Global solutions for one dimensional dispersive models

Daniel Tataru

University of California, Berkeley

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This is joint work with Mihaela Ifrim

Dispersive problems in 1D:

$$i\partial_t u - A(D_x)u = N(u), \qquad u(0) = u_0$$

Dispersion relation:

$$\tau = -a(\xi)$$

Group velocity:

$$v_\xi = a'(\xi)$$

Dispersive models:

$$a''(\xi) \neq 0$$

- NLS: $a(\xi) = \xi^2$
- KdV: $a(\xi) = \xi^3$
- Deep gravity waves $a(\xi) = |\xi|^{\frac{1}{2}}$
- Deep gravity waves $a(\xi) = |\xi|^{\frac{3}{2}}$
- Shallow gravity waves $a(\xi) = \sqrt{\xi \tanh \xi}$
- Shallow capillary waves: $a(\xi) = \sqrt{\xi^3 \tanh \xi}$

The nonlinearity

a) Classified by strength:

- semilinear (e.g. NLS3, KdV), Lipschitz dependence on data
- quasilinear (e.g. water waves), continuous dependence on data

b) Classified by leading homogeneity:

- quadratic,

$$N(u) = Q_1(u,u) + Q_2(u,\bar{u}) + Q_3(\bar{u},\bar{u})$$

- cubic, e,g.

$$N(u) = Q(u,\bar{u},u)$$

- higher order

c) Classified by leading order nonlinear effect (cubic case):

- defocusing
- focusing

The fundamental solution for the linear flow

Self-similar asymptotic behavior,

$$K(t,x) \approx \frac{1}{\sqrt{ta''(\xi_v)}} e^{it\phi}$$

where ϕ solves the eikonal equation

$$\phi_t + a(\phi_x) = 0$$

Based on the dispersion relation and the group velocity of waves

$$\tau + a(\xi) = 0, \qquad v = a'(\xi),$$

the (unique) selfsimilar solution $\phi(t,x) = t\psi(x/t)$ satisfies

$$\psi'(v) = \xi_v := [a']^{-1}(v)$$

Asymptotic solutions for the linear flow with localized data:

$$u(t,x) \approx \frac{1}{\sqrt{t}} \gamma(x/t) e^{it\psi(x/t)}$$

Asymptotic profile:

$$\gamma(v) = c(v)\hat{u}_0(\xi_v)$$

Dispersive decay for the linear equation

- Dispersive bounds:

$$\|u(t)\|_{L^\infty} \lesssim t^{-\frac{1}{2}} \|u(0)\|_{L^1}$$

- Strichartz estimates:

$$\|u\|_{L^6_{x,t}} + \|u\|_{L^4_t L^\infty_x} \lesssim \|u(0)\|_{L^2}$$

  - better for data in L^2 based spaces

- Bilinear L^2 bounds for frequency separated solutions:

$$\|uv\|_{L^2_{x,t}} \lesssim \|u_0\|_{L^2} \|v_0\|_{L^2}$$

  - transversality rather than dispersive bounds

Main Question: Does the nonlinear problem have global, *dispersive* solutions for small initial data ?

Scenario 1: The initial data is *small* and localized. Do we then have global solutions with dispersive, $t^{-\frac{1}{2}}$ decay ?

Scenario 2: The initial data is *small* but unlocalized (e.g. H^s). Do we then have global solutions with L^6 Strichartz decay ?

