Liquid Crystal Droplet and Its Orientation Configuration

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Abstract

Liquid crystal droplets are of interest from both the theory and applications. They are important in the study of the topological defects (in a bulk or on a surface), in understanding of surface energies and anchoring conditions. Rigorous mathematical analysis are challenging. In this lecture I shall describe some recent work (joint with Changyou Wang and Zhiyuan Geng) concerning liquid crystal droplets and the associated orientation configurations.

Mathematical Models

There are three basic mathematical models describing the marcoscopic behavior of liquid crystals: the classical Oseen-Frank model, Ericksen and de Genne-Landau models. These models are of increasing orders of complexity and also flexibility for studying properties of liquid crystal droplets and topological defects.

Recently, Changyou Wang and I have studied nematic droplets as sharp interface limits in the Ericksen model. The advantages of this model can be summarized as follows:

- Mathematically it has a self-contained and consistent theory. The model is natural and relatively simple.
- It can accommodate point defects, disclinations and domain walls in liquid crystals with rigorous analysis.
- It keeps the classical Oseen-Frank model (favored by many practitioners) intact.
- For nematic-isotropic sharp interface transitions, the biaxiality of defects seems to have less effect on both the variational energy and the shape of the smooth droplets.

De Genne-Landau Model

- It is a more general and consistent theory from both mathematical and physical point of views. For example, it can be derived rigorously from microscopic (molecular/kinetic) models.
- It can be used to describe more complex defect structures, both uniaxial and biaxial. In fact, purely uniaxial solutions are very rare in the de Genne-Landau model.
- The de Genne-Landau model may also result in anisotropic surface energies. This would lead to different shapes of (or even non smooth) droplets and defect patterns within it.

Some Related Works

(a) There are many works concerning phase transitions and shape interfaces in liquid crystals (by experiments, simulations and modelings).
Rigorous analysis are few, for examples, P.W.Zhang et al for 1D profiles and high-D special flows; some symmetric cases by E. Virga and recent studies by P.Sternberg et al.
(b)For the analysis of phase transitions in de Genne-Landau model the complexity are formidable. If energy density functions are quadratic functions of grad Q with coefficients are quadratic polynomials in Q, then there are 22 invariants (and 13 surface terms) along with 4 null-Lagrangians. If one would consider additional chrial effects one may need to add a couple more terms.

In the Oseen-Frank model, based on theory developed in works of Friedel, Chandrasekhar, Ericksen ... for the surface energy, a liquid crystal droplet can be described by a pair (Ω, n), where n is the optical director field and Ω is the region occupied by the liquid crystal. Determining the shape and the orientation of a liquid crystal droplet leads to finding a minimizing pair for a functional of the form

$$\int_\Omega W(n, \nabla n)\, dx + \int_{\partial\Omega} f(n \cdot \nu)\, dA,$$

under the constraint that the volume of Ω in $\mathbb{R}^3$ is (pre)fixed and $n : \Omega \to \mathbb{S}^2$ is a unit vector field. Here ν is the unit normal along $\partial\Omega$.

The bulk energy density associated with such a director field n is, via Oseen-Frank, given by

$$W(n, \nabla n) = k_1 |\mathrm{div}\, n|^2 + k_2 |n \cdot \mathrm{curl}\, n|^2 + k_3 |n \wedge \mathrm{curl}\, n|^2 + (k_2 + k_4) \left[\mathrm{tr}(\nabla n)^2 - |\mathrm{div}\, n|^2 \right],$$

k_1, k_2, k_3 are positive constants. The last term is a so-called Null-Lagrangian,

$$\mathrm{tr}(\nabla n)^2 - (\mathrm{div}\, n)^2 = \mathrm{div}\left[(\nabla n) n - (\mathrm{div}\, n) n \right].$$

Remark

If n satisfies a strong anchoring condition on $\partial\Omega$ (i.e.,a Dirichlet boundary condition), then one can always adjust the last term (by adding a term of form $\alpha \left[\mathrm{tr}(\nabla n)^2 - (\mathrm{div}\, n)^2 \right]$) so that the resulting $W(n, \nabla n)$ satisfies

$$\lambda |\nabla n|^2 \leq W(n, \nabla n) \leq \Lambda |\nabla n|^2,$$

for some positive constants λ, Λ.

