Partial regularity of nematic liquid crystal flow with kinematic transport effects

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Brief Background

Liquid crystal phases are states between liquids and solids, and exhibit both fluid (unordered) and crystalline (ordered) structures. At a spatial location, liquid crystal molecules prefer to align in the same direction, which leads to the optical property and profound applications in the LCD.

Two approaches to describe the LC orientational distributions:

- Vector Fields (Oseen-Frank-Ericksen Model): unit vector field $d : \Omega \mapsto \mathbb{S}^2$.
- Tensor Fields (Landau-De Gennes Model): traceless symmetric 3×3 matrix $Q : \Omega \mapsto \mathcal{S}_0^3$. Q describes the second moment of probability distribution function of molecule orientations:

$$Q(x) = \int_{\mathbb{S}^2} (p \otimes p - \frac{1}{3} I_3)\, d\mu(x, p), \;\; \mu(x, \cdot) \in \mathcal{P}(\mathbb{S}^2).$$

- Landau-De Gennes (LDeG) reduces to Oseen-Frank-Ericksen (OFE): When Q is uniaxial or $eigen(Q) = (\lambda, \lambda, -2\lambda)$, it reduces to $Q = s(d \otimes d - \frac{1}{3} I_3)$ $(s = \frac{3\lambda}{2})$.

Oseen-Frank energy functional

The elastic energy of Oseen-Frank: $E_{OF}(d) = \int_\Omega W(d, \nabla d)\, dx$,

$$\begin{aligned} 2W(d, \nabla d) &= k_1(div\ d)^2 + k_2(d \cdot curl\ d)^2 + k_3|d \times curl\ d|^2 \\ &\quad +(k_2 + k_4)(tr(\nabla d)^2 - (div\ d)^2), \qquad (1) \end{aligned}$$

where Frank coefficients $k_1, k_2, k_3 > 0$ are called splay, twist, and bending constants, and k_4 is the null-Lagrangian constant.

Static theory: Study equilibrium of E_{OF}:

$$\inf\{E_{OF}(d) :\ d \in H^1(\Omega, \mathbb{S}^2),\ d\big|_{\partial\Omega} = g \in H^{\frac{1}{2}}(\partial\Omega, \mathbb{S}^2)\}.$$

- (Hardt-Kinderlerhrer-Lin): $\mathcal{H}$-dim(Sing$d$)< 1.
- (Schoen-Uhlenbeck, Brezis-Coron-Lieb, Lin, Simon, etc):

$$k_1 = k_2 = k_3 = k_{24} = 1 \Rightarrow E_{OF} = Dirichlet\ energy,$$

$$d\ minimizing\ harmonic : \Delta d + |\nabla d|^2 d = 0,$$

$$d(x) \sim R(\frac{x-a}{|x-a|}),\quad R \in O(3),\ a \in Sing(d).$$

Hydrodynamics: Ericksen-Leslie system

Treating LC as an anisotropic fluid, Ericksen and Leslie (1958-1968) derived the constitutive equation for nematic liquid crystals.

Assume LC is incompressible and homogeneous (e.g., $\rho = 1$). Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be the velocity field. The EL system is

$$\begin{cases} u_t + u \cdot \nabla u + \nabla P = \nabla \cdot \hat{\sigma}; \quad \nabla \cdot u = 0, \\ (\frac{\delta W}{\delta d} - \gamma_1 N - \gamma_2 A d) \times d = 0. \end{cases} \tag{2}$$

- $\hat{\sigma}$ (total stress tensor): $\hat{\sigma} = \frac{\partial W}{\partial(\nabla d)} \otimes \nabla d + \sigma$,
- σ (Leslie viscous stress tesnor): (Leslie coefficients μ_i's),

$$\sigma = \mu_1 (A : d \otimes d) d \otimes d + \mu_2 N \otimes d + \mu_2 d \otimes N + \mu_4 A \\ + \mu_5 A : d \otimes d + \mu_6 d \otimes d : A.$$

- $A = 1/2(\nabla u + (\nabla u)^t); \quad N = d_t + u \cdot \nabla d - \omega d;$
  $\omega = \frac{1}{2}(\nabla u - (\nabla u)^t); \quad \gamma_1 = \mu_3 - \mu_2, \; \gamma_2 = \mu_6 - \mu_5.$

