On the role of the α_1-viscosity in the Ericksen-Leslie equations with Ginzburg-Landau penalisation and molecular stretching

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Joint work with H. Wu

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Analysis of Nematic Liquid Crystals Flows
Nonlinear Partial Differential Equations in Fluid Dynamics

CIRM, Marseille
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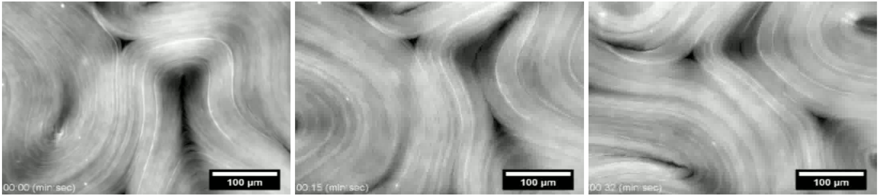
Outline

- The Ericksen-Leslie equations:
  - a quick overview of the model,
  - the Leslie coefficients and the values of α_1,
  - Ginzburg-Landau penalisation.
- Established results and contribution:
  - definition of weak solutions,
  - uniqueness,
  - non-triviality of the problem.
- Some details on the result:
  - the double-logarithmic inequality,
  - frequencies decomposition,
  - conclusion.

The Ericksen-Leslie model

The fluid behaviour



2D **Active** Nematic [1]

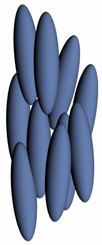
The **center of mass** of each constituent molecule has a **freely degree of translation**, as a common particle in a liquid.

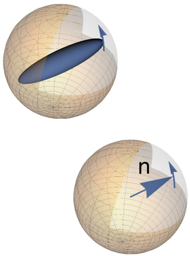
Widespread formalisms:

- Ericksen-Leslie: **Oseen-Frank directors**
- Beris-Edwards: **De Gennes order tensors**

[1] . J. DeCamp, G. S. Redner, A. Baskaran, M. F. Hagan, and Z. Dogic, Nat Mater

Molecular alignment





- Privileged direction modeled through a **director** field

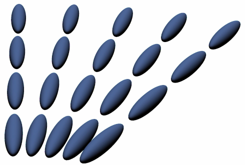
$$n : (0,T) \times \Omega \to \mathbb{S}^2 \subseteq \mathbb{R}^3 \qquad \text{(i.e.} \quad |n(t,x)| = 1).$$

- **Ericksen-Leslie**: constitutive equations for the director $n(t,x)$ and the velocity field $u(t,x)$.

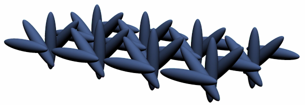
The Oseen-Frank energy density

The mathematical theory of liquid crystals goes back to the seminal works by Frank (1958) and Oseen (1933). **Distortion free energy density:**

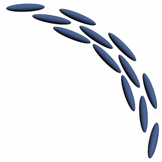
$$w_F(n,\nabla n) := \frac{K_1}{2}|\operatorname{div} n|^2 + \frac{K_2}{2}|n \cdot \operatorname{curl} n|^2 + \frac{K_3}{2}|n \wedge \operatorname{curl} n|^2 + \\ + \frac{K_2+K_4}{2}\left(\operatorname{tr}(\nabla n)^2 - |\operatorname{div} n|^2\right)$$



splay



twist



bend

When the resulting equilibrium equations are complicated: **one-constant approximation**

$$w_F(n,\, \nabla n) = w_F(\nabla n) = \frac{K}{2}|\nabla n|^2.$$

The Ericksen-Leslie model

The most frequently used form of the equations consist of the **constrain** $|n| = 1$ together with

Conservation of mass: $\operatorname{div} u = 0$

Balance of linear momentum: $\partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{T}$

Balance of angular momentum: $n \wedge (\mathrm{g} + \mathrm{h}) = 0$

- p pressure in $\mathbb{R}$
- u velocity field in $\mathbb{R}^{\mathrm{d}}$
- n director field
- g kinematic transport
- h molecular field
- $\mathbb{T}$ total stress tensor

General Ericksen-Leslie system

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$$\mathbb{T} = -\mathrm{p}\,\mathrm{Id} + \sigma^{\mathrm{E}} + \sigma^{\mathrm{L}} \quad \text{where} \quad \begin{cases} \sigma^{\mathrm{E}} & \text{Ericksen stress tensor,} \\ \sigma^{\mathrm{L}} & \text{Leslie stress tensor,} \end{cases}$$

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$\sigma^{\mathrm{E}} = -{}^t\nabla n \dfrac{\partial w_F}{\partial \nabla n}$ where $w_F(n, \nabla n)$ is the Oseen-Frank energy density.