Quadratic vs. cubic nonlinearities

a) Quadratic problems

$$i\partial_t u - A(D_x)u = Q_2(u, u)$$

- stronger nonlinear interactions for small data
- resonance analysis for two wave resonances
- Two favourable cases:
  - nonresonant structure
  - null structure (resonant interactions are killed by the nonlinearity)

b) Cubic problems

$$i\partial_t v - A(D_x)v = Q_2(v, v, v)$$

- weaker nonlinear interactions for small data
- resonance analysis for three wave resonances: many resonant interactions

Normal form methods

From (good) quadratic to cubic analysis:

1. Normal form transformations (Shatah)

$$v = u + B(u, u)$$

   - works for some semilinear problems, but unbounded for quasilinear problems

2. (Quasilinear) Modified energy methods (Hunter-Ifrim-T.): Modify the energy functionals rather than the solutions, to produce cubic energies

$$\frac{d}{dt} E^s(u) \lesssim \|u\|^2 E^s(u)$$

   - works for some quasilinear problems, e.g. gravity waves, capillary waves

3. Paradiagonalization (Alazard-Delort) Apply a partial NFT combined with a paradifferential symmetrization
   - e.g. for gravity waves

A cubic model

Divide and conquer strategy: avoiding quasilinear features and quadratic normal forms, we consider a cubic model

$$i\partial_t u - A(D_x)u = Q(u, \bar{u}, u)$$

with a cubic, translation and phase rotation invariant nonlinearity:

$$\widehat{Q(u, \bar{u}, u)}(\xi) = \int_{\xi=\xi_1-\xi_2+\xi_3} q(\xi_1, \xi_2, \xi_3)\hat{u}(\xi_1)\overline{\hat{u}(\xi_2)}\hat{u}(\xi_3)d\sigma$$

- Conservative nonlinearity:

$$q(\xi, \xi, \xi) \in \mathbb{R}$$

- Focusing/defocusing character given by sign of $q(\xi, \xi, \xi)$ relative to the sign of $a''(\xi)$.
- Semilinear/quasilinear character given by the size of $q(\xi_1, \xi_2, \xi_3)$ relative to the size of a for unbalanced ξ's.

Part I: the localized data result

Theorem (Small and localized data)

Suppose q is conservative, and

$$\|u_0\|_{H^{s_0}} + \|xu_0\|_{H^{s_1}} \ll 1$$

Then the solution u is global, and has global dispersive decay

$$\|u(t)\|_{L^\infty} \lesssim t^{-\frac{1}{2}}$$

- Recent expository notes, Ifrim-T.
  - Simplest case: q has compact support, $s_0 = s_1 = 0$
  - General case, q global, bounded, $a'' \approx |\xi|^\sigma$, $\sigma \in \mathbb{R}$
- Many contributions over time: Hayashi-Naumkin, Lindblad-Soffer, Kato-Pusateri, Alazard-Delort, Ifrim-T., Delort, etc.

Linear dispersion via vector fields

$$iu_t - A(D)u = 0$$

Conserved quantity

$$\|u\|_{L^2} = \text{const}$$

Vector field (via Egorov)

$$L(x,D) = x - ta_\xi(D), \qquad [L,P] = 0$$

Second conserved quantity:

$$\|Lu\|_{L^2} = \text{const}$$

Dispersion via energy estimates:

$$\|u(t)\|_{L^\infty} \lesssim t^{-\frac{1}{2}} \|u\|_{L^2} \|Lu\|_{L^2}$$

Nonlinear dispersion via vector fields

1. Bootstrap assumption:

$$\|u\|_{L^\infty} \lesssim \epsilon t^{-\frac{1}{2}}$$

2. Energy estimates for u

$$\frac{d}{dt}\|u\|_{L^2}^2 \lesssim \|u\|_{L^\infty}^2 \|u\|_{L^2}^2$$

3. Energy estimates for $L^{NL}u$

$$\frac{d}{dt}\|L^{NL}u\|_{L^2}^2 \lesssim \|u\|_{L^\infty}^2 \|Lu\|_{L^2}^2$$

where L^{NL} is a nonlinear correction of L,

$$L^{NL}u = Lu + tB(u, \bar{u}, u)$$

4. Gronwall:

$$\|u\|_{L^2} + \|Lu\|_{L^2} \lesssim t^{C\epsilon^2}$$

5. (Possibly nonlinear) vectorfield bound:

$$\|u\|_{L^\infty} \lesssim t^{-\frac{1}{2}}(\|u\|_{L^2} + \|L^{NL}u\|_{L^2})$$

The asymptotic profile and the asymptotic equation

Objective: Close the bootstrap for $\|u\|_{L^\infty}$.