$k_1 = k_2 = k_3 = k_{24} = 1$

$$W = \frac{1}{2}|\nabla d|^2, \ \frac{\partial W}{\partial \nabla d} \otimes \nabla d = \nabla d \otimes \nabla d,$$

EL(2) reduces to

$$\begin{cases} u_t + u \cdot \nabla u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d) + \nabla \cdot \sigma, \\ \nabla \cdot u = 0, \\ N + \frac{\gamma_2}{\gamma_1} Ad = -\frac{1}{\gamma_1}(\Delta d + |\nabla d|^2 d) + \frac{\gamma_2}{\gamma_1} d^t Ad. \end{cases} \tag{3}$$

Question:

- Global existence of physically meaningful weak solutions of EL(3) under IBC.
- Regularity/singularity, uniqueness properties of certain weak solutions of EL(3) under IBC.

Energy dissipation for El(3)

For $\Omega = \mathbb{R}^n$ $(n = 2, 3)$, set

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} (|u|^2 + |\nabla d|^2)(x,t)\, dx.$$

Assume Parodi's condition: $\boxed{\mu_2 + \mu_3 = \mu_6 - \mu_5}$

$$E'(t) + \int_{\mathbb{R}^n} (\mu_4 |\nabla u|^2 - \frac{2}{\gamma_1}|\Delta d + |\nabla d|^2 d|^2)$$
$$= -\int_{\mathbb{R}^n} [(\mu_1 - \frac{\gamma_2^2}{\gamma_1})|A : d \otimes d|^2 + (\mu_5 + \mu_6 + \frac{\gamma_2^2}{\gamma_1})|Ad|^2] \leq 0 \quad (4)$$

provided[1]

$$\boxed{\gamma_1 < 0,\ \mu_4 > 0,\ (\mu_1 - \tfrac{\gamma_2^2}{\gamma_1}) \geq 0,\ (\mu_5 + \mu_6 + \tfrac{\gamma_2^2}{\gamma_1}) \geq 0} \quad (5)$$

[1] F. Lin, C. Liu (ARMA 154, 2000); H. Wu, X. Xu, C. Liu (ARMA 208, 2013); W. Wang, P. W. Zhang, Z. F. Zhang (ARMA 210, 2013); J. Huang, F. Lin, C. Wang (CMP 331, 2014)

Previous works on EL(3)

There have been considerable activities on a simplified version of EL(3) proposed by Lin[2]: neglecting fluid stretching and rigid rotation, and interacting Leslie stress terms, i.e.

$$\sigma = \mu_4 A, \ N = d_t + u \cdot \nabla d, \ \gamma_1 = -1, \ \gamma_2 = 0,$$

EL(3) reduces to

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d), \\ \nabla \cdot u = 0; \quad d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d. \end{cases} \tag{6}$$

- ($n = 2$) Existence and uniqueness of a global solution with finitely many singular times for EL(6) under IBC; Existence of finite time singularity.[3]
- ($n = 3$) Global weak solutions of EL(6), provided $d_0(\Omega) \subset \mathbb{S}^2_+$; Existence of finite time singularity.[4]

[2] F. Lin, CPAM 42 (1989)

[3] Lin-Lin-Wang, ARMA 197(2010); Lin-Wang, CAM 31(2010); Lai-Lin-Wang-Wei-Zhou, CPAM 75(2022)

[4] Lin-Wang, CPAM 69(2016); Huang-Lin-Wang, CMP 331(2014)

EL system with variable degrees of orientation

According to either De Gennes in Q-tensor theory or Ericksen in OF theory, aside the director d there is another parameter $s \in \mathbb{R}$ measuring the degree of molecules aligning in direction d at a spatial point so that liquid crystals may exhibit different phases (isotropic, nematic, smectic).

Ericksen[5] modified the EL system by incorporating this degree parameter s and replacing E_{OF} by

$$\hat{E}_{OF}(s, \hat{d}) = \int_\Omega (W(s, \hat{d}, \nabla s, \nabla \hat{d}) + F(s))\, dx.$$

A typical choice of bulk function $F \in C^\infty((-1/2, 1))$ satisfies

$$\lim_{s \downarrow -\frac{1}{2}} F(s) = \lim_{s \uparrow 1} F(s) = \infty.$$

[5] J. Ericksen: ARMA, 113 (1990), 97-120

Set $d = s\hat{d}$ and consider a simplified form of $\hat{E}_{OF}$:

$$E_\epsilon(d) = \int_\Omega W_\epsilon(d, \nabla d) = \int_\Omega (\frac{1}{2}|\nabla d|^2 + \frac{1}{4\epsilon^2}(1-|d|^2)^2)\, dx. \quad (7)$$