$\sigma^{\mathrm{E}} = -\nabla n \odot \nabla n$ where $w_F(\nabla n) = \dfrac{|\nabla n|^2}{2}$

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corotational time flux $\mathcal{N} = \partial_t n + u \cdot \nabla n - \frac{1}{2} \operatorname{curl} u \wedge n$

stretching due to $\mathbb{D}n$, where $\mathbb{D} = \frac{1}{2}\left(\nabla u + {}^t\nabla u\right)$

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$$\sigma^{\mathrm{L}} = \alpha_1 (n \cdot \mathbb{D} n) n \otimes n + \alpha_2 \mathscr{N} \otimes n + \alpha_3 \, n \otimes \mathscr{N} + \\ + \alpha_4 \mathbb{D} + \alpha_5 \mathbb{D} n \otimes n + \alpha_6 n \otimes \mathbb{D} n.$$

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- High powers interacting with the flow.

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- Maximal derivatives.

- p — pressure in $\mathbb{R}$
- u — velocity field in $\mathbb{R}^{\mathrm{d}}$
- n — director field
- g — kinematic transport
- h — molecular field
- $\mathbb{T}$ — total stress tensor

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$$\mathrm{g} = \lambda_1 \mathscr{N} + \lambda_2 \mathbb{D} n$$

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$$\mathrm{h} = \frac{\delta w_F}{\delta n} = \frac{\partial w_F}{\partial n} - \operatorname{div} \frac{\partial w_F}{\partial \nabla n}$$

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$$
\begin{cases}
\operatorname{div} u = 0, \\
\partial_t u + \operatorname{div}(u \otimes u) - \nu \Delta u + \nabla p = -\operatorname{div}(\nabla n \odot \nabla n) + \alpha_1 \operatorname{div}(n \otimes n (n \otimes n : \nabla u)), \\
\partial_t n + u \cdot \nabla n - n \cdot \nabla u = \Delta n + |\nabla n|^2 n.
\end{cases}
$$

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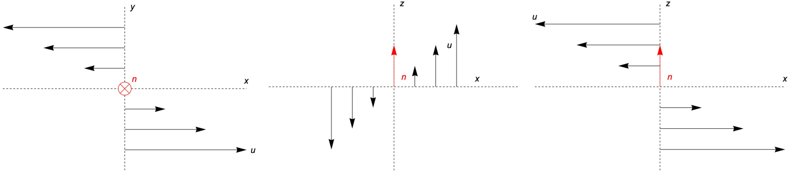
Measuring the Leslie coefficients

- **Laminar flow** under a strong orienting Field [Miesowicz, 1946], the molecules are firmly aligned in one direction by a **constant magnetic field**.



- When the shear stress $(\sigma_L)_{\alpha\beta}$ is known, one can derive the effective viscosity η

$$\eta = \frac{(\sigma_L)_{\alpha\beta}}{2\mathbb{D}_{\alpha\beta}}.$$

- Kneppe & Schneider; Stephen & Straley; Stewart;

Quantities	MBBA near 25°	PAA near 122°	5CB near 26°
α_1	-0.0181	0.0043	-0.0060
α_2	-0.1104	-0.0069	-0.0812
α_3	-0.001104	-0.0002	-0.0036
α_4	0.0826	0.0068	0.0652
α_5	0.0779	0.0047	0.0640
α_6	-0.0336	-0.0023	-0.0208

Viscous dissipation and Parodi relation

- The total energy is dissipative along time evolution. Basic energy law

$$\frac{d}{dt}\int_\Omega \Big(\frac{1}{2}|u|^2 + w_F(n,\nabla n)\Big)dx =$$
$$-\int_\Omega \Big(\alpha_1 |n\cdot \mathbb{D}n|^2 + (\alpha_2+\alpha_3+\lambda_2)\mathscr{N}\cdot \mathbb{D}n + \alpha_4|\mathbb{D}|^2 + (\alpha_5+\alpha_6)|\mathbb{D}n|^2 - \lambda_1|\mathscr{N}|^2\Big)dx$$

- We require

$$\alpha_3-\alpha_2\geq 0,\quad \alpha_4\geq 0,\quad 2\alpha_4+\alpha_5+\alpha_6\geq 0,$$
$$\alpha_1+3\alpha_4+2\alpha_5+2\alpha_6\geq 0,\quad -4\lambda_1(2\alpha_4+\alpha_5+\alpha_6)\geq(\alpha_2+\alpha_3-\lambda_2)^2.$$

