Uniqueness of weak solutions to the 2D Ericksen-Leslie system

Jinkai Li

South China Normal University

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with Z. Xin and E. S. Titi

Outline

Introduction to the system

The Ericksen-Leslie system

The Ericksen-Leslie system, formulated by Ericksen and Leslie in the 1960s, is one of the most successful models for the nematic liquid crystals, which in 3D reads as

$$\partial_t \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \nabla \mathrm{p} = \operatorname{div}(\sigma^{\mathrm{E}} + \sigma^{\mathrm{L}}),$$
$$\operatorname{div}\mathrm{u} = 0, \quad |\mathrm{d}| = 1,$$
$$(\lambda_1 \mathrm{N} + \lambda_2 \mathrm{A}\cdot\mathrm{d} + \mathrm{h}) \times \mathrm{d} = 0.$$

- Unknowns: u (velocity), p (pressure), $\mathrm{d} \in \mathrm{S}^2$ (director)
- Constants: $\lambda_1 < 0, \lambda_2$
- Notations: σ^{E} Ericksen tensor, σ^{L} Leslie tensor, h molecular field, N co-rotational derivative,

$$\mathrm{N} = \partial_t \mathrm{d} + (\mathrm{u}\cdot\nabla)\mathrm{d} - \Omega\cdot\mathrm{d}$$
$$\mathrm{A} = \tfrac{1}{2}(\nabla\mathrm{u} + (\nabla\mathrm{u})^{\mathrm{T}}), \quad \Omega = \tfrac{1}{2}(\nabla\mathrm{u} - (\nabla\mathrm{u})^{\mathrm{T}})$$

Constitutive equations

- Molecular field h:

$$\mathrm{h} = \mathsf{div}\left(\frac{\partial \mathrm{W}(\mathrm{d}, \nabla \mathrm{d})}{\partial(\nabla \mathrm{d})}\right) - \frac{\partial \mathrm{W}(\mathrm{d}, \nabla \mathrm{d})}{\partial \mathrm{d}}.$$

- Oseen-Frank density $\mathrm{W}(\mathrm{d}, \nabla \mathrm{d})$:

$$\begin{aligned} \mathrm{W}(\mathrm{d}, \nabla \mathrm{d}) = & \mathrm{k}_1(\mathsf{div}\,\mathrm{d})^2 + \mathrm{k}_2(\mathrm{d} \cdot \mathsf{curl}\,\mathrm{d})^2 + \mathrm{k}_3|\mathrm{d} \times \mathsf{curl}\,\mathrm{d}|^2 \\ & + (\mathrm{k}_2 + \mathrm{k}_4)[\mathrm{tr}(\nabla \mathrm{d})^2 - (\mathsf{div}\mathrm{d})^2] \end{aligned}$$

  Remark: $\mathrm{tr}(\nabla \mathrm{d})^2 - (\mathsf{div}\mathrm{d})^2$ does not contribute to h

- Frank elastic constants k_1 (splay), k_2 (twist), k_3 (bend); $\mathrm{k}_2 + \mathrm{k}_4$ (saddle-splay constant).

Constitutive equations (continue)

- The Ericksen tensor σ^{E} and Leslie tensor σ^{L}:

$$\sigma^{\mathrm{E}} = -(\nabla \mathrm{d})^{\intercal} \frac{\partial \mathrm{W}(\mathrm{d}, \nabla \mathrm{d})}{\partial(\nabla \mathrm{d})}$$

$$\sigma^{\mathrm{L}} = (\mu_1 \mathrm{d} \otimes \mathrm{d} : \mathrm{A}) \mathrm{d} \otimes \mathrm{d} + \mu_2 \boxed{\mathrm{N} \otimes \mathrm{d}} + \mu_3 \boxed{\mathrm{d} \otimes \mathrm{N}} + \mu_4 \mathrm{A} + \mu_5 (\mathrm{A} \cdot \mathrm{d}) \otimes \mathrm{d} + \mu_6 \mathrm{d} \otimes (\mathrm{A} \cdot \mathrm{d})$$

- The Leslie coefficients $\mu_{\mathrm{i}}, \mathrm{i} = 1, \cdots, 6$, satisfy

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6,$$

$$\mu_2 + \mu_3 = \mu_6 - \mu_5 \quad \text{(Parodi's relation)}.$$

Remark: these are crucial for cancellation structure

Some properties of Ericksen-Leslie system

- The director equations can be rewritten as

$$\partial_t d + (u\cdot\nabla)d + \Big(\frac{\lambda_2}{\lambda_1}A - \Omega\Big)\cdot d = -\frac{1}{\lambda_1}(h - (d\cdot h)d) + \frac{\lambda_2}{\lambda_1}(d\cdot A\cdot d)d$$

- $|d| = 1$ is preserved by the system: $n = |d|^2 - 1$ satisfies

$$\partial_t n + u\cdot\nabla n - \frac{2}{\lambda_1}(\lambda_2 d\cdot A\cdot d + h\cdot d)n = 0.$$

- Nonstandard parabolic of the director equation

$$\partial_t d + \frac{1}{\lambda_1}(I - d\otimes d)\text{div}\Big(\frac{\partial W(d,\nabla d)}{\partial(\nabla d)}\Big) = \text{ other terms}$$

- High order (as high as the leading terms) coupling terms:

$\mu_2 N\otimes d$ and $\mu_3 d\otimes N$ in the momentum equations

$A\cdot d$ and $\Omega\cdot d$ in the director equations.