First order variation of W_ϵ gives

$$-\frac{\delta W_\epsilon}{\delta d} = \Delta d + f(d), \quad f(d) = \frac{1}{\epsilon^2}(1-|d|^2)d.$$

EL(2) becomes

$$\begin{cases} u_t + u\cdot\nabla u + \nabla P = -\frac{\delta W_\epsilon}{\delta d}\cdot\nabla d + \nabla\cdot\sigma, \\ \nabla\cdot u = 0; \quad -\gamma_1 N - \gamma_2 A d = -\frac{\delta W_\epsilon}{\delta d}. \end{cases} \quad (8)$$

Here σ equals to

$$\mu_1(A : d\otimes d)d\otimes d + \mu_2 N\otimes d + \mu_3 d\otimes N + \mu_4 A + \mu_5 A : d\otimes d + \mu_6 d\otimes d : A.$$

Motivation to study EL(8):

- For $\epsilon > 0$, EL(8) has a degree of nonlinearities lower than EL(2). As $\epsilon \to 0$, EL(8) "converges" to EL(2), hence paves ways to EL(2).
- EL(8) is related to Beris-Edwards system in De Gennes Q-tensor model: when $Q = s(\hat{d} \otimes \hat{d} - \frac{1}{n} I_n)$, Beris-Edwards system reduces to EL(8). Studying EL(8) sheds light to the non co-rotational Beris-Edwards system. There have been several studies on the co-rotational Beris-Edwards system[6]

Lin-Liu[7] initiated the study of a simplified version of EL(8):

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla P = -(\Delta d + f(d))\nabla d, \\ \nabla \cdot u = 0; \quad d_t + u \cdot \nabla d = \Delta d + f(d). \end{cases} \tag{9}$$

The existence of a global suitable weak solution of EL(9), whose singular set has zero 1-dimensional Hausdorff measure.

[6]Paicu-Zarnescu: ARMA 203 (2012), SIMA 43 (2011); M. Wilkinson: ARMA 218 (2015); Du-Hu-Wang: ARMA 238 (2020).

[7]C. Liu, F. Lin: CPAM 48 (1995), DCDS 2 (1996).

Beris-Edwards system

Let $u : \Omega \mapsto \mathbb{R}^3$ be the fluid velocity and $Q : \Omega \mapsto \mathcal{S}_0^{(3)}$. The Beris-Edwards system of (u, Q), modeling the hydrodynamics of nematic liquid crystals, is

$$\begin{cases} \partial_t Q + u \cdot \nabla Q - S(\nabla u, Q) = \Gamma H(Q), \\ \partial_t u + u \cdot \nabla u + \nabla P = \mu \Delta u + \nabla \cdot (\tau + \sigma), \\ \nabla \cdot u = 0, \end{cases} \tag{10}$$

where $\Gamma > 0, \mu > 0$, and

$$H(Q) = -\frac{\delta \psi}{\delta Q} = L\Delta Q - \langle \nabla_Q \psi_B(Q) \rangle \quad (\psi(Q) = \frac{1}{2} L |\nabla Q|^2 + \psi_B(Q)),$$

$$\langle \nabla_Q \psi_B(Q) \rangle = \nabla_Q \psi_B(Q) - \frac{1}{3} \mathrm{tr}(\nabla_Q \psi_B(Q)) I_3,$$

$$\sigma = QH(Q) - H(Q)Q \quad (\textit{part of Ericksen tensor}).$$

Here $\psi_B(Q)$ is a bulk potential function that can be either De Gennes type of polynomials or Ball-Majumdar singular type.

$$\tau = -\xi\big(Q + \frac{1}{3}I_3\big)H(Q) - \xi H(Q)\big(Q + \frac{1}{3}I_3\big) + 2\xi\big(Q + \frac{1}{3}I_3\big)QH(Q) - L\nabla Q \otimes \nabla Q,$$

$$S(\nabla u, Q) = (\xi D + \omega)\big(Q + \frac{1}{3}I_3\big) + \big(Q + \frac{1}{3}I_3\big)(\xi D - \omega) - 2\xi\big(Q + \frac{1}{3}I_3\big)\mathrm{tr}(Q\nabla u), \text{ and}$$

$$D = \frac{1}{2}(\nabla u + (\nabla u)^\top),\ \omega = \frac{1}{2}(\nabla u - (\nabla u)^\top).$$