- **Onsager relations** between flows and forces in irreversible thermodynamic systems. Parodi suggested that the Leslie coefficients shall be further restricted:

$$\lambda_1=\alpha_2-\alpha_3,\qquad \lambda_2=\alpha_5-\alpha_6,\qquad \alpha_6-\alpha_5=\alpha_2+\alpha_3.$$

Quantities	MBBA near 25°	PAA near 122°	5CB near 26°
$\alpha_2+\alpha_3$	-0.111504	-0.0071	-0.0848
$\alpha_6-\alpha_5$	-0.1115	-0.007	-0.0848

Unitary constraint vs Ginzburg–Landau relaxation

- Balance of angular momentum:

$$\partial_t n + u \cdot \nabla n - n \cdot \nabla u - \Delta n = |\nabla n|^2 n.$$

- Relaxing the unitary constraint $|n(t,x)| = 1$ through

$$\int_\Omega \frac{|\nabla n(t,x)|^2}{2} dx \quad \to \quad \int_\Omega \frac{|\nabla n(t,x)|^2}{2} dx + \int_\Omega \frac{(|n(t,x)|^2 - 1)^2}{4\varepsilon} dx$$

Natural physical interpretations, attributed to the extensibility of liquid crystal molecules.

- Balance of angular momentum:

$$\partial_t n + u \cdot \nabla n - n \cdot \nabla u - \Delta n = -\frac{|n|^2 - 1}{\varepsilon} n.$$

- In general, loss of an uniform bound for $|n(t,x)|$. Related difficulties on the momentum equation:

$$\ldots \operatorname{div} \Big(n \otimes n (n \otimes n : \nabla u) \Big) \ldots$$

Questions on the analysis of EL

- Is the Ericksen-Leslie model solvable in two and three dimensions?
- Which type of solutions can we build?
- Under which restrictions on the Leslie viscous coefficients?
- What can we say about uniqueness?

Overview of related literature and contribution

Overview of the literature

- *C. Liu & F.-H. Lin (2000)*: First analysis result on a **three dimensional** smooth bounded domain subject to **Dirichlet** boundary conditions. Any initial data with finite total energy generates a global-in-time weak solution. Restriction on the Leslie coefficients, corotational framework ($\lambda_2 = 0$) and **weak maximum principle**.

- *D. Coutand & S. Shkoller ('01)*: First attempt for the general case $\lambda_2 \neq 0$. Local existence and uniqueness of a local-in-time classical solution. Total energy of the system is not dissipative, small-data global existence result even in two dimensions.

- *H. Sun & C. Liu ('09)*: Setting of Leslie coefficients, energy dissipation property is guaranteed (in particular $\alpha_1 = 0$). Existence of global strong solutions provided that the dimension is two or the viscosity $\alpha_4 > 0$ is sufficiently large.

- *H. Wu & X. Xu & C. Liu ('13)*: three dimensional periodic setting, global strong solutions and long-time behavior, under large viscosity α_4; With Parodi's relation, global well-posedness and Lyapunov stability for the system near local energy minimizers.

Overview of the literature

- *C. Cavaterra & E. Rocca & H. Wu ('13)*: Three dimensional bounded domain, natural boundary conditions and suitable requirements on the Leslie coefficients that ensure the energy dissipation. **Existence** of global-in-time weak solutions, **new formulation of weak solutions** that took into account the low regularity of those highly nonlinear stress. Double-level Faedo-Galerkin approximation scheme. Uniqueness was still an open problem.

$$\lambda_1 = \alpha_2 - \alpha_3 < 0, \quad \alpha_1 \geq 0, \quad \alpha_4 > 0, \quad \alpha_5 + \alpha_6 \geq 0, \quad \lambda_2 = \alpha_5 - \alpha_6.$$

$$\begin{cases} \textbf{with Parodi:} & \frac{\lambda_2^2}{-\lambda_1} \leq \alpha_5 + \alpha_6, \\ \textbf{without Parodi:} & |\lambda_2 - \alpha_2 - \alpha_3| < 2\sqrt{-\lambda_1}\sqrt{\alpha_5 + \alpha_6} \end{cases}$$

Life can always be more complicated...

- Unitary constraint: M.-C. Hong ('10), M.-C. Hong & J.-K. Li & Z.-P. Xin ('14), T. Huang & F.-H. Lin & C. Liu & C.-Y. Wang ('16), F.-H. Lin & C.-Y. Wang ('16)...
- Thermal effects: E. Feireisl & M. Frémond & E. Rocca & G. Schimperna ('12), M. Hieber & M. Nesensohn & J. Prüss, K. Schade ('14), M. Hieber & J. Prüss ('16),....
- Inertia: F.D.A & A. Zarnescu ('16), C. Yuan & W. Wei ('20),...
- ...