Remark

However, the contact angle ($\cos\theta = n \cdot \nu$) is not a priori given for a droplet. In addition, one does not know if Ω is a smooth domain in $\mathbb{R}^3$ on which one can perform the usual "integration by parts". Moreover, for a general energy density, one may have also terms of the form: $L_1 \int_{\partial\Omega} (\operatorname{div} n) n \cdot \nu \, dA + L_2 \int_{\partial\Omega} \langle (\nabla n) n, \nu \rangle \, dA$.
Many researchers assume θ is a constant, i.e. $n \cdot \nu =$ constant. Then it is possible to calculate the last two terms explicitly so long as the droplet is sufficiently smooth, see a recent work of [A.Zarnescu].

Following Friedel, Oseen, Chandrasekhar and Ericksen, the interfacial energy is of the form $\int_{\partial\Omega} f(n \cdot \nu)\, dA$. A typical choice of f is

$$f(\cos\theta) = \mu(1 + \lambda \cos^2\theta),$$

where $\mu > 0$ and $\lambda > -1$ (Ericksen, Virga). Thus we lead to the following:

Problem A

Find a pair (Ω, n) such that

$$(\Omega, n) \text{ minimizes } \left(\int_\Omega W(n, \nabla n)\, dx + \int_{\partial\Omega} f(n \cdot \nu)\, dA \right)$$

subject to the constraint $\mathrm{vol}(\Omega) = V_0 > 0$.

Theorem (L-Poon, 96)

Assume $f : [-1, 1] \to \mathbb{R}$ *is continuous with* $f \geq c_0 > 0$. *Then there is a pair* (Ω_A, n_A) *in* $\Gamma_A = \{(\Omega, n) : \Omega \in \mathbb{R}^3$ *is convex with* $|\Omega| = V_0 > 0$ *and* $n \in H^1(\Omega, \mathbb{S}^2)\}$ *that solves the Problem A.*

Remark

In case $W(n, \nabla n) = |\nabla n|^2$, then n_A has a bounded number of singularities in $\overline{\Omega}$. $\nu_{\partial\Omega}, n_A$ has a bounded number of singularities on $\partial\Omega$.

When $\lambda > 0$ and $\mu \to +\infty$, it leads to the following:

Problem B

Find a pair (Ω, n) such that

$$(\Omega, n) \text{ minimizes } \left(\int_\Omega W(n, \nabla n)\, dx + \mu \operatorname{Area}(\partial\Omega) \right)$$

and such that (i) $\mathrm{Vol}(\Omega) = V_0$ and (ii) $n \cdot \nu = 0$ on $\partial\Omega$.

When $\lambda \in (-1, 0)$ and $\mu \to +\infty$, then one has

Problem C

Find a pair (Ω, n) such that

$$(\Omega, n) \text{ minimizes } \left(\int_\Omega W(n, \nabla n)\, dx + \mu \operatorname{Area}(\partial\Omega) \right)$$

subject to (i) $\mathrm{Vol}(\Omega) = V_0$ and (ii) $n \cdot \nu = 1$ on $\partial\Omega$.

Theorem (L-Poon 1996)

There are minimizing pairs (Ω_b, n_B) and (Ω_C, n_C) for the problems B and C among convex domains Ω.

<u>Remark</u> The proofs use various properties of convex domains Ω in $\mathbb{R}^m$ with a fixed Volume $|\Omega| = V_0 > 0$ and bounded surface areas $|\partial\Omega| \leq C_o$.

1. The family $\{H^{n-1} L \partial\Omega\}$ is compact. In fact, they are restrictions of $\{H^{n-1}$ to uniformly Lipschitz graphs over $S^{n-1}_{r_0}$.
2. $\{\nu_\Omega H^{n-1} L \partial\Omega\}$ is compact, say, in $L^1(S^{n-1}_{r_0})$. Here $\{\nu_\Omega$ is the unit normal vector field along the boundary of Ω.
3. $\{S_m(k_1, \ldots, k_{n-1}) H^{n-1} L \partial\Omega\}$ is also compact in L^1, for $m = 1, \ldots, n-1$. Here $k_i' s$ are principle curvatures.

