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Stochastic Ericksen-Leslie Equations

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joint works with

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I

$D \subset \mathbb{R}^2$ bounded, smooth domain

$(\text{or } D = T^2)$

NSE's

$$(1.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla p = 0, & t \geq 0, \ x \in D \\ \operatorname{div} u(t,\cdot) = 0 & \text{in } D \\ u \cdot \vec{n} = 0 & \text{on } \partial D \\ u(0,\cdot) = u_0 & \end{cases}$$

$\nu = 1$

Notation (standard):

$$\mathcal{V} = C_0^\infty(D, \mathbb{R}^2)$$

$$H = \{u \in \mathbb{L}^2 = L^2(D, \mathbb{R}^2) : \operatorname{div} u = 0, \; u \cdot \vec{n} = 0 \text{ on } \partial D\}$$

$$V = \{u \in \mathbb{H}^1 = H^1(D, \mathbb{R}^2) : \operatorname{div} u = 0, \; u|_{\partial D} = 0\}$$

$$= H_0^1(D, \mathbb{R}^2) \cap H$$

$P : \mathbb{L}^2 \longrightarrow H$ orthogonal projection

(Leray - Helmholtz ....)

$$D(A) = \mathbb{H}^2 \cap V, \quad A(u) = -P(\Delta u), \; u \in D(A)$$

$$B(u) \; B(u,u) = P(u \cdot \nabla u)$$

$$V \subset H \cong H' \subset V'$$

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$(1.1) \iff (1.2)$

$$(1.2) \quad \frac{\partial u}{\partial t} + Au + B(u) = 0 \qquad u(0) = u_0 \in H$$

Observation 1. If $B = 0$ then (1.2) is formally
$-$grad flow w.r.t. to Hilbert space $V' = D(A^{-1/2})$
of the energy

$$\frac{1}{2} |u|_H^2$$

i.e.

$$\frac{\partial u}{\partial t} = -\nabla_{V'} \Phi(u)$$

Observation 2. $B(u) \perp Au$ w.r.t. V'

(1.3) $$\langle Bu, Au \rangle_{V'} = \langle A^{-1/2} B(u), A^{-1/2} Au \rangle_H$$
$$= \langle B(u), u \rangle_H = 0$$

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II The S^2-valued heat flow on D

$$(2.1) \quad \frac{\partial n}{\partial t} = -n \times (n \times \Delta n) \quad \boxed{\left(= \Delta n + |\nabla n|^2 n\right)}$$

$$\left.\frac{\partial n}{\partial \vec{n}}\right|_{\partial D} = 0 \qquad n(0,\cdot) = n_0$$

(2.1) is formally a $-$ grad flow w.r.t. "riemannian" structure on

$$M = \{ u \in H^1(D, \mathbb{R}^3) : \ u \in S^2 \ \text{a.e.} \}$$

induced by $L^2(D, \mathbb{R}^3)$ inner product of energy

$$\Psi(n) = \frac{1}{2} \|\nabla n\|^2_{L^2}, \quad n : D \longrightarrow S^2$$

(2.2) $$\frac{\partial n}{\partial t} = -\nabla_{L^2} \Psi(n)$$

There is a more general system than (2.1):

(2.3) $$\frac{\partial n}{\partial t} = -n \times (n \times \Delta n) + \lambda \boxed{n \times \Delta n}$$

gyroscopic force

(2.3) ≡ Landau - Lifshitz - Gilbert Equations (LLGE)

Observation 3. $n \times \Delta n \perp -n(n \times \Delta n) = -\nabla_{L^2} \psi(n)$

Hence LLGEs (2.3) have a similar structure to the one of NSEs:

$-$ grad $+$ perp. term

This implies a similar a'priori estimates

$$(1.4) \quad \frac{1}{2} |u(t)|^2_{L^2} + \int_0^t |\nabla u(s)|^2_{L^2} \, ds \le \frac{1}{2} |u_0|^2_{L^2}$$

$$(2.4) \quad \frac{1}{2} |\nabla n(t)|^2_{L^2} + \int_0^t |n \times \Delta n(s)|^2_{L^2} \, ds \le \frac{1}{2} |\nabla n_0|^2_{L^2}$$

<u>Remark</u>. Such types of finite dimensional problems were studied by

- Freidlin - Wentzel (Springer book)

- van den Eijden, Kohn, Reznikov (2005)

III

Ericksen – Leslie Equations

$$(3.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} + Au + Bu + \operatorname{div}(\nabla n \odot \nabla n) = 0 \\ \dfrac{\partial n}{\partial t} + n \times (n \times \Delta n) + \lambda\, n \times \Delta n \\ \qquad + u \cdot \nabla n = 0 \end{cases}$$