- $\xi = 0$ (co-rotational case): Better understood. See Paicu-Zarnescu (De Gennes potential), Wilkinson (Ball-Majumdar potential), and Du-Hu-Wang.
- $\xi \neq 0$ (non co-rotational cases): Less known. See Paicu-Zarnescu for $|\xi| << 1$; Cavaterra-Rocca-Wu-Xu (SIMA 2016) for all ξ in $\mathbb{R}^2$; Abels-Dolzmann-Liu (SIMA 2014) for all ξ in bounded domain Ω.

EL system with kinematic transport effects

Question: What if all 6 Leslie coefficients occur in σ?

Answer: Can handle σ with $\mu_2, \cdots, \mu_6$. It's open for $\mu_1 \neq 0$.

I will discuss EL(8) for σ with μ_2, μ_3, μ_4:

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d + S_\alpha[\Delta d + f(d), d]), \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d - T_\alpha[\nabla u, d] = \Delta d + f(d), \end{cases} \tag{11}$$

where $\alpha \in [0, 1]$, and

$$S_\alpha[\Delta d + f(d), d] = \alpha(\Delta d + f(d)) \otimes d - (1-\alpha) d \otimes (\Delta d + f(d)),$$
$$T_\alpha[\nabla u, d] = \alpha(\nabla u) \otimes d - (1-\alpha)(\nabla u)^t \otimes d.$$

[$\alpha = 0, \frac{1}{2}, 1$ corresponds to disk-like, spherical, and rod-like shape of LC molecules][8]

[8] H. Wu, X. Xu, C. Liu: CVPDE 45 (2012); ARMA 208 (2013)

Remark on strong solutions of EL(8) or EL(11)

- Hieber-Nesensoln-Pruss-Shade[9], Hieber-Pruss[10]:

  Local well-posedness of strong solutions and long-time convergence near equilbrium of simplified /or general Ericksen-Leslie system with general Leslie stress tensors and isotropic Oseen-Frank energy.
- Hong-Xin[11], Hong-Li-Xin [12], Feng-Hong-Mei[13]:

  Convergence results of GL approximation of simplified EL system with general Oseen-Frank energy in the class of strong solutions.
- Hieber-Wilke-Li[14]: Local strong solutions of general Ericksen-Leslie system with general Oseen-Frank energy in $\mathbb{R}^3$.

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[9] AIHP-AN 33 (2016) 397-408.

[10] Math Ann 369 (2016) 977-996; ARMA 233 (2019) 1441-1468.

[11] Adv. Math. 231 (2012) 1364-1400.

[12] CPDE 39 (2014) 1284-1328.

[13] SIMA 52 (2020) 481-423.

[14] Talk in this workshop

Main theorems on suitable weak solutions of EL (11)

Theorem (Hengrong Du-Wang, nonlinearity (2021)) *For* $u_0 \in L^2_{div}$, $d_0 \in \dot{H}^1(\mathbb{R}^3)$ *with* $F(d_0) \in L^1(\mathbb{R}^3)$, *there exists a global suitable weak solution* $(u, d, P) : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ *of EL(11) such that* $(u, d) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}_+ \setminus \Sigma)$, *where* Σ *is a closed set satisfying*

$$\mathcal{P}^{\frac{15}{7}+\alpha}(\Sigma) = 0, \ \forall \alpha > 0 \ (\textit{or equivalently} \ \dim_H(\Sigma) \leq \frac{15}{7})$$

Structural cancellation property

$$\int S_\alpha[\Delta d + f(d), d] : \nabla u \phi = \int T_\alpha[\nabla u, d] : d\phi, \ \forall \phi \in C_0^\infty(\mathbb{R}^3).$$

Lemma 1. *A reasonable solution* (u, d, P) *of EL(11) satisfies*

$$\int_{\mathbb{R}^3} (\frac{1}{2}|u|^2 + (\frac{1}{2}|\nabla d|^2 + F(d)))(t) + \int_0^t \int_{\mathbb{R}^3} (|\nabla u|^2 + |\Delta d + f(d)|^2)$$
$$\leq \int_{\mathbb{R}^3} (\frac{1}{2}|u_0|^2 + (\frac{1}{2}|\nabla d_0|^2 + F(d_0)). \qquad (12)$$

