On the Navier-Stokes equations on surfaces

Gieri Simonett

Vanderbilt University

Joint work with Jan Prüss† and Mathias Wilke

Non-linear PDEs in Fluid Dynamics

CIRM, Luminy

May 9-13, 2022

The setting:

- Σ is a smooth, compact, embedded (oriented) hypersurface in $\mathbb{R}^{d+1}$ without boundary;

Setting I

The setting:

- Σ is a smooth, compact, embedded (oriented) hypersurface in $\mathbb{R}^{d+1}$ without boundary;
- $\mathsf{T}\Sigma$ = the tangent bundle; ν_Σ = unit normal field;
- $\mathcal{P}_\Sigma$ = the orthogonal projection onto $\mathsf{T}\Sigma$;
- ∇_Σ, div_Σ = the surface gradient and surface divergence, respectively;

Setting I

The setting:

- Σ is a smooth, compact, embedded (oriented) hypersurface in $\mathbb{R}^{d+1}$ without boundary;
- $\mathsf{T}\Sigma =$ the tangent bundle; $\nu_\Sigma =$ unit normal field;
- $\mathcal{P}_\Sigma =$ the orthogonal projection onto $\mathsf{T}\Sigma$;
- ∇_Σ, $\mathrm{div}_\Sigma =$ the surface gradient and surface divergence, respectively;
- $\varrho =$ density (constant); $u =$ velocity field; $\mathcal{T}_\Sigma =$ surface stress tensor.

Setting I

The setting:

- Σ is a smooth, compact, embedded (oriented) hypersurface in $\mathbb{R}^{d+1}$ without boundary;
- $\mathsf{T}\Sigma$ = the tangent bundle; ν_Σ = unit normal field;
- $\mathcal{P}_\Sigma$ = the orthogonal projection onto $\mathsf{T}\Sigma$;
- ∇_Σ, div_Σ = the surface gradient and surface divergence, respectively;
- ϱ = density (constant); u = velocity field; $\mathcal{T}_\Sigma$ = surface stress tensor.

Surface Navier-Stokes equations (incompressible):

$$\begin{aligned} \varrho\big(\partial_t u + \mathcal{P}_\Sigma(u \cdot \nabla_\Sigma u)\big) - \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma &= 0 \quad \text{on } \Sigma \\ \operatorname{div}_\Sigma u &= 0 \quad \text{on } \Sigma \\ u(0) &= u_0 \quad \text{on } \Sigma. \end{aligned} \tag{1}$$

Setting II

Surface stress tensor $\mathcal{T}_\Sigma$ via Boussinesq-Scriven:

$$\mathcal{T}_\Sigma = \mathcal{T}_\Sigma(u,\pi) = 2\mu_s \mathcal{D}_\Sigma(u) + (\lambda_s - \mu_s)(\mathrm{div}_\Sigma u)\mathcal{P}_\Sigma - \pi \mathcal{P}_\Sigma,$$

where $\mu_s > 0$ is the surface shear viscosity, $\lambda_s > 0$ is the surface dilatational viscosity, π the pressure, and

$$\mathcal{D}_\Sigma(u) := \frac{1}{2}\mathcal{P}_\Sigma \left(\nabla_\Sigma u + [\nabla_\Sigma u]^{\mathsf{T}}\right)\mathcal{P}_\Sigma$$

is the *surface rate-of-strain tensor.*

The model I

$$\begin{aligned} \varrho\big(\partial_t u + \mathcal{P}_\Sigma(u\cdot\nabla_\Sigma u)\big) - \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma = 0 &\quad \text{on } \Sigma \\ \operatorname{div}_\Sigma u = 0 &\quad \text{on } \Sigma \\ u(0) = u_0 &\quad \text{on } \Sigma. \end{aligned}$$

- If $u_0 \in \mathsf{T}\Sigma$, then $u(t) \in \mathsf{T}\Sigma$, $t > 0$.