Some simplifications

Some simplifications are made by adopting one or several of the following assumptions:

(i) Ginzburg-Landau approximation (introduced by Lin-Liu): replacing $\mathrm{h} - (\mathrm{d} \cdot \mathrm{h})\mathrm{d}$ by $\mathrm{h} + \frac{1-|\mathrm{d}|^2}{\varepsilon^2}\mathrm{d}$ and dropping $|\mathrm{d}| = 1$ (such constraint is not preserved by the resulting Ginzburg-Landau approximate system);

(ii) With simplified Oseen-Frank density, say, elastically isotropic case, i.e. $\mathrm{k}_1 = \mathrm{k}_2 = \mathrm{k}_3$;

(iii) With simplified σ^{L}, say, $\sigma^{\mathrm{L}} = \frac{\mu}{2}(\nabla \mathrm{u} + (\nabla \mathrm{u})^{\intercal})$;

(iv) Ignoring the stretching terms involving $\mathrm{A} \cdot \mathrm{d}$ and $\Omega \cdot \mathrm{d}$ to weaken the high order coupling.

Existence of weak solutions

- Lin–Lin–Wang (ARMA 2010, 2D, bdd domains, (ii)–(iv));
- Hong (CVPDE 2011, 2D, whole space, (ii)–(iv)), using Ginzburg-Landau approximation;
- Hong–Xin (Adv. Math 2012, 2D, whole space, (iii)–(iv)), using Ginzburg-Landau approximation;
- Huang–Lin–Wang (CMP 2014, 2D, whole space, (ii));
- Wang–Wang (CVPDE 2014, 2D, whole space);
- Li–Xin (2013 (unpublished), 2D, bdd domains, (iii)–(iv));
- Lin–Wang (CPAM 2015, 3D, $\mathrm{d}_0^3 \geq 0$, (ii)–(iv)).

Uniqueness of weak solutions

- Lin–Wang (Chin. Ann. Math 2010, 2D, (ii)–(iv)):

$$\begin{aligned} \partial_t \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} - \Delta \mathrm{u} &= -\mathrm{div}(\nabla \mathrm{d} \odot \nabla \mathrm{d}), \\ \mathrm{div}\, \mathrm{u} &= 0, \\ \partial_t \mathrm{d} + (\mathrm{u}\cdot\nabla)\mathrm{d} &= \Delta \mathrm{d} + |\nabla \mathrm{d}|^2 \mathrm{d}, \\ |\mathrm{d}| &= 1, \end{aligned}$$

by using the semigroup approach.

Question : uniqueness of the general case?

Main result: uniqueness of weak solutions

On the notations for 2D Ericksen-Leslie system

Consider the 2D Ericksen-Leslie system. In this case, $\mathrm{A} \cdot \mathrm{d}$ and $\Omega \cdot \mathrm{d}$ are understood as

$$\mathrm{A} \cdot \mathrm{d} = \begin{pmatrix} \mathrm{A} & 0 \\ 0 & 0 \end{pmatrix} \mathrm{d} = \begin{pmatrix} \mathrm{A}\hat{\mathrm{d}} \\ 0 \end{pmatrix}, \quad \Omega \cdot \mathrm{d} = \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} \mathrm{d} = \begin{pmatrix} \Omega\hat{\mathrm{d}} \\ 0 \end{pmatrix}$$

with $\hat{\mathrm{d}} = (\mathrm{d}^1, \mathrm{d}^2)$, the first two components of d, and consequently

$$\mathrm{d} \cdot \mathrm{A} \cdot \mathrm{d} = \hat{\mathrm{d}} \cdot \mathrm{A} \cdot \hat{\mathrm{d}}, \quad \mathrm{d} \cdot \Omega \cdot \mathrm{d} = \hat{\mathrm{d}} \cdot \Omega \cdot \hat{\mathrm{d}}.$$

Remark

Such understanding is natural when supposing that the Ericksen-Leslie system (in 3D) depends only on the first two spatial variables and the flow motion is in the plane (but the director d still takes values in $3\mathrm{D}$).

The director equation in 2D

With the above notations, the director equation can be rewritten in the component form as

$$\begin{aligned}
\partial_t \hat{d} + (u \cdot \nabla)\hat{d} &+ \left(\frac{\lambda_2}{\lambda_1} A - \Omega\right) \cdot \hat{d} \\
&= -\frac{1}{\lambda_1}(\hat{h} - (d \cdot h)\hat{d}) + \frac{\lambda_2}{\lambda_1}(\hat{d} \cdot A \cdot \hat{d})\hat{d}, \\
\partial_t d^3 + u \cdot \nabla d^3 &= -\frac{1}{\lambda_1}(h^3 - (d \cdot h)d^3) + \frac{\lambda_2}{\lambda_1}(\hat{d} \cdot A \cdot \hat{d})d^3,
\end{aligned}$$

where $\hat{h}$ is the first two components of h, i.e. $\hat{h} = (h^1, h^2)$, and all terms in the above are now understood in the usual way.

Therefore, the Ericksen-Leslie system in 2D reads as

$$\begin{aligned}
\partial_t u + (u \cdot \nabla) u &= \nabla p + \operatorname{div}(\sigma^E + \sigma^L), \\
\operatorname{div} u &= 0, \\
\partial_t \hat{d} + (u \cdot \nabla)\hat{d} + \left(\frac{\lambda_2}{\lambda_1} A - \Omega \right) \cdot \hat{d} & \\
&= -\frac{1}{\lambda_1}(\hat{h} - (d \cdot h)\hat{d}) + \frac{\lambda_2}{\lambda_1}(\hat{d} \cdot A \cdot \hat{d})\hat{d}, \\
\partial_t d^3 + u \cdot \nabla d^3 &= -\frac{1}{\lambda_1}(h^3 - (d \cdot h) d^3) + \frac{\lambda_2}{\lambda_1}(\hat{d} \cdot A \cdot \hat{d}) d^3, \\
|d| &= 1,
\end{aligned}$$

where $\hat{d} = (d^1, d^2)$ and $\hat{h} = (h^1, h^2)$.