Weak solutions

A pair (u, n) is called a weak solution to the Ericksen-Leslie system if it is a distributional solution and for any $\zeta, \zeta' \in (0,1)$:

$$\begin{aligned} & u \in L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;\dot{H}^1(\mathbb{T}^2)) && \text{with} && \partial_t u \in L^2(0,T;W^{-1,\frac{2}{1+\zeta}}(\mathbb{T}^2)), \\ & n \in L^\infty(0,T;H^1(\mathbb{T}^2)) \cap L^2(0,T;\dot{H}^2(\mathbb{T}^2)) && \text{with} && \partial_t n \in L^2(0,T;L^{\frac{2}{1+\zeta'}}(\mathbb{T}^2)), \\ & n \cdot \mathbb{D}n \in L^2((0,T)\times\mathbb{T}^2), && \text{and} && \mathbb{D}n \in L^2((0,T)\times\mathbb{T}^2). \end{aligned}$$

- With Parodi

$$\mathcal{E}(t) + \int_0^t \int_{\mathbb{T}^2} \alpha_1 |n \cdot \mathbb{D}n|^2 + \frac{\alpha_4}{2}|\nabla u|^2 - \frac{1}{\lambda_1}\left|\Delta n - \frac{n^2-1}{\varepsilon} n\right|^2 + \left(\alpha_5 + \alpha_6 + \frac{\lambda_2^2}{\lambda_1}\right)|\mathbb{D}n|^2 \leq \mathcal{E}(0),$$

- Without Parodi

$$\mathcal{E}(t) + \int_0^t \int_{\mathbb{T}^2} \alpha_1 |n \cdot \mathbb{D}n|^2 + \frac{\alpha_4}{2}|\nabla u|^2 - \frac{1}{\lambda_1}\left|\Delta n - \frac{n^2-1}{\varepsilon} n\right|^2 + \eta\left(|\mathbb{D}n|^2 + |\mathcal{N}|^2\right) \leq \mathcal{E}(0).$$

Theorem (C. Cavaterra, E. Rocca, H. Wu (2013))

There exists a global-in-time weak solution, satisfying the energy inequality.

Contribution

F. D.A. & H. Wu (2021)

Consider Ericksen Leslie in $(0,T)\times\mathbb{T}^2$. Let (u_1,n_1) and (u_2,n_2) be two global weak solutions, subject to the same initial data $(u_0,n_0)\in L^2(\mathbb{T}^2)\times H^1(\mathbb{T}^2)$. Then we have $(u_1(t),n_1(t))=(u_2(t),n_2(t))$ for all $t\in(0,T)$ (Uniqueness).

- For the proof, ansatz on the Leslie coefficients:

$$\alpha_1=1,\quad \alpha_2=-1,\quad \alpha_3=0,\quad \alpha_4=2\nu>0,\quad \alpha_5=3,\quad \alpha_6=1,$$

  in particular we do not require the Parodi's relation. In general $\alpha_1\geq 0$.

- To clarify:
  - what we mean with weak solutions,
  - why the problem is not trivial even in two dimension,
  - philosophy behind the proof.

Weak solutions á la Leray

The **Ericksen-Leslie** model inherits the major challenges of the **Navier-Stokes** equations:

$$\partial_t u + \text{div}\,(u \otimes u) - \nu \Delta u + \nabla p = 0, \qquad \text{div}\, u = 0, \qquad (t,x) \in (0,T) \times \mathbb{T}^d.$$

Leray (1934)

For any initial data $u_0 \in L^2(\mathbb{T}^d))$ with $\text{div}\, u_0 = 0$, there exists a Leray solution on $(0,T) \times \mathbb{T}^d$.

A **Leray solution** u is a distributional solution, such that

- $u \in L^\infty(0,T; L^2(\mathbb{T}^d))$,
- $u \in L^2(0,T; \dot{H}^1(\mathbb{T}^d))$,
- $\lim_{t \to 0} \|u(t) - u_0\|_{L^2} = 0$,
- Energy inequality: $\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2$.

In **two dimensions** $d = 2$ one has the **energy equality** and the **uniqueness** of these solutions.
Why does this result not work for us?