The model I

$$
\begin{aligned}
\varrho\big(\partial_t u + \mathcal{P}_\Sigma(u\cdot\nabla_\Sigma u)\big) - \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma = 0 \quad &\text{on } \Sigma \\
\operatorname{div}_\Sigma u = 0 \quad &\text{on } \Sigma \\
u(0) = u_0 \quad &\text{on } \Sigma.
\end{aligned}
$$

- If $u_0 \in \mathsf{T}\Sigma$, then $u(t) \in \mathsf{T}\Sigma$, $t > 0$.
- It holds

$$
-\mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma = -\mu_s(\Delta_\Sigma u + \operatorname{Ric}_\Sigma u) + \nabla_\Sigma \pi,
$$

$$\varrho(\partial_t u + \mathcal{P}_\Sigma(u \cdot \nabla_\Sigma u)) - \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma = 0 \quad \text{on } \Sigma$$
$$\operatorname{div}_\Sigma u = 0 \quad \text{on } \Sigma$$
$$u(0) = u_0 \quad \text{on } \Sigma.$$

- If $u_0 \in \mathsf{T}\Sigma$, then $u(t) \in \mathsf{T}\Sigma$, $t > 0$.
- It holds
$$-\mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma = -\mu_s(\Delta_\Sigma u + \mathrm{Ric}_\Sigma u) + \nabla_\Sigma \pi,$$
where
  - Δ_Σ is the (negative) Bochner-Laplacian and
  - Ric_Σ is the Ricci curvature tensor.

The model I

$$\begin{aligned} \varrho\big(\partial_t u + \mathcal{P}_\Sigma(u \cdot \nabla_\Sigma u)\big) - \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma &= 0 \quad \text{on } \Sigma \\ \operatorname{div}_\Sigma u &= 0 \quad \text{on } \Sigma \\ u(0) &= u_0 \quad \text{on } \Sigma. \end{aligned}$$

- If $u_0 \in \mathsf{T}\Sigma$, then $u(t) \in \mathsf{T}\Sigma$, $t > 0$.
- It holds

$$-\mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma = -\mu_s(\Delta_\Sigma u + \operatorname{Ric}_\Sigma u) + \nabla_\Sigma \pi,$$

  where
  - Δ_Σ is the (negative) Bochner-Laplacian and
  - $\operatorname{Ric}_\Sigma$ is the Ricci curvature tensor.

  In case $d = 2$, $\operatorname{Ric}_\Sigma u = K_\Sigma u$, where K_Σ is the Gaussian curvature of Σ (the product of the principal curvatures).

$$\begin{aligned} \varrho(\partial_t u + \mathcal{P}_\Sigma(u \cdot \nabla_\Sigma u)) - \mu_s(\Delta_\Sigma u + \mathrm{Ric}_\Sigma u) + \nabla_\Sigma \pi = 0 &\quad \text{on } \Sigma \\ \mathrm{div}_\Sigma u = 0 &\quad \text{on } \Sigma \\ u(0) = u_0 &\quad \text{on } \Sigma. \end{aligned}$$

Remarks:

- $\mathcal{P}_\Sigma(u \cdot \nabla_\Sigma u) = \nabla_u u$, where ∇ is the Levi-Civita connection of Σ;
- $\mathcal{D}_\Sigma u = \frac{1}{2}\left(\nabla u + [\nabla u]^{\mathsf{T}}\right)$,
- $\Delta_\Sigma = (\Delta_H + \mathrm{Ric}_\Sigma)$,

  where Δ_H is the Hodge Laplacian (also called Laplace-de Rham operator), acting on 1-forms.