Suitable weak solutions

Definition 1. *A weak solution (u, d, P) of the system EL(11) is a suitable weak solution if it satisfies the local energy inequality:*

$$
\begin{aligned}
&\frac{d}{dt}\int[\frac{1}{2}|u|^2+(\frac{1}{2}|\nabla d|^2+F(d))]\phi+\int(|\nabla u|^2+|\Delta d|^2+|f(d)|^2)\\
&\leq \int[\frac{1}{2}(|u|^2+|\nabla d|^2)(\phi_t+\Delta\phi)+F(d)\phi_t]\\
&+\int[\frac{1}{2}(|u|^2+2P)u\cdot\nabla\phi+\nabla d\otimes\nabla d : u\otimes\nabla\phi]\\
&+\int(\nabla d\otimes\nabla d-|\nabla d|^2 I_3) : \nabla^2\phi\\
&+\int[S_\alpha[\Delta d+f(d), d] : u\otimes\nabla\phi+T_\alpha[\nabla u, d](\nabla\phi\cdot\nabla d)]\\
&+\int[f(d)(\nabla\phi\cdot\nabla d)+2\nabla f(d) : \nabla d\phi] \qquad (13)
\end{aligned}
$$

$\forall 0\leq\phi\in C_0^\infty(\mathbb{R}^3\times(0,\infty))$.

Remark

- (12) can be derived by multiplying equation of u by $u\phi$, equation of d by $(\Delta d + f(d))\phi$, and utilizing the structural cancellation property.
- A similar local energy inequality for the NSE has played critical roles in CKN's regularity theory for NSE.
- A simpler version of (12) was first introduced by Lin-Liu in the simplified EL system (9).
- A key technical advantage of EL(9) is that

$$d_t + u \cdot \nabla d = \Delta d + f(d)$$

  enjoys the maximum principle for $|d|$.
- There is no maximum principle of d in

$$d_t + u \cdot \nabla d - T_\alpha[\nabla u, d] = \Delta d + f(d)$$

  from EL(10).

ϵ_0-regularity criteria

To obtain decay of d, we adapt ideas by Giaquinta, Hildebrandt, and Evans in the study of parabolic systems associated with Morrey's quasiconvex energy functionals into our setting.

For $r > 0$, denote $P_r = B_r \times (-r^2, 0)$.

Lemma 2. *$\forall M > 0$, $\exists \epsilon_0 > 0$ and $0 < \tau_0 < 1$ and $C_0 > 1$ so that if*

$$|d_r| := \Big|\frac{1}{|P_r|}\int_{P_r} d\Big| \le M, \tag{14}$$

$$\Phi(u, d, P; r) := r^{-2}\int_{P_r}(|u|^3 + |\nabla d|^3) + (r^{-3}\int_{P_r}|P|^{\frac{3}{2}})^2 + (r^{-5}\int_{P_r}|d - d_r|^6)^{\frac{1}{2}} \le \epsilon_0^3, \tag{15}$$

then

$$\Phi(u, d, P; \tau_0 r) \le \frac{1}{2}\max\{\Phi(u, d, P; r),\ C_0 r^3\}. \tag{16}$$

Proof of Lemma 2 via blowing-up argument

Suppose that Lemma 2 false. Then there would exist M_0 such that $\forall \tau \in (0,1)$, $\exists \epsilon_i \to 0, C_i \to \infty$, and $r_i > 0$ so that

$$|\bar{d}_{r_i}| \leq M_0,\ \Phi(u,d,P;r_i) = \epsilon_i^3, \&\ \Phi(u,d,P;\tau r_i) > \frac{1}{2}\max\{\epsilon_i^3, C_i r_i^3\}.$$

$$\Longrightarrow r_i^3 \leq \frac{\epsilon_i^3}{C_i \max\{\tau^{-4}, 8\tau^{-5/2}\}} \to 0.$$

Define blow-up sequence

$$(u^i, d^i, P^i)(x,t) := \Big(\frac{r_i u}{\epsilon_i}, \frac{d - d_{r_i}}{\epsilon_i}, \frac{r_i^2 P}{\epsilon_i}\Big)(r_i x, r_i^2 t).$$