Theorem (L-Poon, 96)

There are minimizers among convex Ω's for both Problem B and Problem C. The only solution to Problem C (up to Euclidean motion) is $(B_R, \frac{x}{|x|})$, $|B_R| = V_0$.

Note: when $\lambda = 0$, no anchoring condition for $n\big|_{\partial\Omega}$. The minimizing solution is given by: Ω is a ball; and n is a constant.

Remark

1. For a convex domain Ω in $\mathbb{R}^3$, and let $n \in H^1(\Omega, S^2)$, one has
$$\int_{\Omega} |\nabla n|^2 \, dx \geq \int_{\partial\Omega} [(divn)n \cdot \nu - (\nabla n)n \cdot \nu] \, dH^2$$
Here ν is the unit normal along $\partial\Omega$.
2. If in addition $n = \nu$ on $\partial\Omega$, then
$$\int_{\Omega} |\nabla n|^2 \, dx \geq 2 \int_{\partial\Omega} H \, dH^2$$
3. $\min\{|\partial\Omega| + \beta \int_{\partial\Omega} H \, dH^{n-1} : \; |\Omega| = V_0$ and convex in $\mathbb{R}^n$ in achieved a ball.

Open Question

What regularity of Ω and n can one obtain for a minimizing pair (Ω, n) of the problem B? In particular, could an "olive-core" shaped Ω be a minimizer for the problem B? Could an "apple" shaped Ω be a minimizer for problem B?

Remark

In both problem B and C, and Ω convex (or C^2 or ...), one can add terms (to the surface energy) like $L_1 \int_{\partial\Omega} (\mathrm{div}\, n)(n \cdot \nu)\, dA$ and $L_2 \int_{\partial\Omega} \langle (\nabla n) n, \nu \rangle \, dA$.
Recently C.Y.Wang et al extended the theory by allowing non-convex droplets, but the curvature is bounded below by a negative constant.

2D elongated droplets with two cusps

This is a recent joint work with Zhiyuan Geng.
We are particularly interested in the elongated droplets known as **tactoids**, which usually possess a characteristic eye shape. In 2D, its boundary consists of two smooth curves that meet at singular points and form two cusps.
We consider a 2D free boundary problem with **tangential anchoring boundary condition**, which is Problem B. We take $w(n, \nabla n) = |\nabla n|^2$. And we assume the droplet domain Ω is **symmetric with respect to x-axis**.

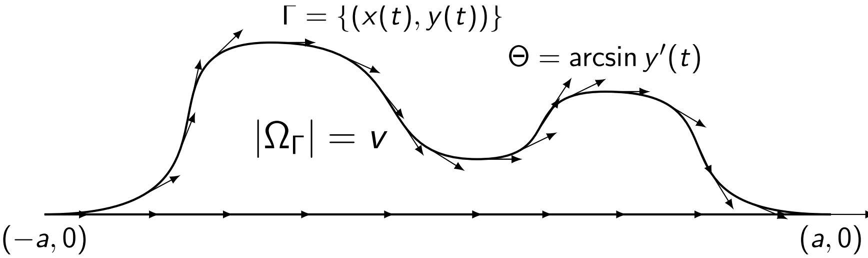

Figure: Curve Γ, domain Ω_Γ and Θ

Let $\Gamma : [0, 1] \to \mathbb{R}^2_+$ be a rectifiable Jordan curve, and Ω_Γ is the region enclosed by Γ and x-axis. Γ satisfies the following:

(i) $\Gamma = \{(x(t), y(t)) : x(t),\ y(t) \in AC([0, l(\Gamma)])\}$, where $l(\Gamma) =$ length(Γ).

(ii) $(x(0), y(0)) = (0, -a)$, $(x(l), y(l)) = (0, a)$ for some $a > 0$.

