210.01053

Thanks for your attention