Here

$$(\nabla n \odot \nabla n)_{i,j} := \langle \partial_i n, \partial_j n \rangle_{\mathbb{R}^3}$$

for $n : D \longrightarrow S^2 \subset \mathbb{R}^3$

Because

$$(3.2)\quad \langle \operatorname{div}(\nabla n \odot \nabla n), u \rangle_{L^2} + \langle u \cdot \nabla n, \Delta n \rangle_{L^2} = 0$$

the ELEs (3.1) have a similar structure of

$-$ grad $+$ perp. term

on the "space" $H \times \mathcal{M}$

with energy

$$(3.3)\quad E(u,n) = \frac{1}{2} |u|^2_{L^2} + \frac{1}{2} |\nabla n|^2_{L^2}$$

This will be used later!

IV The Ginzburg-Landau approximation to ECLEs $(\lambda = 0)$

$$(4.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} + Au + Bu + \operatorname{div}(\nabla n \odot \nabla n) = 0 \\[2ex] \dfrac{\partial n}{\partial t} + \Delta n + \dfrac{1}{\varepsilon^2} F'(n) + (u \cdot \nabla n) = 0 \end{cases}$$

$$F'(y) = \frac{1}{4}\left(|y|^2 - 1\right)^2$$

has a similar structure with energy

$$(4.2)\quad \mathcal{E}(u,n) = \frac{1}{2}|u|^2_{L^2} + |\nabla n|^2_{L^2} + \int_{\mathcal{D}} F(n(x))\,dx$$

System (4.1) was studied by Lin et al (1996)

The ELEs (3.1) have been studied

by Lin, Lin & Wang (2010)

and Hong (Min-chun) (2011)

The former used approximation of initial data by smooth data, the latter used approximation by (3.1).

The first step was to get a'priori estimates:

$$(4.4)\quad |u(t)|^2_{L^2} + \int_0^t |\nabla u(s)|^2 ds \le |u_0|^2$$

$$(4.5)\quad |n(t)|^2 + \int_0^t \underbrace{|n \times \Delta n|^2}_{|\Delta n(s) - |\nabla n(s)|^2 n(s)|^2_{L^2}} ds \le |n_0|^2$$

or

$$(4.5')\quad |n^\varepsilon(t)|^2_{L^2} + \int_0^t |\Delta n_\varepsilon + \frac{1}{\varepsilon^2} n_\varepsilon (1 - |n_\varepsilon|^2)|^2 ds$$

$$\le |n^\varepsilon(0)|^2$$

Such estimates are not sufficient to pass to the limit.

Difficult terms: $\operatorname{div}(\nabla n \odot \nabla n)$

and $n|\nabla n|^2$

Both papers used a method of Struwe (1985) originally applied to heat flow equation (with a general target manifold instead of S^2).

Ladyzhenskaya inequality: $\exists\, c_1 > 0 : \forall R > 0$

(4.6) $$\iint_{[0,T]\times D} |f(t,x)|^4 \, dx\, dt \le c_1 \operatorname*{ess\,sup}_{t\in[0,T]} \sup_{x} \int_{B(x,R)} |f(t,x)|^2 dx \times \left[\iint_{[0,T]\times D} |\nabla f(t,x)|^2 dx\, dt + \frac{1}{R^2} \iint_{[0,T]\times D} | f(t,x)|^2 dx\, dt \right]$$

V There are many reasons to introduce randomness, e.g.

Landau-Lifshitz v. VI, ch. XVII (1958) "Fluid Mechanics"

(see recent papers by G. Eyink)

or

Faris-Jona Lasinio (1982):

"An example of primary importance in physics is provided by hydrodynamics. As is well known, the behaviour of an impompressible viscous fluid is usually described in terms of the Navier Stokes Equations. However, this equation is known to be approximate in more than one aspect. It takes into account only approximately the microscopic nature of a classical field. In addition, quantum effects and other sources of fluctuations are completely ignored. It is therefore of interest to know which properties described by the NSEs survive perturbations, in particular small stochastic perturbations which imitate some of the neglected effects.

This latter problem is also of special importance in connection with modern theories of turbulence, where one would like to determine physically interesting measures invariant under the flow generated by the NSEs and stable under small perturbations."

According to Landau-Lifshitz the noise has to be added to the Δ terms

$$\Delta u + dW$$

$$\Delta n + d\tilde{W}$$

where $W, \tilde{W}$ two independent Wiener processes.