1) <u>Eqns of $\{u^i, d^i, P^i\}$</u> :

$$\begin{cases} u_t^i + \epsilon_i u^i \cdot \nabla u^i + \nabla P^i - \Delta u^i \\ = -\epsilon_i \nabla d^i \cdot \Delta d^i + \frac{r_i^2}{\epsilon_i} \nabla d^i \cdot f(\hat{d}^i) - \nabla \cdot S_\alpha[\Delta d^i - \frac{r_i^2}{\epsilon_i} f(\hat{d}^i), \hat{d}^i], \\ \nabla \cdot u^i = 0; \quad d_t^i - \epsilon_i u^i \cdot \nabla d^i - T_\alpha[\nabla u^i, \hat{d}^i] = \Delta d^i + \frac{r_i^2}{\epsilon_i} f(\hat{d}^i). \end{cases}$$

Here $\hat{d}^i(x,t) = d(r_i x, r_i^2 t)$.

2) Bounds of $\{u^i, d^i, P^i\}$:

$$\begin{cases} \int_{P_1} d^i = 0; \quad |\bar{d}_1^i| \leq M_0; \quad \Phi(u^i, d^i, P^i; 1) = 1, \\ \qquad \Phi(u^i, d^i, P^i; \tau_0) > \frac{1}{2} \max\left\{1, C_i(\frac{r_i}{\epsilon_i})^3\right\}, \\ \int_{P_1} \left(|d_i|^6 + F^{\frac{3}{2}}(d^i) + |f(d^i)|^2 + |\nabla_d f(d^i)|^3\right) \lesssim 1 + \epsilon_i^6 + M_0^6. \end{cases}$$

3) Blow-up limit $\{v, e, Q\}$: $\exists$ $\{v, e, Q\}$ such that

$$\begin{cases} \{u^i, d^i, P\} \rightharpoonup \{v, e, Q\} \ \text{ in } \ L^3 \times L_t^3 W_x^{1,3} \times L^{\frac{3}{2}}, \\ \Phi(v, e, Q; 1) \leq 1 = \lim_{i \to \infty} \Phi(u^i, d^i, P^i; 1), \end{cases}$$

and

$$\begin{cases} v_t - \Delta v + \nabla Q = -\nabla \cdot S_\alpha[\Delta e, \bar{d}], \\ \nabla \cdot v = 0; \quad e_t - \Delta e = T_\alpha[\nabla v, \bar{d}], \end{cases} \tag{17}$$

where $\bar{d} = \lim_{i \to \infty} \bar{d}_{r_i}$ satisfies $|\bar{d}| \leq M_0$.

4) Regularity of the blow-up limit $\{v, e\}$:

Lemma 3. *$\{v, e\} \in C^\infty$, and for any $\tau \in (0, \frac{1}{2})$,*

$$\tau^{-2} \int_{P_\tau} (|v|^3 + |\nabla e|^3 + |Q|^{\frac{3}{2}}) \lesssim \tau^3 \int_{P_{\frac{1}{2}}} (|v|^3 + |\nabla e|^3 + |Q|^{\frac{3}{2}}).$$

Proof. Follows from local energy inequalities of (16) of arbitrary orders and $L^{\frac{3}{2}}$-estimate of

$$-\Delta Q = \nabla^2 \cdot S_\alpha[\Delta e, \bar{d}].$$

5) Strong convergence of $\{u^i, d^i, P^i\}$ to $\{v, e, Q\}$:

Lemma 4. *$\{u^i,\ d^i,\ \nabla d^i\} \to \{v,\ e,\ \nabla e\}$ in $L^3 \times L^6 \times L^3(P_{\frac{1}{2}})$.*

Proof. Follows from local energy inequality for $\{u^i, d^i, P^i\}$:

$$\sup_{t\in[-1/4,0]} \int_{B_{\frac{1}{2}}} (|u^i|^2 + |\nabla d^i|^2) + \int_{P_{\frac{1}{2}}} (|\nabla u^i|^2 + |\nabla^2 d^i|^2)$$
$$\lesssim \int_{P_1} [(|u^i|^2 + |\nabla d^i|^2) + \frac{r_i^2}{\epsilon_i^2} F(\hat{d}^i)] + \epsilon_i (|u^i|^2 + |\nabla d^i|^2 + |P^i|)|u^i|$$
$$+ \int_{P_1} [|\hat{d}^i|^2 (|u^i|^2 + |\nabla d^i|^2) + |\nabla d^i|^2 + r_i^2 |\nabla d^i|^2 |\nabla_d f(\hat{d}^i)|] \le C.$$