The model II

$$
\begin{aligned}
\varrho\big(\partial_t u + \mathcal{P}_\Sigma(u\cdot\nabla_\Sigma u)\big) - \mu_s(\Delta_\Sigma u + \mathrm{Ric}_\Sigma u) + \nabla_\Sigma \pi = 0 \quad &\text{on } \Sigma\\
\mathrm{div}_\Sigma u = 0 \quad &\text{on } \Sigma\\
u(0) = u_0 \quad &\text{on } \Sigma.
\end{aligned}
$$

Remarks:

- $\mathcal{P}_\Sigma(u\cdot\nabla_\Sigma u) = \nabla_u u$, where ∇ is the Levi-Civita connection of Σ;
- $\mathcal{D}_\Sigma u = \frac{1}{2}\left(\nabla u + [\nabla u]^{\mathsf{T}}\right)$,
- $\Delta_\Sigma = (\Delta_H + \mathrm{Ric}_\Sigma)$,

  where Δ_H is the Hodge Laplacian (also called Laplace-de Rham operator), acting on 1-forms.
- Surface Navier-Stokes equations:

$$
\begin{aligned}
\varrho\big(\partial_t u + \nabla_u u\big) - \mu_s(\Delta_H u + 2\mathrm{Ric}\, u) + \mathrm{grad}\, \pi = 0 \quad &\text{on } \Sigma\\
\mathrm{div} u = 0 \quad &\text{on } \Sigma\\
u(0) = u_0 \quad &\text{on } \Sigma.
\end{aligned}
$$

Selected Literature I

$$\begin{aligned} \varrho\big(\partial_t u + \mathcal{P}_\Sigma(u\cdot\nabla_\Sigma u)\big) - \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma &= 0 \quad \text{on } \Sigma \\ \operatorname{div}_\Sigma u &= 0 \quad \text{on } \Sigma \\ u(0) &= u_0 \quad \text{on } \Sigma. \end{aligned}$$

Justification of model:

- [Jankuhn-Olshanskii-Reusken, '18] Incompressible fluid problems on embedded surfaces: modeling and variational formulations.
- [Koba-Liu-Giga, '17] Energetic variational approaches for incompressible fluid systems on an evolving surface.

Numerical Analysis:

- [Jankuhn, Olshanskii, Quaini, Reusken, Voigt, Yushutin, '18-...]

Selected Literature II

$$\varrho(\partial_t u + \nabla_u u) - \mu_s(\Delta_H u + \boxed{2\mathrm{Ric}\, u}) + \mathrm{grad}\,\pi = 0 \quad \text{on } \Sigma$$
$$\mathrm{div}\, u = 0 \quad \text{on } \Sigma$$
$$u(0) = u_0 \quad \text{on } \Sigma.$$

- [Ebin-Marsden, '70]: Note added in Proof.
- [Taylor, '92]: Existence and uniqueness for initial data in Morrey spaces, global existence for 2d surfaces.
- [Mazzucato, 03]: Existence and uniqueness for initial data in Besov-Morrey spaces.
- [Chan-Czubak, '13-15] Navier-Stokes on hyperbolic spaces.
- [Chan-Czubak-Disconzi, '17] Discussion and evaluation of different models for NS on manifolds.

Related work:

$$\begin{aligned} \varrho(\partial_t u + \nabla_u u) - \mu_s \Delta_H u + \operatorname{grad} \pi &= 0 \quad \text{on } \Sigma \\ \operatorname{div} u &= 0 \quad \text{on } \Sigma \\ u(0) &= u_0 \quad \text{on } \Sigma. \end{aligned}$$

- [Il'in, Il'in-Filatov, '89-'94]: $\dim \Sigma = 2$: Existence and uniqueness of generalized solutions.
- [Cao-Rammaha-Titi, '99]:

$$\begin{aligned} \varrho(\partial_t u + \nabla_u u) - \mu_s \Delta_H u + \gamma\, \nu_{\mathbb{S}^2} \times u + \operatorname{grad} \pi &= f \quad \text{on } \mathbb{S}^2 \\ \operatorname{div} u &= 0 \quad \text{on } \mathbb{S}^2 \\ u(0) &= u_0 \quad \text{on } \mathbb{S}^2. \end{aligned}$$

  NS on rotating $\mathbb{S}^2$: Gevrey regularity for $t > 0$, degrees of freedom.
- [Foias-Temam, '89]: 2d NS on periodic domains: Gevrey regularity for $t > 0$.