The initial condition

Consider the Cauchy problem to the Ericksen-Leslie system in 2D. The initial condition reads as

$$(u, d)|_{t=0} = (u_0, d_0), \tag{1}$$

with

$$u_0 \in H, \quad d_0 \in H^1_b,$$

where

$$\mathcal{D}(\mathbb{R}^2) = \{\varphi \in C_0^\infty(\mathbb{R}^2) \,|\, \operatorname{div} \varphi = 0\},$$
$$H = \text{the closure of } \mathcal{D}(\mathbb{R}^2) \text{ in } L^2(\mathbb{R}^2),$$
$$H^k_b = \{d \,|\, d - b \in H^k(\mathbb{R}^2), |d| = 1\},$$

where b is a given unit constant vector.

Weak solutions

Definition: weak solutions

Given a positive time $T \in (0, \infty)$, and the initial data (u_0, d_0), with $u_0 \in H$ and $d_0 \in H^1_b$. A pair (u, d) is called a weak solution to the Ericksen-Leslie system in $\mathbb{R}^2 \times (0, T)$, subject to the initial condition (1), if the following statements hold

(i) (u, d) has the regularities that

$$u \in C([0, T]; H) \cap L^2(0, T; H^1),$$
$$d \in C([0, T]; H^1_b) \cap L^2(0, T; H^2_b),$$

(ii) the Ericksen-Leslie system is satisfied in the sense of distribution,

(iii) (u, d) satisfies the initial condition (1).

Main result

Theorem (JL-Titi-Xin 2016, Uniqueness of weak solutions)

Let T be a positive time. Set $\mathrm{a} = \min\{\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3\}$, $\delta = \max\{\mathrm{k}_1 - \mathrm{a}, \mathrm{k}_2 - \mathrm{a}, \mathrm{k}_3 - \mathrm{a}\}$. Assume that the Leslie coefficients $\mu_\mathrm{i}, \mathrm{i} = 1, \cdots, 6$, satisfy, in addition that

$$\mu_1 - \frac{\lambda_2^2}{\lambda_1} \geq 0, \quad \mu_4 > 0, \quad \mu_5 + \mu_6 \geq -\frac{\lambda_2^2}{\lambda_1}. \tag{2}$$

Then, for any $(\mathrm{u}_0, \mathrm{d}_0) \in \mathrm{H} \times \mathrm{H}^1_\mathrm{b}$, the Ericksen-Leslie system on $\mathbb{R}^2 \times (0, \mathrm{T})$, subject to (1), has at most one weak solution, if

$$\delta \leq \delta_0 := \min\left\{1, \frac{|\lambda_1|}{|\lambda_1| + |\lambda_2|}\sqrt{\frac{\mu_4}{-2\lambda_1}}\right\}\frac{\mathrm{a}}{\mathrm{C}_0},$$

where C_0 is an absolute positive constant.

Some remarks

Remark

The conclusion still holds if we replace (2) by the following slightly weaker assumption

$$\min\left\{\mu_4, \mu_1+\mu_4+\mu_5+\mu_6, \mu_4+\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right\} > 0.$$

This generalization relies on the fact that

$$\mathcal{Q}(\mathrm{d}, \mathrm{A}) \geq \mu|\mathrm{A}|^2,$$

where

$$\mathcal{Q}(\mathrm{d}, \mathrm{A}) = \left(\mu_1 - \frac{\lambda_2^2}{\lambda_1}\right)(\mathrm{d}\cdot\mathrm{A}\cdot\mathrm{d})^2 + \mu_4|\mathrm{A}|^2 + \left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)(\mathrm{A}\cdot\mathrm{d})^2.$$

Remark

Uniqueness of weak solutions to the general Ericksen-Leslie system was also obtained by Wang–Wang–Zhang (DCDS 2016), by using the Littlewood-Paley decomposition method. We note that our method is completely different from theirs, moreover, our method can be applied to the case of bounded domains.

Remark

Weak solutions established in the existing articles only have the regularity that $d \in L^2_{loc}(\cup_{i=0}^{N}[T_i, T_{i+1}); H^2_b)$, for some $0 =: T_0 < T_1 < \cdots < T_N < T_{N+1} := \infty$ and some integer N. However, by our result, uniqueness holds up to the first singular time T_1, then thanks to the continuity in time of weak solutions, uniqueness also holds at time T_1, next starting from time T_1, repeating this argument yields uniqueness on the time interval $[0, T_2]$, and finally to any time.

Analysis on the proof

A usual way to prove the uniqueness

A usual way for proving the uniqueness:

step 1 Consider the **Subtracted System**

Subtracted System = System for solution 1 − System for solution 2

step 2 Perform energy estimates for the subtracted System **at the level of the basic energy** of the system.

Remark

One may try the usual way stated above to prove the uniqueness of weak solutions to the 2D Ericksen-Leslie system; however,this **does not work because of the critical term like** $|\nabla \mathrm{d}|^2 \mathrm{d}$ (see the case of harmonic heat flow, below).

Harmonic heat flow

To clarify the ideas, we start from the extensively simplified case, the harmonic heat flow, as the example:

$$\partial_t d = \Delta d + |\nabla d|^2 d, \quad |d| = 1, \qquad \text{in } \mathbb{R}^2.$$

- **Basic energy identity for the harmonic heat flow**

$$\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx + \int |\Delta d + |\nabla d|^2 d|^2 dx = 0.$$

- **Regularities of weak solutions to the harmonic heat flow** (before the first singular time):

$$d \in L^\infty(0, T; H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \cap L^2(0, T; H^2(\mathbb{R}^2)).$$

First try for uniqueness of harmonic heat flow

Let d_1 and d_2 be two weak solutions to the harmonic heat flow, and set $\mathrm{d} = \mathrm{d}_1 - \mathrm{d}_2$. Then, trying the usual way stated above:

(i) The subtracted system for d is

$$\partial_t \mathrm{d} = \Delta \mathrm{d} + |\nabla \mathrm{d}_1|^2 \mathrm{d} + \nabla \mathrm{d} : \nabla(\mathrm{d}_1 + \mathrm{d}_2)\mathrm{d}_2.$$

(ii) Performing energy estimates at **basic energy level**:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int |\nabla \mathrm{d}|^2 + \int |\Delta \mathrm{d}|^2 = -\int |\nabla \mathrm{d}_1|^2 \mathrm{d} \cdot \Delta \mathrm{d} + \cdots .$$

Deal with critical term $\int |\nabla d_1|^2 d \cdot \Delta d$

Critical term $\int |\nabla d_1|^2 d \cdot \Delta d dx$:

$$
\begin{aligned}
-\int |\nabla d_1|^2 d \cdot \Delta d dx &\leq \|\nabla d_1\|_{L^q}^2 \|d\|_{L^{\frac{2q}{q-4}}} \|\Delta d\|_{L^2} \\
&\leq \frac{1}{2}\|\Delta d\|_{L^2}^2 + \frac{1}{2}\|\nabla d_1\|_{L^q}^4 \|d\|_{L^{\frac{2q}{q-4}}}^2, \quad \forall q \geq 4
\end{aligned}
$$

$\Longrightarrow$

$$\text{need} \quad \nabla d_i \in L^4(0, T; L^q), \quad q \geq 4$$

Recall $d_i \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \Longrightarrow$

$$\nabla d_i \in L^4(0, T; L^4), \quad \nabla d_i \notin L^4(0, T; L^{4^+})$$

$\Longrightarrow$

The only choice $q = 4$ $\quad \rightsquigarrow \|d\|_{L^\infty}^2$

Fault at the basic energy level

We are forced to

$$\frac{d}{dt}\int |\nabla d|^2 + \int |\Delta d|^2 \le \|\nabla d_1\|_{L^4}^4 \|d\|_{L^\infty}^2 + \cdots$$

Note that

$$H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty, \quad \|\nabla d_1\|_{L^4}^4 \in L^1((0,T))$$

$\Longrightarrow$

Performing estimates at basic energy level is NOT suitable, due to the insufficient regularities for dealing with $|\nabla d|^2 d$

Second try for uniqueness of harmonic heat flow

$$\partial_t d = \Delta d + |\nabla d_1|^2 d + \nabla d : \nabla(d_1 + d_2) d_2.$$

Performing estimates at **level one order below the basic energy**:

$$\boxed{\frac{d}{dt}\|d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \leq C\|(\nabla d_1, \nabla d_2)\|_{L^4}^4 \|d\|_{L^2}^2},$$

from which, recalling that

$$\nabla d_i \in L^4(0, T; L^4(\mathbb{R}^2)),$$

by the Gronwall inequality, one obtains the uniqueness.

$$\Downarrow$$

Performing estimates at **one order lower than the basic energy** is suitable for proving the uniqueness of the harmonic heat flow

Simplified Ericksen-Leslie system in $\mathbb{R}^2$:

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p &= -\mathrm{div}(\nabla d \odot \nabla d), \\ \mathrm{div}\, u &= 0, \\ \partial_t d + (u\cdot\nabla)d &= \Delta d + |\nabla d|^2 d, \\ |d| &= 1, \end{aligned}$$

where $\nabla d \odot \nabla d = (\partial_i d \cdot \partial_j d)_{2\times 2}$

Remark:
$\int \left\{ \mathrm{Eq.u} \cdot u \right\} dx \Rightarrow$ uniqueness of 2D Navier-Stokes equations

Some facts for the simplified Ericksen-Leslie system

- **Regularities of weak solutions** (before first singular time)

$$u \in L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;H^1(\mathbb{R}^2)),$$
$$d \in L^\infty(0,T;H^1_b(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \cap L^2(0,T;H^2_b(\mathbb{R}^2)),$$

which, by the Ladyzhenskaya inequality, imply

$$(u, \nabla d) \in L^4(0,T;L^4(\mathbb{R}^2)).$$

- **Basic energy identity**

$$\frac{1}{2}\frac{d}{dt}\int (|u|^2 + |\nabla d|^2) + \int (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) = 0.$$

A possible try: mixed levels of energy

Subtracted system ($(u, d) = (u_1, d_1) - (u_2, d_2)$):

$$\partial_t u + \mathrm{div}(u_1 \otimes u + u \otimes u_2) + \nabla p - \Delta u = -\mathrm{div}(\nabla d_1 \odot \nabla d + \nabla d \odot \nabla d_2),$$
$$\mathrm{div}\, u = 0,$$
$$\partial_t d + u_1 \cdot \nabla d + u \cdot \nabla d_2 = \Delta d + |\nabla d_1|^2 d + \nabla d : \nabla(d_1 + d_2) d_2.$$

A possible try: $\int \left\{ (\mathrm{Eq.u} \cdot u) + (\mathrm{Eq.d} \cdot d) \right\} dx \Rightarrow$

$$\frac{1}{2}\frac{d}{dt}\int (|u|^2 + |d|^2) + \int (|\nabla u|^2 + |\nabla d|^2)$$
$$= -\int \mathrm{div}(\nabla d_1 \odot \nabla d) \cdot u + \cdots = \int \nabla d_1 \odot \nabla d : \nabla u + \cdots$$
$$\leq \|\nabla d_1\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla d\|_{L^2} + \cdots$$

Does not lead to the uniqueness!