(iii) $\sqrt{|x'(t)|^2 + |y'(t)|^2} = 1, \quad x'(t) \geq 0$ almost everywhere.

Let $n = (n_1, n_2) = (\cos\Theta, \sin\Theta)$, then the **tangential anchoring boundary condition** for the angle function Θ is

$$\Theta(x,y) = \begin{cases} 0 & \text{on } \{(x,0) : x \in [-a,a]\}, \\ \arcsin y'(t) & \text{on } x(t), y(t)) \in \Gamma. \end{cases} \tag{1}$$

Define the **admissible set** for Γ:

$$\mathcal{G}_v := \{\Gamma \text{ satisfies condition (i–iii), and } \Theta\big|_{\partial\Omega_\Gamma} \text{ has a harmonic extension } \Theta$$

$$\text{defined in } \overline{\Omega_\Gamma} \text{ such that } \int_{\Omega_\Gamma} |\nabla\Theta|^2\, dxdy < \infty \text{ and } |\Omega_\Gamma| = v\}$$

One can easily check that the admissible set $\mathcal{G}_v$ is non-empty.

Variational problem

Consider the following variational problem

Problem D

Find $\Gamma \in \mathcal{G}_v$ that minimizes the following functional

$$E(\Gamma) = \int_{\Omega_\Gamma} |\nabla\Theta|^2 \, dxdy + l(\Gamma), \tag{2}$$

where Θ is determined by Γ in the following way

$$\begin{cases} \Delta\Theta = 0, & \text{in } \Omega_\Gamma, \\ \Theta\big|_{\partial\Omega_\Gamma} & \text{is defined as in } (\textbf{??}). \end{cases}$$

Goal: Study the existence and properties of energy the minimizers of Problem D.

An observation

Assume $v = 1$, and Problem D admits a global minimizer $(\Gamma_m, \Omega_m, \Theta_m)$.

Observation

Then the minimizing curve Γ_m will not touch x-axis besides two end points, which means Ω is simply connected.

Prove by contradiction: Assume $y(t_0) = 0$ for some $t_0 \in (0, l(\Gamma_m))$. Then the point $(x(t_0), y(t_0))$ naturally cuts Γ_m and Ω_m into two parts, denoted by (Γ_1, Ω_1) and (Γ_2, Ω_2). By rescaling each curve $\tilde{\Gamma}_i = \frac{1}{\sqrt{|\Omega_i|}}\Gamma_i$ for $i = 1, 2$, we get two energy competitors for Problem D. By comparing energies and minimizing property of Γ_m, Ω_m, we get a contradiction.

Three point property

We don't require Γ to be a minimizer in the following several lemmas.

Lemma

If $\Gamma \in \mathcal{G}_1$ *and* $E(\Gamma) \leq M$, *then there exists a a constant* $C = C(M)$ *such that for any three points* $z_1 = (x(t_1), y(t_1))$, $z_2 = (x(t_2), y(t_2))$ *and* $z_3 = (x(t_3), y(t_3))$ *on* Γ *such that* $t_1 < t_2 < t_3$, *it holds that*

$$\max\{\mathrm{dist}(z_1, z_2), \mathrm{dist}(z_2, z_3)\} \leq C\mathrm{dist}(z_1, z_3). \tag{3}$$

Ω_Γ is a Sobolev extension domain

A consequence of the lemma is that we can **extend Θ to a H^1 function defined on a larger domain that contains Ω_Γ**. According to Ahlfors (1963), the Jordan curve Γ is a quasicircle if and only if it satisfies the "three-points condition". And in 2D a quasidisk is equivalent to a Sobolev extension domain, see ([P.W. Jones, 1981]). Therefore we have

Lemma (Sobolev extension domain)

Assume $\Gamma \in \mathcal{G}_1$ and $E(\Gamma) \le M$, then there exists a constant C_2 that depends on M such that $\Theta|_{\overline{\Omega}_\Gamma}$ can be extend to the whole plane with the norm control

$$\|\Theta\|_{H^1(\mathbb{R}^2)} \le C_2 \|\Theta\|_{H^1(\Omega_\Gamma)}.$$

Γ is a chord-arc curve

Lemma

Let $\Gamma \in \mathcal{G}_1$ and $E(\Gamma) \leq M$. There exists a constant $C_3(M)$ such that for any two points $z_1, z_2 \in \Gamma$ and the arc $\gamma := \Gamma_{z_1 z_2}$, we have

$$l(\gamma) < C_3 |z_1 - z_2|.$$

In other words, Γ is a chord-arc curve.