There are other theories about noise, e.g. transport noise

$$(5.1)\begin{cases} \dfrac{\partial u}{\partial t} + Au + Bu + \operatorname{div}(\nabla n \odot \nabla n) = dW \\ \dfrac{\partial n}{\partial t} + n\times(n\times\Delta n) + \lambda n\times\Delta n \\ \qquad + u\cdot\nabla n = \end{cases}$$

$$-n\times(n\times d\tilde{w}) \quad - \lambda n \times d\tilde{w}$$

Very often the terms $-n\times(n\times d\tilde{w})$

and $\lambda n \times \Delta n$

are omitted

Hence we get

$$(5.2)\quad \begin{cases} \dfrac{\partial u}{\partial t} + Au + Bu + \operatorname{div}(\nabla n \odot \nabla n) = dW \\[2ex] \dfrac{\partial n}{\partial t} + n \times (n \times \Delta n) + + u \cdot \nabla n \\ \qquad = -\lambda\, n \times d\tilde{w} \end{cases}$$

One can also study the GL approximation of (5.2):

$$(5.3)\begin{cases} \dfrac{\partial u}{\partial t} + Au + Bu + \operatorname{div}(\nabla n \odot \nabla n) = dW \\ \dfrac{\partial n}{\partial t} + \dfrac{1}{\varepsilon^2} F'(n) - \Delta n + \quad + u \cdot \nabla n \\ \qquad = -\lambda n \times d\tilde{w} \end{cases}$$

<u>Note</u>: The term

$$-\lambda n \times d\tilde{w}$$

has to be understood in the Stratonovitch sense, in order to be able to prove the constraint condition

$$n(t, x) \in S^2$$

Existence of strong maximal solutions $(d=3)$ and global solutions $(d=2)$ was proved by

Z. B., P. Razafimandimby
and E. Hausenblas (2013, 2021)

However, uniqueness and existence of strong global solutions ($d=2$) was only proved after a breakthrough paper by A. Hocquet (2018) on stochastic heat flow; who generalized Struwe's approach.

- de Bouard et all (2021)
- ZB + P. Razafimandimby

  - general domain
  - general noise

(torus T^2 only noise in director equation)

The solution to this problem is based on approximation by more regular data and using a family of "Lyapunov" functions:

$$\mathcal{E}_R(u,n) := \sup_{x \in D} \left[\frac{1}{2} \int_{B(x,2R)} |u|^2 + |\nabla n|^2 \right]$$

A solution (u,n) is $H \times H^1$-valued weakly continuous and $V \times H^2$-valued measurable between finite number $\tau_1, \dots \tau_N$ of stopping times at which $H \times H^1$ norm jumps down!

Moreover in a joint paper with G. Deugoue and P. Razafimandimby we prove that Hong's approximation also leads to a unique solution.

Remark: Our uniqueness proof uses $H^{-1} \times L^2$ norm of (u, n).

We learnt this method from

an old ('95) PhD thesis
about quasi-geostrophic equations
(and used also for Stochastic
Navier-Stokes Equ.)

VI Large Deviations Principle:

Behaviour as the noise becomes weak

Consider a model problem

$$(6.1)\quad \begin{cases} du^\varepsilon = F(u^\varepsilon)\,dt + \sqrt{\varepsilon}\,G(u^\varepsilon)\,dW \\ u^\varepsilon(0) = u_0 \end{cases}$$

$T > 0$, u_0 fixed.

X_T – a space of trajectories

$u^\varepsilon(\cdot, \omega) \in X_T$, $\omega \in \Omega$.

(6.2) $\mu^\varepsilon := \mathrm{Law}(u^\varepsilon)$ – a Borel prob. measure on the top. space X_T.

(D1) $I : X_T \longrightarrow [0, \infty]$ is a good rate function if $\forall\, R > 0$

$\{ I \le R \}$ is closed (lower sc)

and $\{ I \le R \}$ is compact

Example 1 $\mathcal{X}_T := C_0([0,T]; \mathbb{R})$

$$I(x) = \begin{cases} \int_0^T |\dot{x}(s)|^2 ds & , \text{if } x \in {}_0H^{1,2} \\ \infty & , \text{otherwise} \end{cases}$$

<u>D2</u>. Family $(\mu^\varepsilon)_{\varepsilon > 0}$ satisfies LDP on $\mathcal{X}_T$ with a good rate function I iff $(\sim)$

$$(6.3) \qquad \mu_\varepsilon(A) \sim \exp\left[-\frac{1}{\varepsilon} \inf_{x \in A} I(x)\right]$$

<u>Example 2</u> If w is a BM on $[0,T]$

and $\mu^\varepsilon = \text{Law}(\sqrt{\varepsilon}\, w)$

then μ^ε satisfies LDP on X_T

with I as in Example 1.