Another try: one order below the basic energy

- Idea: perform the energy estimate to the subtracted system at the level of **one order lower than the basic energy**:

$$\begin{array}{rll} \text{basic energy:} & u & \nabla d \\ \text{one order below basic:} & (\nabla)^{-1}u & d \end{array}$$

- Introduce

$$\boxed{\xi_i = (I - \Delta)^{-1} u_i, \quad \xi = \xi_1 - \xi_2}$$

$\Longrightarrow$ div $\xi = 0$ and

$$\partial_t(\xi - \Delta\xi) + \text{div}(u_1 \otimes u + u \otimes u_2) + \nabla p - \Delta\xi + \Delta^2\xi$$
$$= -\text{div}(\nabla d_1 \odot \nabla d + \nabla d \odot \nabla d_2)$$

Energy estimates for the velocity

$$\partial_t(\xi - \Delta\xi) + \mathrm{div}(u_1 \otimes u + u \otimes u_2) + \nabla p - \Delta\xi + \Delta^2\xi$$
$$= -\mathrm{div}(\nabla d_1 \odot \nabla d + \nabla d \odot \nabla d_2)$$

$\int \Big\{ \mathrm{Eq}.\xi \cdot \xi \Big\} dx \Longrightarrow$

$$\frac{1}{2}\frac{d}{dt}\int(|\xi|^2 + |\nabla\xi|^2) + \int(|\nabla\xi|^2 + |\Delta\xi|^2)$$
$$= \int(\nabla d_1 \odot \nabla d + \nabla d \odot \nabla d_2 + u_1 \otimes u + u \otimes u_2) : \nabla\xi$$
$$\leq (\|(\nabla d_1, \nabla d_2)\|_{L^4}\|\nabla d\|_{L^2} + \|(u_1, u_2)\|_{L^4}\|(\xi, \Delta\xi)\|_{L^2})\|\nabla\xi\|_{L^4}$$
$$\leq C(\|\nabla d_1, \nabla d_2\|_{L^4}\|\nabla d\|_{L^2} + \|u_1, u_2\|_{L^4}\|\xi, \Delta\xi\|_{L^2})\|\nabla\xi\|_{L^2}^{\frac{1}{2}}\|\Delta\xi\|_{L^2}^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\|\nabla d, \nabla\xi, \Delta\xi\|_{L^2}^2 + C(\|\nabla d_1, \nabla d_2, u_1, u_2\|_{L^4}^4 + 1)\|\xi, \nabla\xi\|_{L^2}^2.$$

Energy estimate for the director

$$\partial_t d + u_1 \cdot \nabla d + u \cdot \nabla d_2 = \Delta d + |\nabla d_1|^2 d + \nabla d : \nabla(d_1 + d_2) d_2$$

$\int \left\{ \mathrm{Eq.}d \cdot d \right\} dx \Longrightarrow$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |d|^2 + \int |\nabla d|^2 \\
=&\int (|\nabla d_1|^2 |d|^2 + \nabla d : \nabla(d_1 + d_2) d_2 \cdot d - u \cdot \nabla d_2 \cdot d) \\
\leq& \|\nabla d_1\|_{L^4}^2 \|d\|_{L^4}^2 + \|\nabla d\|_{L^2} \|\nabla d_1, \nabla d_2\|_{L^4} \|d\|_{L^4} \\
&+ (\|\xi\|_{L^2} + \|\Delta \xi\|_{L^2}) \|\nabla d_2\|_{L^4} \|d\|_{L^4} \\
\leq& \frac{1}{4} \|\nabla d, \Delta \xi\|_{L^2}^2 + C(\|\nabla d_1, \nabla d_2\|_{L^4}^4 + 1) \|d, \xi\|_{L^2}^2.
\end{aligned}$$

Conclusion for the simplified Ericksen-Leslie system

$\Longrightarrow$

$$\frac{d}{dt}\|(\xi, \nabla\xi, d)\|_{L^2}^2 + \|(\nabla\xi, \Delta\xi, \nabla d)\|_{L^2}^2$$
$$\leq C(1 + \|(\nabla d_1, \nabla d_2, u_1, u_2)\|_{L^4}^4)\|(\xi, \nabla\xi, d)\|_{L^2}^2,$$

$\Longrightarrow$ uniqueness (recall that $(u_i, \nabla d_i) \in L^4(0, T; L^4)$).

$\Downarrow$

Performing estimates at **one order lower than basic energy** is suitable for proving uniqueness of simplified Ericksen-Leslie system

Proof of the main result

New difficulty and idea

We now consider the general Ericksen-Leslie system.

- **New difficulty**: high order coupling terms

$$\mathrm{div}(N \otimes d), \mathrm{div}(d \otimes N), A \cdot d, \Omega \cdot d.$$

Remark: these coupling terms are as high as the leading terms, from the point view of the level of the energy

$$\begin{aligned} u \sim \nabla d \quad &\Longrightarrow \quad \left(\frac{\lambda_2}{\lambda_1} A - \Omega\right) \cdot d \sim \nabla u \sim \nabla^2 d \\ N \sim h \sim \Delta d \sim \nabla u \quad &\Longrightarrow \quad \mathrm{div}(N \otimes d) \sim \Delta u \end{aligned}$$

- **Idea**: cancellation structure between these high order terms when performing the energy estimates.

Equivalent form of σ^{L}

$$\begin{aligned}\sigma^{\mathrm{L}} = &\left(\mu_1 - \frac{\lambda_2^2}{\lambda_1}\right)(\mathrm{d}\cdot\mathrm{A}\cdot\mathrm{d})\mathrm{d}\otimes\mathrm{d} + \mu_4\mathrm{A} \\ &+ \left(\mu_5 - \frac{\lambda_2}{\lambda_1}\mu_2\right)(\mathrm{A}\cdot\mathrm{d})\otimes\mathrm{d} + \left(\mu_6 - \frac{\lambda_2}{\lambda_1}\mu_3\right)\mathrm{d}\otimes(\mathrm{A}\cdot\mathrm{d}) \\ &\underbrace{-\frac{1}{\lambda_1}[\mu_2(\mathrm{h}-(\mathrm{d}\cdot\mathrm{h})\mathrm{d})\otimes\mathrm{d} + \mu_3\mathrm{d}\otimes(\mathrm{h}-(\mathrm{d}\cdot\mathrm{h})\mathrm{d})]}_{=:\sigma^{\mathrm{L,d}},\ \text{high order coupling term}} \\ =:&\sigma^{\mathrm{L,u}} + \sigma^{\mathrm{L,d}} \qquad (\sigma^{\mathrm{L,u}} \text{ is symmetric}, \sigma^{\mathrm{L,d}} \text{ is not})\end{aligned}$$