The proof relies on a technical argument to show that if the arc length are way too long compared with the chord length, then the Dirichlet energy "generated" by this part of boundary will be huge. Co-area formula are used repeatedly in the proof.

Γ is a vanishing chord-arc curve

Lemma

Let $\Gamma \in \mathcal{G}_1$ *and* $E(\Gamma) \leq M$. *For any* $\varepsilon > 0$, *there exists a* $r = r(\varepsilon, \Gamma)$ *such that for any two points* $z_1, z_2 \in \Gamma$ *that satisfy* $|z_1 - z_2| \leq r$, *we have*

$$l(\gamma) \leq (1 + \varepsilon)|z_1 - z_2|, \quad \text{for } \gamma = \Gamma_{z_1 z_2}$$

Remark

The results of Kenig & Toro (1997) implies that the normal vector $\nu \in \mathrm{VMO}$, i.e.

$$\lim_{r \to 0} \left(\frac{1}{l(B(x,r) \cap \Gamma)} \int_{B(x,r)\cap\Gamma} |\nu - \nu_{B(x,r)}| \, dl \right) = 0 \;\; x \in \Gamma,$$

where

$$\nu_{B(x,r)} = \frac{1}{l(B(x,r) \cap \Gamma)} \int_{B(x,r)\cap\Gamma} \nu \, dl$$

Weil-Petersson curve

Consider a conformal mapping $f : \mathbb{D} \to \Omega$, where Ω is a quasidisk bounded by Γ. Γ is called a **Weil-Petersson curve** if and only if $(\log f')' \in L^2(\mathbb{D})$.

In a recent work [Bishop, 2019], Christopher Bishop gives 26 equivalent characterizations of the Weil-Petersson class. In particular, he shows that a curve Γ is Weil-Petersson if and only if it **has arclength parameterization in $H^{3/2}(\mathbb{T})$, has finite Möbius energy or can be well approximated by polygons** in some precise sense. Another equivalent characterization is that Weil-Peterssson curve **has local curvature that is square integrable over all locations and scales**, where local curvatures are measured using various quantities such as Peter Jone's β-numbers, conformal welding and Menger curvature.

- **Similarity:** In our problem, the curves in $\mathcal{G}_v$ resemble a lot to the Weil-Petersson curves. Both are related to the interior L^2 extension of $\arg \nu \ldots$

- **Difference:** Γ is not a closed curve and the domain Ω_Γ is not a quasidisk.

Properties generalized from Weil-Petersson curve

1. The arc-length parameterization $z(t) : [0, l] \to \Gamma$ is in the Sobolev space $H^{3/2}([0, l])$.
2. $$\int_\Gamma \int_\Gamma \left(\frac{|\nu(z) - \nu(w)|}{|z - w|} \right)^2 |dz|\,|dw| < \infty.$$
3. Γ has finite Möbius energy, i.e.
$$M\ddot{o}b(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|z - w|^2} - \frac{1}{l(z, w)^2} \right) dz\,dw < \infty.$$
4. For each n, let $z_j^n = z\left(\frac{jl}{2^n}\right)$ for $j = 0, 1, \ldots, 2^n$. Let Γ_n be the curve that consists of all the line segments $z_j^n z_{j+1}^n$ for $j = 0, \ldots, 2^n - 1$.
$$\sum_{n=1}^{\infty} 2^n [l(\Gamma) - l(\Gamma_n)] < \infty.$$

Theorem

There exists a $\Gamma \in \mathcal{G}_1$ minimizing the functional $E(\Gamma)$. Γ and x-axis will form two cusps at their intersection points.