Surface Navier-Stokes equations:

$$\begin{aligned} \varrho(\partial_t u + \nabla_u u) - \mu_s(\Delta_\Sigma u + \mathrm{Ric}_\Sigma\, u) + \nabla_\Sigma \pi = 0 \quad &\text{on } \Sigma \\ \mathrm{div}_\Sigma u = 0 \quad &\text{on } \Sigma \\ u(0) = u_0 \quad &\text{on } \Sigma. \end{aligned}$$

In local coordinates:

Existence and Uniqueness I

Surface Navier-Stokes equations:

$$\begin{aligned} \varrho\big(\partial_t u + \nabla_u u\big) - \mu_s(\Delta_\Sigma u + \mathrm{Ric}_\Sigma\, u) + \nabla_\Sigma \pi = 0 &\quad \text{on } \Sigma \\ \mathrm{div}_\Sigma u = 0 &\quad \text{on } \Sigma \\ u(0) = u_0 &\quad \text{on } \Sigma. \end{aligned}$$

In local coordinates:

$$\begin{aligned} \Delta_\Sigma u &= g^{ij}(\nabla_i \nabla_j - \Lambda_{ij}^k \nabla_k)u, \\ \mathrm{Ric}_\Sigma u &= R^i_j u^j \frac{\partial}{\partial x^i}, \\ \nabla_\Sigma \pi &= g^{ij}\partial_j \pi \frac{\partial}{\partial x^i}, \end{aligned}$$

where

- ∇_i are covariant derivatives,
- Λ_{ij}^k are the Christoffel symbols.

Locally, this results in *generalized Stokes equations* in $\mathbb{R}^d$.

Existence and Uniqueness II

- The surface Stokes operator

$$Au := -2\,\mathbb{P}\,\mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u) = -\mu_s \mathbb{P}(\Delta_\Sigma + \mathrm{Ric}_\Sigma)u, \quad u \in H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma).$$

A has maximal regularity in $L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$. [Prüss, S, Wilke '21].

Existence and Uniqueness II

- The surface Stokes operator

$$Au := -2\,\mathbb{P}\,\mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u) = -\mu_s \mathbb{P}(\Delta_\Sigma + \mathrm{Ric}_\Sigma)u, \quad u \in H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma).$$

  A has maximal regularity in $L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$. [Prüss, S, Wilke '21].

- By the contraction mapping principle, for any $u_0 \in B^{2-2/p}_{qp}(\Sigma, \mathsf{T}\Sigma)$ with $\mathrm{div}_\Sigma\, u_0 = 0$ there exists a number $a = a(u_0) > 0$ and a unique solution

$$u \in H^1_p((0,a); L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap L_p((0,a); H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)),$$

$$\pi \in L_p((0,a); \dot{H}^1_q(\Sigma))$$

  of the surface Navier Stokes equations (1), provided p, q are subject to additional conditions.

- Time weights of Muckenhoupt type allow to decrease the initial regularity.

- A admits a bounded H^∞-calculus in $L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$. [S, Wilke '21].

Existence and Uniqueness III

- A admits a bounded H^∞-calculus in $L_{q,\sigma}(\Sigma, \mathrm{T}\Sigma)$. [S, Wilke '21].
- **Critical spaces:**

$$u_0 \in B^{d/q-1}_{qp,\sigma}(\Sigma, \mathrm{T}\Sigma) \quad \text{where} \quad \begin{cases} 2/p + d/q \le 3, \;\; q \in (d/3, d) & \text{strong setting,} \\ 2/p + d/q \le 2, \;\; q \in (d/2, \infty) & \text{weak setting.} \end{cases}$$

The Sobolev index is always -1, independent of q, p.

- A admits a bounded H^∞-calculus in $L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$. [S, Wilke '21].

- **Critical spaces:**

$$u_0 \in B^{d/q-1}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma) \quad \text{where} \quad \begin{cases} 2/p + d/q \le 3, \ \ q \in (d/3, d) & \text{strong setting,} \\ 2/p + d/q \le 2, \ \ q \in (d/2, \infty) & \text{weak setting.} \end{cases}$$

The Sobolev index is always -1, independent of q, p.

- $d = 3, \quad u_0 \in H^{1/2}_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$. 'Fujita-Kato'.