Non triviality of the problem

Navier-Stokes:

- In dimension two the problem is critical. Scaling invariance.
- Energy equality and continuity $u \in C([0,T], L^2(\mathbb{T}^2))$.
- $\partial_t u$ belongs to $L^2(0,T;\dot{H}^{-1}(\mathbb{T}^2))$ since $u \in L^4((0,T)\times\mathbb{T}^2)$ and

$$\partial_t u - \nu\Delta u + \nabla p = -\mathrm{div}\left(\underbrace{u \otimes u}_{\in L^2((0,T)\times\mathbb{T}^2)} \right)$$

Ericksen-Leslie:

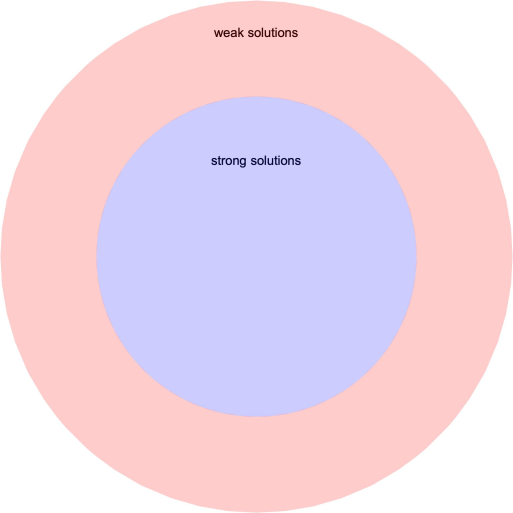
- The velocity could lose energy at high modes: $u \notin C([0,T], L^2(\mathbb{T}^2))$:

$$\partial_t u - \nu\Delta u + \nabla p = -\mathrm{div}\left(\underbrace{n \otimes n(n \otimes n : \nabla u)}_{\notin L^2((0,T)\times\mathbb{T}^2)} \right) + \dots$$

- The stretching prevents the director to be bounded $n \notin L^\infty((0,T)\times\mathbb{T}^2)$.

Uniqueness

Difference between two solutions $(\delta u,\ \delta n) = (u_1, n_1) - (u_2, n_2)$.



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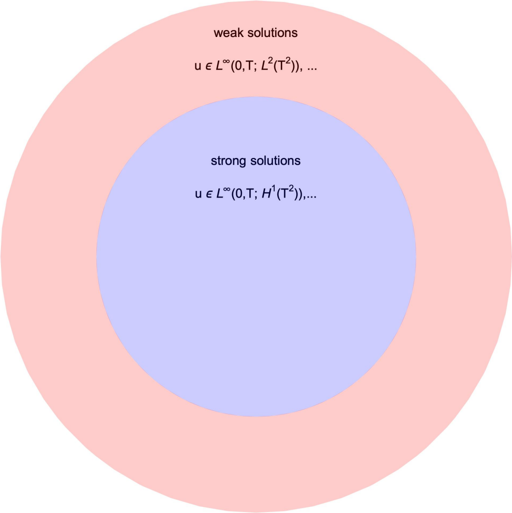

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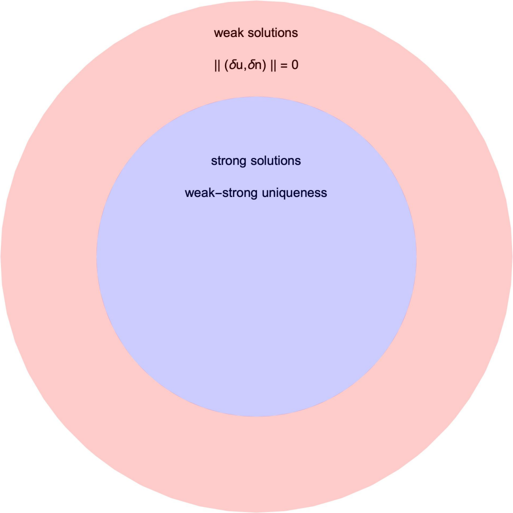

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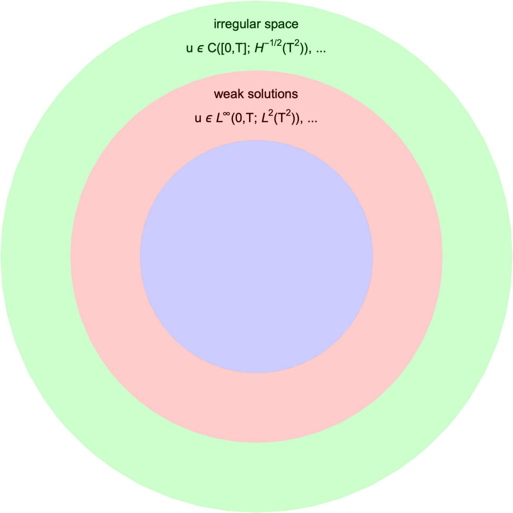