Proof: Let $\{\Gamma_i\}_{i=1}^{\infty}$ be a minimizing sequence in $\mathcal{G}_1$.

$$\lim_{i\to\infty} E(\Gamma_i) = M_0 := \inf_{\Gamma\in\mathcal{G}_1} E(\Gamma).$$

Let Ω_i, Θ_i denote the corresponding $\Omega_{\Gamma_i}, \Theta_{\Gamma_i}$. Let B_R be a large ball such that $\Omega_i \subset B_R$ for any i. Then Θ_i can be extended to a H^1 function on $B(0,R)$ such that $\|\Theta_i\|_{H^1(B(0,R))} \le C$ for some universal constant C.

Proof of the existence

There is a subsequence, still denoted by $\{(\Gamma_i, \Omega_i, \Theta_i)\}$ that converges in the following sense:

1. $\chi_{\Omega_i} \to \chi_\Omega$ weakly in $BV(B_R)$ and strongly in $L^1(B_R)$, for some Ω which is a set of finite perimeter in B_R with volume equal to 1.
2. $\Theta_i \to \Theta$ weakly in $H^1(B_R)$ and strongly in $L^2(B(0, 2M))$ for some $\Theta \in H^1(B_R)$.
3. $\Gamma_i \to \Gamma$ in Hausdorff distance for a chord-arc curve Γ. Γ coincides with $\partial\Omega$ in $\mathbb{R}^2_+$.
4. (Tangential anchoring condition is preserved.) $\nu \cdot (\cos\Theta, \sin\Theta) = 0$ a.e. on $\partial^*\Omega$.

Singularities near two ends

The following lemma indicates that Γ and $x-$axis form approximately cusps near two ends.

Lemma

Let $\Gamma \in \mathcal{G}_1$ satisfy $E(\Gamma) \leq M$. Γ intersects with $x-$axis at $z_1 = (-a, 0)$ and $z_2 = (a, 0)$. For any $k > 0$, there exists a constant r that depends on k and Γ such that

$$\text{If } z = (x, y) \in \Gamma \cap B(z_1, r), \text{ then } \frac{y}{x+a} \leq k,$$
$$\text{If } z = (x, y) \in \Gamma \cap B(z_2, r), \text{ then } \frac{y}{a-x} \leq k.$$

Open problems

Here we give the following open problems:

Problem

1. Is Γ a C^∞ curve, or at least C^1?

Problem

2. Can one write Γ as a curve of function $f(x)$, such that $|\frac{df}{dx}| \leq C$ for some constant $C < \infty$?

The difficulty in answering these questions is due to the strong non-local character of the tangential anchoring boundary condition.

Euler-Lagrange equation

Let $\Gamma = \{(x, f(x)) : x \in [-a, a]\}$ such that

$$f(-a) = f(a) = 0, \quad f(x) > 0 \text{ for } x \in (-a, a), \quad \text{and} \int_{-a}^{a} f(x)\,dx = 1.$$

Direct calculation gives

$$\lambda = -\frac{d}{dx}\left(\frac{f'}{\sqrt{1+|f'|^2}}\right) - 2\frac{d}{dx}\left(\frac{\partial\Theta}{\partial\nu}\frac{1}{\sqrt{1+|f'(x)|^2}}\right) - \frac{\partial\Theta}{\partial\nu}\frac{\partial\Theta}{\partial y}\sqrt{1+|f'|^2} + |\nabla\Theta|^2$$

where λ is the Lagrange multiplier and the derivative of Θ is taking value at $(x, f(x))$.

Goal: study the behavior of the minimizer as the volume v tends to be extremely large or small.
Look at this problem from a scaling point of view. The curve length term is of dimension one while the Dirichlet energy term is of dimension zero. Therefore, when the volume is very large, the first term will be the dominating term and the minimizer is expected to be close to a semicircle (minimizes length of graph under fixed volume constraint). On the other hand, when the volume is very small, the domain is energy preferable to be very thin to avoid large elastic energy.