  $d = 2, \quad u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Energy dissipation and equilibria

$$\begin{aligned} \partial_t u + \nabla_u u - 2\mu_s \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{D}_\Sigma(u) + \nabla_\Sigma \pi = 0 \quad &\text{on } \Sigma \\ \operatorname{div}_\Sigma u = 0 \quad &\text{on } \Sigma \\ u(0) = u_0 \quad &\text{on } \Sigma. \end{aligned}$$

Energy dissipation:

$$\mathsf{E}(t) := \int_\Sigma \frac{1}{2}|u(t)|^2 \, d\Sigma, \qquad \frac{d}{dt}\mathsf{E}(t) = -2\mu_s \int_\Sigma |\mathcal{D}_\Sigma(u(t))|^2 \, d\Sigma.$$

If u is an equilibrium, then $\mathcal{D}_\Sigma(u) = 0$.

Energy dissipation and equilibria

$$
\begin{aligned}
\partial_t u + \nabla_u u - 2\mu_s \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{D}_\Sigma(u) + \nabla_\Sigma \pi = 0 \quad &\text{on } \Sigma \\
\operatorname{div}_\Sigma u = 0 \quad &\text{on } \Sigma \\
u(0) = u_0 \quad &\text{on } \Sigma.
\end{aligned}
$$

Energy dissipation:

$$
\mathsf{E}(t) := \int_\Sigma \frac{1}{2}|u(t)|^2 \, d\Sigma, \qquad \frac{d}{dt}\mathsf{E}(t) = -2\mu_s \int_\Sigma |\mathcal{D}_\Sigma(u(t))|^2 \, d\Sigma.
$$

If u is an equilibrium, then $\mathcal{D}_\Sigma(u) = 0$.

This readily implies $\nabla_\Sigma \pi = \frac{1}{2}\nabla_\Sigma |u|^2$.

Set of equilibria:

$$
\mathfrak{E} = \Big\{ (u,\pi) : \operatorname{div}_\Sigma u = 0,\ \mathcal{D}_\Sigma(u) = 0,\ \pi = \frac{1}{2}|u|^2 + c \Big\}.
$$

Set of equilibria:

$$\mathfrak{E} = \Big\{(u, \pi) : \mathrm{div}_\Sigma u = 0,\ \mathcal{D}_\Sigma(u) = 0,\ \pi = \frac{1}{2}|u|^2 + c\Big\}.$$

A tangential field is called a Killing field if

$$(\nabla_v u|w) + (\nabla_w u|v) = 0 \quad \text{for all tangential fields } v, w \text{ on } \Sigma,$$

where ∇ is the Levi-Civita connection.

Equilibria

Set of equilibria:

$$\mathfrak{E} = \Big\{(u,\pi) : \mathrm{div}_\Sigma u = 0,\ \mathcal{D}_\Sigma(u) = 0,\ \pi = \frac{1}{2}|u|^2 + c\Big\}.$$

A tangential field is called a Killing field if

$$(\nabla_v u|w) + (\nabla_w u|v) = 0 \quad \text{for all tangential fields } v, w \text{ on } \Sigma,$$

where ∇ is the Levi-Civita connection.

It holds

$$(\mathcal{D}_\Sigma(u)v|w) + (\mathcal{D}_\Sigma(u)w|v) = (\nabla_v u|w) + (\nabla_w u|v),$$

hence

$$\mathcal{D}_\Sigma(u) = 0 \iff u \text{ is a Killing field.}$$

Killing fields I

Some properties of Killing fields:

- Killing fields on a Riemannian manifold form a sub Lie-algebra of the Lie-algebra of all tangential fields.

Some properties of Killing fields:

- Killing fields on a Riemannian manifold form a sub Lie-algebra of the Lie-algebra of all tangential fields.
- Killing fields of a Riemannian manifold (M, g) are the infinitesimal generators of the isometries $I(M, g)$ on (M, g), that is, the generators of flows that are isometries on (M, g).