Theorem 6.1 (ZB + U Manna + A Panda, 2019)

$$(6.4)\quad \begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + A u^\varepsilon + B u^\varepsilon + \operatorname{div}(\nabla n^\varepsilon \odot \nabla n^\varepsilon) = \sqrt{\varepsilon}\, dW \\ \dfrac{\partial n^\varepsilon}{\partial t} + F'(n^\varepsilon) - \Delta n^\varepsilon + u^\varepsilon \cdot \nabla n^\varepsilon \\ \qquad = \sqrt{\varepsilon}\left(-\lambda\, n^\varepsilon \times d\tilde{w}\right) \end{cases}$$

The family

$$\mu^\varepsilon = \mathrm{Law}(u^\varepsilon, \eta^\varepsilon)$$

satisfies LDP on

$$X_T = [C([0,T];H) \cap L^2(0,T;V)] \times C([0,T], H^1) \cap L^2(0,T;H^2)$$

with rate functional I defined as follows.

Let K, $\tilde{K}$ be RKHS associated with W, $\tilde{W}$. Put

(6.5) $$S = L^2(0,T;K) \times L^2(0,T,\tilde{K})$$

If $(f,g) \in S$, $J^0(f,g) = (u,n)$

iff

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Skeleton equation

$$(6.6)\quad \begin{cases} \dfrac{\partial u}{\partial t} + Au + Bu + \operatorname{div}(\nabla n \odot \nabla n = f \\ \dfrac{\partial n}{\partial t} + F'(n - \Delta n) + \ + u \cdot \nabla n \\ \qquad = \ (-\lambda n \times g) \end{cases}$$

$u(0) = u_0 \ , \ \eta(0) = n_0$

$$I((u,n)) := \inf\left\{ \frac{1}{2}\|(f,g)\|^2_{L^2(0,T;\mathbb{R})\times L^2(0,T;\mathbb{R})} \; : \; (u,n) = J^0(f,g) \right\} \tag{6.7}$$

Corollary If $A \subset X_T$ is a good set then $\mu_\varepsilon(A) > 0 \quad \forall\, \varepsilon > 0$.

Proof is based on the Laplace principle version of LDP proved by Budhiraja + Dupuis:

Two results are needed.

Lemma B. A stochastic version of Lemma A

Lemma A. If $(f_n, g_n) \to (f, g)$ weakly in $L^2(0,T;K) \times L^2(0,T;\tilde{K})$ then $J(f_n, g_n) \to J(f, g)$ strongly in X_T.

VII. Theorem 6.1 has been generalised (ZB, PR + GD) to the ELEs.

The skeleton equation corresponding to (5.2):

$$(7.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} + Au + Bu + \operatorname{div}(\nabla n \odot \nabla n) = f \\ \dfrac{\partial n}{\partial t} + n \times (n \times \Delta n) + + u \cdot \nabla n \\ \qquad = -\lambda n \times g \end{cases}$$

The second equation in (7.1) can be generalised to contain anisotropy energy $\phi(u)$, i.e.

$$(7.2) \quad \begin{cases} \frac{\partial u}{\partial t} + Au + Bu + \operatorname{div}(\nabla n \odot \nabla n) = f \\ \frac{\partial n}{\partial t} + n \times \left(n \times \left(\Delta n - \phi'(n)\right)\right) + u \cdot \nabla n \\ \qquad = -\lambda n \times g \end{cases}$$

$$\phi : S^2 \to [0, \infty)$$

e.g. $\phi(n) = (n \cdot H)^2, \quad H \in \mathbb{R}^3$

Th. 7.1 If $f \in L^2(0,\infty,V')$
and $g \in L^2(0,\infty,L^2)$
then (7.2) has a unique strong Struwe solution with a finite number of singular times.
If $\|f\|, \|g\|$ are small enough then the solution is regular, i.e. without singular times.

Moreover, if $y_0 \in S^2$ is a strict local minimum of ϕ and $\bar{n}_0 \equiv y_0$ then $\exists\, \varepsilon > 0$: if $\| \bar{n}_0 - n_0 \|_{H^1} < \varepsilon$

$\| u_0 \|_{L^2} < \varepsilon$ then $\exists\, f, g \in L^2(-\infty, 0, \dots)$ and solutions (u, n) of (7.2) on $(-\infty, 0)$ s.th.

$$(u, n)(-\infty) = (0, \bar{n}_0)$$

$$(u, n)(0) = (u_0, n_0)$$

Moreover,

$$\frac{1}{2}\inf\left[\int_{-\infty}^{0}|f(s)|_{V'}^{2}\,ds+\int_{-\infty}^{0}|g(s)|_{L^2}^{2}\,ds\right]$$

$$=\mathcal{E}(u_0,n_0).$$

This result generalizes a result by Z. B, S. Cerrai and M Freidlin (2015) for 2-D SNE on a torus and by ZB, L. Li and EH (2019) for 1-D LLGEs.

Th. 7.2 LDP for Stochastic ELE's holds.

Thank you.