- L^6-convergence follows from $d^i \in L^{10}_{t,x}(P_{\frac{1}{2}})$ and

$$\|d^i\|_{L^{10}_{t,x}} \lesssim \|\nabla d^i\|_{L^{10}_t L_x^{\frac{30}{13}}} \lesssim \|d^i\|^{\frac{4}{5}}_{L_t^\infty H_x^1} \|d^i\|^{\frac{1}{5}}_{L_t^2 H_x^2}.$$

6) Estimate of pressure P^i: P^i solves Poisson's equation

$$-\Delta P^i = \epsilon_i \nabla^2 \cdot [u^i \otimes u^i + \nabla d^i \otimes \nabla d^i - (\frac{1}{2}|\nabla d^i|^2 + \frac{r_i^2}{\epsilon_i^2} F(\hat{d}^i)) I_3] + \nabla^2 \cdot \underbrace{S_\alpha[\Delta d^i + \frac{r_i^2}{\epsilon_i} f(\hat{d}^i), d^i]}. \tag{18}$$

- Note: $\nabla^2 \cdot S_\alpha \neq 0$ unless $\alpha = \frac{1}{2}$.

- New ingredients needed for the decay of $\tau^{-2} \int_{P_\tau} |P^i|^{\frac{3}{2}}$.

7) Higher order convergence of $\{u^i, d^i\}$ to $\{v, e\}$:

$$(\nabla u^i, \nabla^2 d^i) \to (\nabla v, \nabla^2 e) \ \text{ in } \ L^2(P_{\frac{1}{3}}). \tag{19}$$

Ideas: Since $(v, e) \in C^\infty$ solves the limit equation (16), we can show

$$(\widetilde{u}^i, \widetilde{d}^i, \widetilde{P}^i) := (u^i - v, d^i - e, P^i - Q)$$

is a suitable weak solution of the difference equation. Hence it satisfies a modified local energy inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^3} (|\widetilde{u}^i|^2 + |\nabla \widetilde{d}^i|^2)\phi + 2\int_0^t \int_{\mathbb{R}^3} |\nabla \widetilde{u}^i|^2 + |\Delta \widetilde{d}^i|^2)\phi \\
&\le \int_0^t \int_{\mathbb{R}^3} [(|\widetilde{u}^i|^2 + |\nabla \widetilde{d}^i|^2)(\phi_t + \Delta\phi) + C\epsilon_i + o(1) \\
&+ \int_0^t \int_{\mathbb{R}^3} [S_\alpha[\Delta d^i, \hat{d}^i] : \nabla v \phi - S_\alpha[\Delta e, \bar{d}] : \nabla u^i \phi] \\
&+ \int_0^t \int_{\mathbb{R}^3} [T_\alpha[\nabla u^i, \hat{d}^i] : \Delta e \phi - T_\alpha[\nabla v, \bar{d}] : \Delta d^i \phi].
\end{aligned}$$

Since

$$-\Delta Q = \nabla^2 \cdot S_\alpha[\Delta e, \bar{d}],$$

by Calderon-Zygmund's $W^{2,\frac{3}{2}}$-estimate of

$$\Delta(P^i - Q) = O(\epsilon_i) + \nabla^2 \cdot (S_\alpha[\Delta d^i + \frac{r_i^2}{\epsilon_i} f(\hat{d}^i), d^i] - S_\alpha[\Delta e, \bar{d}]),$$

we obtain the decay of $\tau^{-2} \int_{P_\tau} |P^i|^{\frac{3}{2}}$.

- To obtain the dimension estimate of the singular set

$$\Sigma = \Big\{ z : \ \textit{either} \ \liminf_{r \to 0} |d_{z,r}| = \infty \ \textit{or} \ \liminf_{r \to 0} \Phi(u, d, P; z, r) > 0 \Big\},$$

observe the bound from local energy inequality of (u, d, P) and the equation of d yield $d \in W^{2,\frac{1}{2}}_{\frac{20}{7}}$. Hence by Sobolev-Poincaré,

$$\left(r^{-5} \int_{P_r(z)} |d - d_{z,r}|^6\right)^{\frac{1}{6}} \lesssim r^{1-\frac{7}{4}} \|d\|_{W^{2,\frac{1}{2}}_{\frac{20}{7}}(P_r(z))}.$$

THANK YOU FOR LISTENING