Recall $\lambda_1 = \mu_2 - \mu_3, \lambda_2 = \mu_5 - \mu_6$, and $\mu_2 + \mu_3 = \mu_6 - \mu_5$

$$\sigma^{\mathrm{L,u}} : \nabla\mathrm{u} = \mathcal{Q}(\mathrm{d},\mathrm{A}) \quad (\text{nonnegative quadratic form in } \mathrm{A})$$

$$\sigma^{\mathrm{L,d}} : \nabla\mathrm{u} = \underbrace{\left(\frac{\lambda_2}{\lambda_1}\mathrm{A}\cdot\mathrm{d} - \Omega\cdot\mathrm{d}\right)\cdot[\mathrm{h}-(\mathrm{d}\cdot\mathrm{h})\mathrm{d}]}_{\text{high order coupling term}}$$

Cancellation structure

$$\partial_t \frac{|u|^2}{2} + \mathrm{div}\left(u\left(\frac{|u|^2}{2} + p\right)\right) = \mathrm{div}(\sigma^E \cdot u + \sigma^L \cdot u)$$

$$- \underbrace{\sigma^{L,u} : \nabla u}_{=\mathcal{Q}(d,A)} - \boxed{\sigma^{L,d} : \nabla u} - \sigma^E : \nabla u$$

$$\boxed{\sigma^{L,d} : \nabla u} = \left(\frac{\lambda_2}{\lambda_1} A \cdot d - \Omega \cdot d\right) \cdot [h - (d \cdot h)d]$$

$$= \boxed{h \cdot \left(\frac{\lambda_2}{\lambda_1} A - \Omega\right) \cdot d - \frac{\lambda_2}{\lambda_1}(d \cdot A \cdot d)d \cdot h}$$

$$\partial_t W(d, \nabla d) + u \cdot \nabla W(d, \nabla d)$$

$$= \frac{1}{\lambda_1}[h - (h \cdot d)d] \cdot h + \boxed{h \cdot \left(\frac{\lambda_2}{\lambda_1} A - \Omega\right) \cdot d - \frac{\lambda_2}{\lambda_1}(d \cdot A \cdot d)d \cdot h}$$

$$+ \mathrm{div}\left(\frac{\partial W(d, \nabla d)}{\partial \nabla d} \cdot (\partial_t d + (u \cdot \nabla)d)\right) + \sigma^E : \nabla u.$$

Basic energy law

$$\partial_t \left(\frac{|u|^2}{2} + W(d, \nabla d) \right) + u \cdot \nabla \left(\frac{|u|^2}{2} + W(d, \nabla d) \right) + \mathcal{Q}(d, A) - \frac{|h \times d|^2}{\lambda_1}$$

$$= \mathrm{div}(\sigma^E \cdot u + \sigma^L \cdot u - up) + \mathrm{div} \left(\frac{\partial W(d, \nabla d)}{\partial \nabla d} \cdot (\partial_t d + (u \cdot \nabla) d) \right)$$

$$\Longrightarrow$$

$$\frac{d}{dt} \int \left(\frac{|u|^2}{2} + W(d, \nabla d) \right) dx + \int \left(\mathcal{Q}(d, A) - \frac{1}{\lambda_1} |h - (d \cdot h) d|^2 \right) dx = 0$$

Reformulation of the molecular field

Rewrite $W(d, \nabla d)$ as

$$W(d, \nabla d) = \underbrace{a|\nabla d|^2}_{\text{main part}} + \underbrace{V(d, \nabla d)}_{\text{perturbation}}$$

where $a = \min\{k_1, k_2, k_3\}$ and

$$V(d, \nabla d) = (k_1 - a)(\operatorname{div} d)^2 + (k_2 - a)(d \cdot \operatorname{curl} d)^2 + (k_3 - a)|d \times \operatorname{curl} d|^2.$$

Thus, one can rewrite the molecular field h as

$$h = \underbrace{2a\Delta d}_{\text{main part}} + \underbrace{H}_{\text{perturbation}}$$

Reformulation of the director equation

As a result, we have

$$\mathrm{h} - (\mathrm{d}\cdot\mathrm{h})\mathrm{d} = \underbrace{2\mathrm{a}(\Delta\mathrm{d} + |\nabla\mathrm{d}|^2\mathrm{d})}_{\text{main term}} + \underbrace{\mathrm{H} - (\mathrm{d}\cdot\mathrm{H})\mathrm{d}}_{\text{perturbation term}},$$

and the director equation can be rewritten as

$$\partial_t\mathrm{d} + (\mathrm{u}\cdot\nabla)\mathrm{d} + \left(\frac{\lambda_2}{\lambda_1}\mathrm{A} - \Omega\right)\cdot\mathrm{d} - \frac{\lambda_2}{\lambda_1}(\mathrm{d}\cdot\mathrm{A}\cdot\mathrm{d})\mathrm{d}$$
$$= \underbrace{-\frac{2\mathrm{a}}{\lambda_1}(\Delta\mathrm{d} + |\nabla\mathrm{d}|^2\mathrm{d})}_{\text{main term}} - \underbrace{\frac{1}{\lambda_1}(\mathrm{H} - (\mathrm{d}\cdot\mathrm{H})\mathrm{d})}_{\text{perturbation term}}$$