Ansatz inspired by linear behavior:

$$u(t,x) \approx \frac{1}{\sqrt{t}}\gamma(t,v)e^{it\psi(v)}, \qquad v = x/t$$

Then we need a uniform bound for the *asymptotic profile* γ

Modified scattering: We have to allow a slow time dependence on γ, which should approximately solve an *asymptotic equation*

$$\dot{\gamma}(t,v) \approx -iq(\xi_v,\xi_v,\xi_v)t^{-1}\gamma(t,v)|\gamma(t,v)|^2$$

The global boundedness of gamma follows if $q(\xi_v,\xi_v,\xi_v)$ is real.

Difficulty: We need to make a good choice for γ in terms of the solution u.

Asymptotic equations in NLS context

A. Hayashi-Naumkin, refined by Kato-Pusateri; derive an asymptotic equation for the Fourier transform of the solutions,

$$\frac{d}{dt}\hat{u}(t,\xi) = \lambda i t^{-1}\,\hat{u}(t,\xi)|\hat{u}(t,\xi)|^2 + O_{L^\infty}(t^{-1-\epsilon}).$$

B. Lindblad-Soffer; derive an asymptotic equation in the physical space along rays,

$$(t\partial_t + x\partial_x)u(t,x) = \lambda i t u(t,x)|u(t,x)|^2 + O_{L^\infty}(t^{-\epsilon}).$$

C. Deift-Zhou used complete integrability and the inverse scattering method to obtain long range asymptotics

D. Ifrim-T. *wave packet testing*: test the NLS solution with an approximate wave packet type linear wave expanding on the $t^{\frac{1}{2}}$ spatial scale.

Wave packets with time dependent scale

Spatial scales associated to time scale t at velocity v and associated frequency $a'(\xi_v) = v$:

$$\delta x = t^{\frac{1}{2}} a_{\xi\xi}(\xi_v)^{\frac{1}{2}}, \quad \delta\xi = t^{-\frac{1}{2}} a_{\xi\xi}^{-\frac{1}{2}}(\xi_v).$$

Global in time wave packets with time dependent scale:

$$\mathbf{u}_v = a_{\xi\xi}(\xi_v)^{\frac{1}{2}} \chi \left(\frac{x - vt}{t^{\frac{1}{2}} a_{\xi\xi}(\xi_v)^{\frac{1}{2}}} \right) e^{i\phi}$$

Good approximate solutions for the linear flow on dyadic time scales:

$$(i\partial_t - A(D))\mathbf{u}_v = O(t^{-1})$$

Better estimate using quadratic expansion of $a(\xi)$ near $\xi = \xi_v$:

$$(i\partial_t - A(D))\mathbf{u}_v = ct^{-2} L\mathbf{u}_v + O(t^{-\frac{3}{2}})$$

Wave packet testing

The asymptotic profile function γ:

$$\gamma(t,v) = \langle \mathbf{u}_v, u \rangle_{L^2}$$

Main estimates:

- Good approximation for u:

$$u(t,x) = \gamma(t,v) t^{-\frac{1}{2}} e^{i\phi} + O(t^{-\frac{5}{8}})$$

- Asymptotic equation for γ:

$$\dot{\gamma}(t,v) = iq(\xi_v, \xi_v, \xi_v) t^{-1} \gamma(t,v) |\gamma(t,v)|^2 + O(\epsilon t^{-1-}), \quad \gamma(1,\alpha) = O(\epsilon)$$

Further goals:

- Modified scattering:

$$u(t,x) = W(v) t^{-\frac{1}{2}} e^{i\phi} e^{iq(\xi_v,\xi_v,\xi_v)|W(v)|^2 \log t} + O(t^{-\frac{5}{8}})$$

- Asymptotic completeness.