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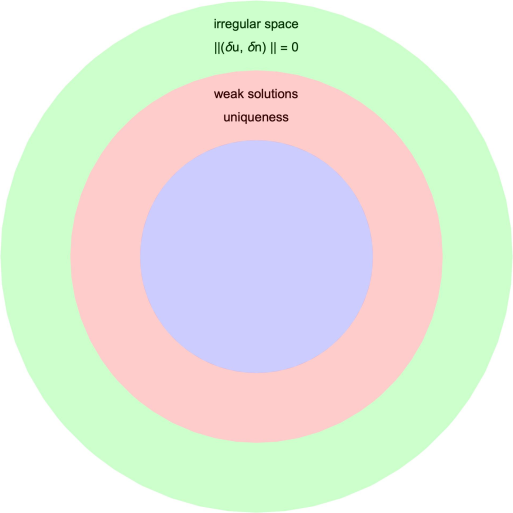


Uniqueness for EL - Conclusion

- Controlling the difference $(\delta u,\, \delta n) = (u_1, n_1) - (u_2, n_2)$ at lower regularities than the energy space:

$$\delta\mathcal{E}(t) = \|\delta u(t)\|^2_{\dot{H}^{-\frac{1}{2}}} + \|\delta n(t)\|^2_{\dot{H}^{\frac{1}{2}}}.$$

- Approach on frequencies decomposition (**paradifferential calculus** and Bony's decomposition); **double logarithmic inequality**

$$\delta\mathcal{E}(t) + \int_0^t \delta\mathfrak{D}(s)ds \leq \int_0^t f(s)\delta\mathcal{E}(s)\Big(-\ln\big(\delta\mathcal{E}(s)\big)\Big)\ln\big(-\ln\big(\delta\mathcal{E}(s)\big)\big)ds.$$

- Osgood lemma $\delta\mathcal{E} \equiv 0 \Rightarrow$ Uniqueness

Some details on the double logarithmic inequality

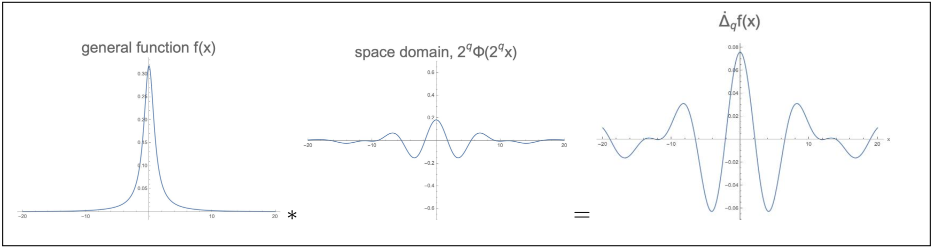
A first identity

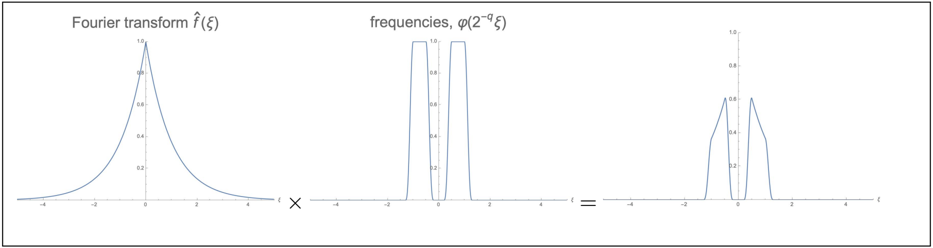
The contribution of the Leslie tensor $\sigma_L : \nabla u$ is not integrable on $(0,T) \times \mathbb{T}^2$. We need to pass to lower regularities for a first identity:

$$\frac{1}{2}\|\delta u(t)\|^2_{H^{-\frac{1}{2}}} + \frac{1}{2}\|\nabla \delta n(t)\|^2_{H^{-\frac{1}{2}}} + \int_0^t \left(\frac{\alpha_4}{2}\|\nabla \delta u(s)\|^2_{H^{-\frac{1}{2}}} + \frac{\lambda_2}{-\lambda_1}\|\nabla \delta n(s)\|^2_{H^{\frac{1}{2}}} \right) ds$$
$$= -\alpha_1 \int_0^t \langle (d_1 \otimes d_1)(d_1 \otimes d_1 : \nabla \delta u), \nabla \delta u \rangle_{H^{-\frac{1}{2}}} ds + \dots$$

- Identity holds, since each inner product is well-defined and time integrable.
- Aim: localise any dissipative term and estimate the remaining ones through a modulus of continuity.
- A standard commutator would provide a dissipative contribution; Further analysis is however inconclusive.
- Alternative approach: better for of the dissipation through Fourier analysis and Littlewood-Paley decomposition.