Reformulation of the Leslie stress tensor

Using the reformulation of the molecular field, the Leslie stress tensor σ^{L} can be rewritten as

$$\sigma^{\mathrm{L}} = \underbrace{\Sigma^{\mathrm{L}}}_{\text{main part}} + \underbrace{\Pi^{\mathrm{L}}}_{\text{perturbation}},$$

where Σ^{L} and Π^{L} are given by

$$\begin{aligned}\Sigma^{\mathrm{L}} = {} & \left(\mu_1 - \tfrac{\lambda_2^2}{\lambda_1}\right)(\hat{\mathrm{d}} \cdot \mathrm{A} \cdot \hat{\mathrm{d}})\hat{\mathrm{d}} \otimes \hat{\mathrm{d}} + \mu_4 \mathrm{A} \\ & + \left(\mu_5 - \tfrac{\lambda_2}{\lambda_1}\mu_2\right)(\mathrm{A} \cdot \hat{\mathrm{d}}) \otimes \hat{\mathrm{d}} + \left(\mu_6 - \tfrac{\lambda_2}{\lambda_1}\mu_3\right)\hat{\mathrm{d}} \otimes (\mathrm{A} \cdot \hat{\mathrm{d}}) \\ & \underbrace{- \tfrac{2\mathrm{a}}{\lambda_1}(\mu_2 \Delta\hat{\mathrm{d}} \otimes \hat{\mathrm{d}} + \mu_3 \hat{\mathrm{d}} \otimes \Delta\hat{\mathrm{d}}) - \frac{2\mathrm{a}\lambda_2}{\lambda_1}(\Delta\mathrm{d} \cdot \mathrm{d})\hat{\mathrm{d}} \otimes \hat{\mathrm{d}}}_{=:\Sigma^{\mathrm{L,d}},\ \text{to be cancelled with the stretching terms in } \mathrm{d} \text{ equation}} \\ =: {} & \Sigma^{\mathrm{L,u}} + \Sigma^{\mathrm{L,d}}, \\ \Pi^{\mathrm{L}} = {} & -\tfrac{1}{\lambda_1}[\mu_2 \hat{\mathrm{H}} \otimes \hat{\mathrm{d}} + \mu_3 \hat{\mathrm{d}} \otimes \hat{\mathrm{H}} + \lambda_2(\mathrm{d} \cdot \mathrm{H})\hat{\mathrm{d}} \otimes \hat{\mathrm{d}}].\end{aligned}$$

Some notations

Let (u_1, d_1) and (u_2, d_2) be two weak solutions. Set

$$(u, d) = (u_1, d_1) - (u_2, d_2).$$

Define

$$\xi_i = (I - \Delta)^{-1} u_i, \quad \xi = \xi_1 - \xi_2$$

Set

$$S = \frac{1}{2}(\nabla \xi + (\nabla \xi)^{\mathsf{T}}), \quad Q = \frac{1}{2}(\nabla \xi - (\nabla \xi)^{\mathsf{T}}).$$

Then, (recall that $A = \frac{1}{2}(\nabla u + (\nabla u)^T), \Omega = \frac{1}{2}(\nabla u - (\nabla u)^T)$)

$$A = (I - \Delta)S, \quad \Omega = (I - \Delta)Q$$

Momentum equation of the subtracted system

$$\partial_t u + \nabla p = \mathrm{div}(\Sigma_1^L - \Sigma_2^L + \Pi_1^L - \Pi_2^L) + \cdots .$$

$\Longrightarrow$

$$\frac{1}{2}\frac{d}{dt}\int (|\nabla\xi|^2 + |\xi|^2)dx + \underbrace{\int (\Sigma_1^L - \Sigma_2^L) : \nabla\xi dx}_{\text{the leading term}}$$

$$= \underbrace{-\int (\Pi_1^L - \Pi_2^L) : \nabla\xi dx}_{\text{perturbation term}} + \text{"other lower order terms"}.$$

Estimate on $\int(\Sigma_1^{\mathrm{L}} - \Sigma_2^{\mathrm{L}}) : \nabla\xi$

$$
\begin{aligned}
&\int(\Sigma_1^{\mathrm{L}} - \Sigma_2^{\mathrm{L}}) : \nabla\xi \mathrm{dx} \\
= \;& \int(\Sigma_1^{\mathrm{L,d}} - \Sigma_2^{\mathrm{L,d}}) : \nabla\xi + \int(\Sigma_1^{\mathrm{L,u}} - \Sigma_2^{\mathrm{L,u}}) : \nabla\xi \\
\geq \;& \underbrace{-\frac{2\mathrm{a}\lambda_2}{\lambda_1}\int(\mathrm{d}\cdot\mathrm{d}_1)\hat{\mathrm{d}}_1 \otimes \hat{\mathrm{d}}_1 : \Delta\mathrm{S} + 2\mathrm{a}\int\hat{\mathrm{d}}\cdot\left(\frac{\lambda_2}{\lambda_1}\Delta\mathrm{S} - \Delta\mathrm{Q}\right)\cdot\hat{\mathrm{d}}_1}_{\Delta\mathrm{S}\sim\Delta\mathrm{Q}\sim\nabla\mathrm{u}\sim\nabla^3\xi,\ \text{bad high order coupling terms}} \\
&+\frac{\mu_4}{4}\int(|\nabla\xi|^2 + |\nabla^2\xi|^2) + \text{“low order terms”},
\end{aligned}
$$

Estimate on $\int(\Pi_1^L - \Pi_2^L) : \nabla\xi$

We have the estimate

$$\begin{aligned} &\left|\int(\Pi_1^L - \Pi_2^L) : \nabla\xi \mathrm{d}x\right| \\ \leq\ & \varepsilon\int(|\nabla d|^2 + |\nabla^2\xi|^2)\mathrm{d}x + \text{“low order terms”} \\ & + C_1\delta\left(\left|\frac{\lambda_2}{\lambda_1}\right| + 1\right)\int|\nabla d||\nabla^2\xi|\mathrm{d}x, \end{aligned}$$

for any $\varepsilon > 0$, where $\delta = \max\{k_1 - a, k_2 - a, k_3 - a\}$, and C_1 is an absolute constant.