Part II: the non-localized data result

Theorem (Small and non-localized data)

Suppose q is conservative and defocusing.

$$\|u_0\|_{L^2} \leq \epsilon \ll 1$$

Then the solution u is global, and satisfies global Strichartz estimates

$$\|u(t)\|_{L^6} \lesssim \epsilon^{\frac{2}{3}}$$

and bilinear estimates

$$\|P_A u P_B u\|_{L^2} \lesssim \epsilon^2 d(A,B)^{-\frac{1}{2}}$$

- work in progress, Ifrim-T.
- First result of this type
- The estimates are new even for NLS3 (weaker bounds by Planchon-Vega in this case)

The bootstrap set-up

Bootstrap assumption based on unit scale frequency decomposition

$$u = \sum u_j$$

with *slowly varying* frequency envelope $\{c_j\}$ so that

$$\|u_{0j}\|_{L^2} \lesssim c_j$$

Then assume that

$$(BOOT1) \qquad \|u_j\|_{L^\infty L^2} \lesssim \epsilon c_j$$

$$(BOOT2) \qquad \|u_j(t)\|_{L^6} \lesssim (\epsilon c_j)^{\frac{2}{3}}$$

$$(BOOT3) \qquad \|\partial_x(u_j \bar{u}_k)\|_{L^2} \lesssim \epsilon^2 (1 + |j-k|)^{\frac{1}{2}}$$

- bootstraping both Strichartz and bilinear: Ifrim-T., Benjamin Ono

Localized density flux identities

a) Linear /nonlinear case:

$$\partial_t m_j(u, \bar{u}) = \partial_x p_j(u, \bar{u}) + \text{quartic}$$

$$\partial_t p_j(u, \bar{u}) = \partial_x e_j(u, \bar{u}) + \text{quartic}$$

b) Nonlinear case, modified energies

$$m_j^\sharp(u, \bar{u}) = m_j(u, \bar{u}) + B^4_{j,p}(u, \bar{u}, u, \bar{u}), \quad p_j^\sharp(u, \bar{u}) = p_j(u, \bar{u}) + B^4_{j,p}(u, \bar{u}, u, \bar{u}),$$

Density flux identities:

$$\partial_t m_j^\sharp(u, \bar{u}) = \partial_x (p_j(u, \bar{u}) + R^4_{j,m}(u, \bar{u}, u, \bar{u})) + R^6_{j,m}(u, \bar{u}, u, \bar{u}, u, \bar{u})$$

$$\partial_t p_j^\sharp(u, \bar{u}) = \partial_x (e_j(u, \bar{u}) + R^4_{j,p}(u, \bar{u}, u, \bar{u})) + R^6_{j,p}(u, \bar{u}, u, \bar{u}, u, \bar{u})$$

- This requires solving a complex division problem
- Energy bounds follow by direct integration

Interaction Morawetz bounds

a) Interaction Morawetz functional:

$$I(u_j, u_j) = \int_{x<y} m_j^\sharp(x) p_j^\sharp(y) - m_j^\sharp(y) p_j^\sharp(x) dx dy$$

Time differentiation:

$$\frac{d}{dt} I(u_j, u_j) \approx \|\partial_x (u_j \bar{u}_j)\|_{L^2}^2 + \|u_j\|_{L^6}^6 + \text{Errors (6,8,10)}$$

This proves the L^6 Strichartz and diagonal bilinear L^2.

b) Transversal Interaction Morawetz functional:

$$I(u_j, u_k) = \int_{x<y} m_j^\sharp(x) p_k^\sharp(y) - m_k^\sharp(y) p_j^\sharp(x) dx dy$$

Time differentiation:

$$\frac{d}{dt} I(u_j, u_k) \approx \|\partial_x (u_j \bar{u}_k)\|_{L^2}^2 + \text{Errors (6,8,10)}$$

This proves the off-diagonal bilinear L^2.

Thank you !