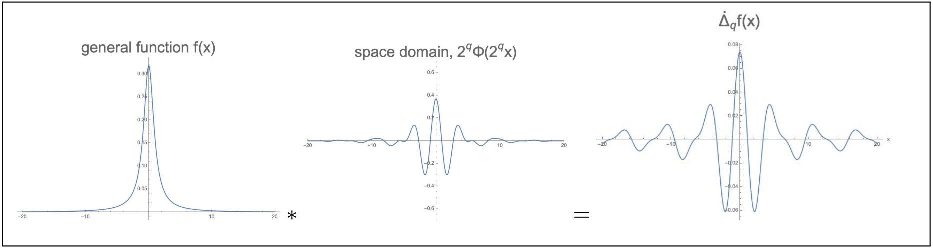
Dyadic decomposition

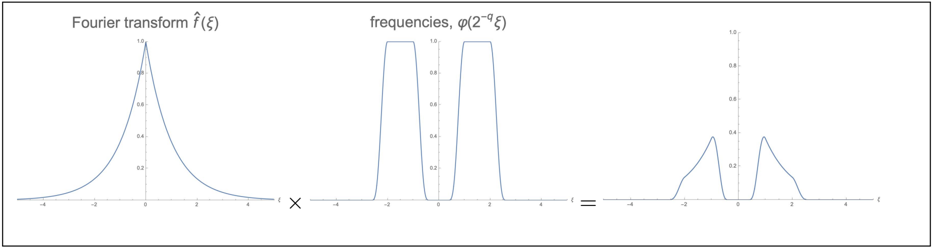




q = 0

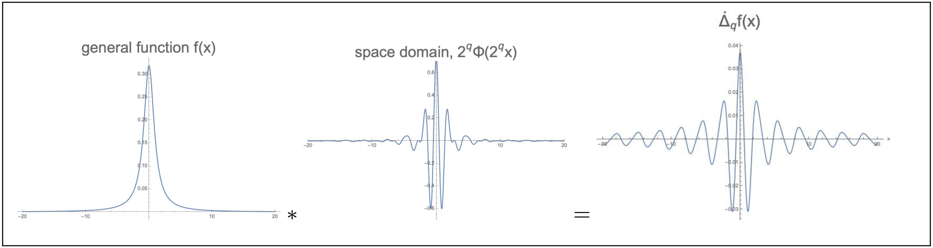
Dyadic decomposition

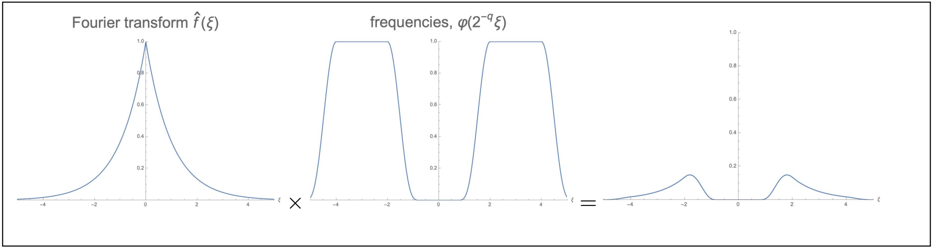




q = 1

Dyadic decomposition





q = 2

A toolbox of Fourier Analysis

- The $H^{-1/2}$-inner product $\approx \sum_q 2^{-q} \langle \dot\Delta_q f,\, \dot\Delta_q g\rangle_{L^2}$.

- Low-frequencies cut-off: $S_{q-1} f := \sum_{j<q-1} \dot\Delta_q f$.

- Bony's paraproduct decomposition:

$$\dot\Delta_q(f\,g) = \sum_{|j-q|\leq 5} \left[\dot\Delta_q,\, \dot S_{j-1} f,\right] \dot\Delta_j g \qquad ☑ \rightarrow \text{ commutator inequality}$$

$$+ \sum_{|j-q|\leq 5} (\dot S_{q-1} - \dot S_{j-1}) f \dot\Delta_q \dot\Delta_j g \qquad ☑ \rightarrow \text{ Bernstein inequality}$$

$$+ \qquad S_{q-1} f \dot\Delta_q g \qquad ☒$$

$$+ \sum_{j\geq q+5} \dot\Delta_q \left(\dot\Delta_j f,\, S_{q+2} g\right) \qquad ☑ \rightarrow \text{ Young inequality}$$

- The third element generates nonlinear challenging terms to control.
  The structure of the Ericksen-Leslie system cancels them or make them dissipative terms.