Equivalent problem by scaling

Problem D

Find $\Gamma \in \mathcal{G}_v$ that minimizes the following functional

$$E(\Gamma) = \int_{\Omega_\Gamma} |\nabla \Theta|^2 \, dxdy + l(\Gamma).$$

$$\Longleftrightarrow$$

Problem D'

Find Γ such that goes from $(-1, 0)$ to $(1, 0)$ and minimizes the following functional

$$E_v(\Gamma) = \frac{1}{\sqrt{v}} \int_{\Omega_\Gamma} |\nabla \Theta|^2 \, dxdy + \frac{l(\Gamma)}{\sqrt{|\Omega_\Gamma|}}.$$

We denote by Γ_v the minimizer of functional $E_v(\Gamma)$.

Large volume limit

As $v \to +\infty$, one expects that Γ_v will converge in some proper sense to $\Gamma^* := \{(x, \sqrt{1-x^2}) : x \in [-1,1]\}$.

$$\lim_{v\to\infty} E_v(\Gamma_v) = \sqrt{2\pi} = \frac{l(\Gamma^*)}{\sqrt{|\Omega_{\Gamma^*}|}}.$$

$$\lim_{v\to\infty} |\Omega_{\Gamma_v} \Delta \Omega_{\Gamma^*}| = 0, \quad \lim_{v\to\infty} \mathrm{d}_{\mathcal{H}}(\Gamma_v, \Gamma^*) = 0.$$

Small volume limit

Now assume $v << 1$.
An easy observation: $\int_{\Omega_{\Gamma_v}} |\nabla\Theta|^2 \, dxdy \sim O(v)$.
If we don't fix two endpoints of Γ, then by a simple scaling analysis we have that the energy-minimizing droplet with volume v will be a elongated drop with length of the order $v^{\frac{1}{3}}$ and the total energy is of order $v^{\frac{1}{3}}$.
If we write $\Gamma_v = \{(x, f(x)) : x \in [-1, 1], \ f = v^{\frac{1}{3}} g\}$ for some C^2 function g. By a Γ-convergence argument, we can get the approximated profile for g is a graph of $1 + \cos(x)$.

Variational equation:

$$f^t(x) = f(x) + t\xi(x), f_x^t(x) = f_x + t\xi_x, \theta^t(x) = \tan^{-1}(f_x + t\xi_x)$$

$$\int (\frac{f_x}{\sqrt{1+f_x^2}}\xi_x + \frac{u_\nu}{1+f_x^2}\xi_x)\, dx = 0$$

$$\frac{\partial}{\partial_x}(\frac{f_x}{\sqrt{1+f_x^2}}) + \Big(\frac{1}{1+f_x^2} \wedge (f, f_x)\Big)_x = \text{const}$$

$$\wedge(f, f_x) = \wedge_{\Omega_f}(\theta) \quad \text{(Dirichlet to Neumann map)}$$

Thin Domain Limit

Set $f(x) = \epsilon g(x)$, $0 < \epsilon << 1$. Here $g > 0$ on $(-a, a)$ is C^1 and $g(a) = g(-a) = g'(a) = g'(-a) = 0$. Note that volume $\Omega_f = 2\epsilon \int_{-a}^{a} g\, dx = 2\epsilon$.

$$\beta \int_{-a}^{a} \sqrt{1 + f_x^2}\, dx \simeq \beta \frac{\epsilon^2}{2} \int_{-a}^{a} g_x^2\, dx$$

Let $v(x, y) = u(x, \epsilon y)/\epsilon$, then $v(x, y) \doteq g_x$, for $y = g(x)$.
$\int\limits_{\Omega_f} |\nabla u|^2\, dydx \doteq \epsilon^3 \int_{-a}^{a} \int_{-g}^{g} \left(v_x^2 + \frac{v_y^2}{\epsilon^2}\, dydx\right.$.

$$\frac{\partial}{\partial x} \wedge_g (g_x) + \epsilon\beta g_{xx} + \epsilon^2 [O(g_{xx})^2 + O(g_x)^2) + \text{Nonlinear }] = \text{const.}$$

Thank you!