Energy estimate for ξ

By the aid of the previous estimates, we have the following energy estimate on ξ:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int(|\nabla\xi|^2+|\xi|^2)\mathrm{dx}+\frac{\mu_4}{4}\int(|\nabla\xi|^2+|\nabla^2\xi|^2)\mathrm{dx}$$

$$\underbrace{-\tfrac{2\mathrm{a}\lambda_2}{\lambda_1}\int(\mathrm{d}\cdot\mathrm{d}_1)\hat{\mathrm{d}}_1\otimes\hat{\mathrm{d}}_1:\Delta\mathrm{S}+2\mathrm{a}\int\hat{\mathrm{d}}\cdot\left(\tfrac{\lambda_2}{\lambda_1}\Delta\mathrm{S}-\Delta\mathrm{Q}\right)\cdot\hat{\mathrm{d}}_1}_{\text{bad high order term, }\Delta\mathrm{S}\sim\Delta\mathrm{Q}\sim\nabla\mathrm{u}\sim\nabla^3\xi}$$

$$\leq\mathrm{C}_1\delta\left(\left|\frac{\lambda_2}{\lambda_1}\right|+1\right)\int|\nabla\mathrm{d}||\nabla^2\xi|\mathrm{dx}+\varepsilon\int(|\nabla\mathrm{d}|^2+|\nabla^2\xi|^2)\mathrm{dx}$$

$$+\text{ “lower order terms”}.$$

Energy estimate for the director

Director equation of the subtracted system for (u, d) reads as

$$\partial_t \mathrm{d} + \frac{2\mathrm{a}}{\lambda_1}\Delta \mathrm{d} + \left(\frac{\lambda_2}{\lambda_1}\mathrm{A} - \Omega\right)\cdot \mathrm{d}_1 - \frac{\lambda_2}{\lambda_1}(\mathrm{A} : \hat{\mathrm{d}}_1 \otimes \hat{\mathrm{d}}_1)\mathrm{d}_1$$
$$= -\frac{1}{\lambda_1}[\mathrm{H}_1 - \mathrm{H}_2 - (\mathrm{d}_1 \cdot \mathrm{H}_1)\mathrm{d}_1 + (\mathrm{d}_2 \cdot \mathrm{H}_2)\mathrm{d}_2] + \cdots .$$

Performing standard energy estimate to the above leads to

$$\frac{\mathrm{d}}{\mathrm{dt}}\int \mathrm{a}|\mathrm{d}|^2 \mathrm{dx} - \frac{4\mathrm{a}^2}{\lambda_1}\int |\nabla \mathrm{d}|^2 \mathrm{dx}$$
$$\underbrace{-\tfrac{2\mathrm{a}\lambda_2}{\lambda_1}\int (\mathrm{d}\cdot \mathrm{d}_1)(\mathrm{A} : \hat{\mathrm{d}}_1 \otimes \hat{\mathrm{d}}_1)\mathrm{dx} + 2\mathrm{a}\int \hat{\mathrm{d}} \cdot \left(\tfrac{\lambda_2}{\lambda_1}\mathrm{A} - \Omega\right)\cdot \hat{\mathrm{d}}_1 \mathrm{dx}}_{\text{bad high order term (from stretching), } \mathrm{A},\Omega\sim\nabla \mathrm{u}\sim\nabla^2 \mathrm{d}\text{, to be cancelled}}$$
$$\leq -\frac{\mathrm{a}C_1\delta}{\lambda_1}\int |\nabla \mathrm{d}|^2 + \varepsilon \int (|\nabla \mathrm{d}|^2 + |\Delta \xi|^2) + \text{“low order terms”}.$$

Cancellation structure of the bad high order coupling terms

Recall the bad terms:

$$-\frac{2a\lambda_2}{\lambda_1}\underbrace{\int (d\cdot d_1)\hat{d}_1\otimes\hat{d}_1 : \Delta S}_{B_{11}} + 2a\underbrace{\int \hat{d}\cdot\left(\frac{\lambda_2}{\lambda_1}\Delta S - \Delta Q\right)\cdot\hat{d}_1}_{B_{12}}$$

$$-\frac{2a\lambda_2}{\lambda_1}\underbrace{\int (d\cdot d_1)(A : \hat{d}_1\otimes\hat{d}_1)}_{B_{21}} + 2a\underbrace{\int \hat{d}\cdot\left(\frac{\lambda_2}{\lambda_1}A - \Omega\right)\cdot\hat{d}_1}_{B_{22}}$$

Recalling $A = -\Delta S + S$ and $\Omega = -\Delta Q + Q$, we have

$$|B_{11} + B_{21}| = \left|\int (d\cdot d_1)(S : \hat{d}_1\otimes\hat{d}_1)dx\right| \leq C\int |d||\nabla\xi|dx,$$

and similarly $|B_{21} + B_{22}| \leq C\int |\nabla\xi||d|$

The full energy estimate

$$\frac{d}{dt}\int(|\xi|^2+|\nabla\xi|^2+2a|d|^2)+\int\left(\frac{\mu_4}{4}(|\nabla\xi|^2+|\nabla^2\xi|^2)-\frac{4a^2}{\lambda_1}|\nabla d|^2\right)$$
$$\leq C_0\delta\left(\left|\frac{\lambda_2}{\lambda_1}\right|+1\right)\int|\nabla d||\nabla^2\xi|-\frac{aC_0\delta}{\lambda_1}\int|\nabla d|^2+\text{“low order terms”},$$

where C_0 is an absolute positive constant.

THANK YOU!