A Brezis-Gallouet inequality

- Similar approach as for proving the **Brezis-Gallouet** inequality:

$$\|f\|_{L^\infty} \leq C\|u\|_{H^1}\left\{1+\sqrt{\ln\left(1+\frac{\|u\|_{H^2}}{\|u\|_{H^1}}\right)}\right\}.$$

- We separately control **low and high frequencies** of the Leslie viscous stress:

$$\alpha_1 \underbrace{\langle (d_2\cdot(\nabla u_2 d_2))\delta d\otimes S_N d_1, \nabla\delta u\rangle_{H^{-\frac{1}{2}}}}_{=:\mathcal{I}} + \alpha_1 \underbrace{\langle (d_2\cdot(\nabla u_2 d_2))\delta d\otimes (\mathrm{Id}-S_N) d_1, \nabla\delta u\rangle_{H^{-\frac{1}{2}}}}_{=:\mathcal{II}}$$

- The first term increases proportionally to the radius N, where the **low frequencies** are localised:

$$\mathcal{I} \leq C\Big(\text{term that is integrable in time}\Big)\|S_N d_1\|_{L^\infty}^2\|\delta d\|_{H^{\frac{1}{2}}}^2 + \text{terms that can be absorbed}$$
$$\leq C\Big(\text{term that is integrable in time}\Big)\|d_1\|_{H^1}^2\|\delta d\|_{H^{\frac{1}{2}}}^2 N + \text{terms that can be absorbed.}$$

- The **high frequencies** fix the value of $N\approx -\ln\delta\mathcal{E}(t)$ and therefore the logarithmic inequality:

$$\mathcal{II} \leq C\Big(\text{term that is integrable in time}\Big)2^{-N}.$$

The double-logarithmic inequality

- We use the following Sobolev interpolation inequality for a general $\varepsilon \in (0, 1/2]$,

$$\|f\|_{L^{\frac{2}{\varepsilon}}} \leq \frac{C}{\sqrt{\varepsilon}} \|f\|_{L^2}^{\varepsilon} \|f\|_{L^2}^{1-\varepsilon}.$$

- **Key tool:** explicit constant of embedding in terms of ε. We aim at choosing an appropriate value of ε, in order to get a modulus of continuity.

$$\left| \alpha_1 \langle (d_1 \cdot (\nabla \delta u d_2)) d_1 \otimes d_1, \nabla \delta u \rangle_{H^{-\frac{1}{2}}} \right| \leq$$

$$(\dots) \left(\frac{N}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \delta\mathcal{E}(t) + (\dots)\, 2^{-N} + \text{terms that can be absorbed}$$

- By choosing a fix value ε, such as $\varepsilon = 1/2$, we lose the modulus of continuity:

$$\left(\frac{N}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \delta\mathcal{E}(t) \leq C N^2 \delta\mathcal{E}(t) \leq C \delta\mathcal{E}(t) \ln\big(-\delta\mathcal{E}(t)\big)^2 \qquad ✗$$

- **Solution:** choose $\varepsilon \approx (1 + \ln N)^{-1}$ and $N \approx -\ln \delta\mathcal{E}(t)$:

$$\left(\frac{N}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \delta\mathcal{E}(t) \leq C \frac{N}{\varepsilon} \delta\mathcal{E}(t) \leq C \delta\mathcal{E}(t) \ln\big(-\delta\mathcal{E}(t)\big) \ln\Big(\ln\big(-\delta\mathcal{E}(t)\big)\Big) \qquad ☑$$

- The uniqueness follows by using Osgood theorem.

Conclusion

- Ericksen-Leslie equations.
- The leslie viscous coefficients, in particular the value of α_1.
- The Ginzbourg-Landau penalisation.
- Model based on assumption $\alpha_1 \geq 0$ (with further conditions to ensure viscous dissipation) admits a unique weak solution.
- Uniqueness follows from suitable frequencies decompositions, leading to a logarithmic inequality of Osgood type.
- For details see: F. D.A. & H. Wu: *Uniqueness of weak solutions for the general Ericksen-Leslie system with Ginzburg-Landau penalization in* $\mathbb{T}^2$, arXiv:2107.02101

Thanks for your attention!