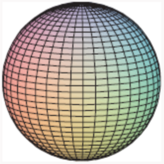
Some properties of Killing fields:

- Killing fields on a Riemannian manifold form a sub Lie-algebra of the Lie-algebra of all tangential fields.
- Killing fields of a Riemannian manifold (M, g) are the infinitesimal generators of the isometries $I(M, g)$ on (M, g), that is, the generators of flows that are isometries on (M, g).
- If (M, g) is compact and the Ricci tensor is negative definite everywhere, then any Killing field on M is equal to zero and $I(M, g)$ is a finite group.

Some properties of Killing fields:

- Killing fields on a Riemannian manifold form a sub Lie-algebra of the Lie-algebra of all tangential fields.
- Killing fields of a Riemannian manifold (M, g) are the infinitesimal generators of the isometries $I(M, g)$ on (M, g), that is, the generators of flows that are isometries on (M, g).
- If (M, g) is compact and the Ricci tensor is negative definite everywhere, then any Killing field on M is equal to zero and $I(M, g)$ is a finite group.
- Dimension is less or equal to $d(d+1)/2$ with equality if and only if Σ is compact and isomorphic to $\mathbb{S}^d$.

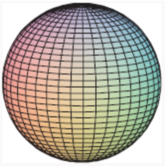
(a) Sphere:



$\Sigma = \mathbb{S}^2$: $\dim \mathcal{E} = 3$.

$$u_* \in \mathcal{E} \iff u_*(x) = \omega \times x, \quad x \in \mathbb{S}^2,\ \omega \in \mathbb{R}^3.$$

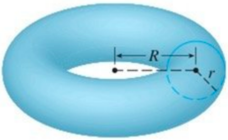
(a) Sphere:



$\Sigma = \mathbb{S}^2$: $\dim \mathcal{E} = 3$.

$$u_* \in \mathcal{E} \Longleftrightarrow u_*(x) = \omega \times x, \quad x \in \mathbb{S}^2,\ \omega \in \mathbb{R}^3.$$

(b) Torus:



$\Sigma = \mathbb{T}^2$: $\dim \mathcal{E} = 1$. Every equilibrium (Killing field) u_* is a rotation w.r.t. the z-axis.

Stability

Theorem (Prüss, S, Wilke, J. Evol. Eq. 2021)

Each solution that starts out close to an equilibrium $u_* \in \mathcal{E}$ exists globally and converges to a (possibly different) equilibrium $u_\infty \in \mathcal{E}$ at an exponential rate.

$u_\infty = P_{\mathcal{E}} u_0$, where $P_{\mathcal{E}}$ is the orthogonal projection of u_0 onto $\mathcal{E}$ (with respect to the L_2-inner product).

Proof:

- Each equilibrium is normally stable.
  Dimension of kernel of linearization at $u_* =$ dimension of $\mathcal{E}$.
- *Generalized principle of linearized stability* [Prüss, S, Zacher, '09].

Global existence for two-dimensional surfaces

Theorem (S, Wilke. 2021)

Suppose $d = 2$ *and* $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Theorem (S, Wilke. 2021)

Suppose $d = 2$ *and* $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Then the solution exists globally and converges exponentially fast to the equilibrium $u_* = P_{\mathcal{E}} u_0$ *in the topology of* $H^2_q(\Sigma, \mathsf{T}\Sigma)$ *for any fixed* $q \in (1, \infty)$.

Theorem (S, Wilke. 2021)

Suppose $d = 2$ and $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Then the solution exists globally and converges exponentially fast to the equilibrium $u_* = P_{\mathcal{E}} u_0$ in the topology of $H^2_q(\Sigma, \mathsf{T}\Sigma)$ for any fixed $q \in (1, \infty)$.

Proof:

- Critical spaces (critical weights),

Theorem (S, Wilke. 2021)

Suppose $d = 2$ *and* $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Then the solution exists globally and converges exponentially fast to the equilibrium $u_* = P_{\mathcal{E}} u_0$ *in the topology of* $H^2_q(\Sigma, \mathsf{T}\Sigma)$ *for any fixed* $q \in (1, \infty)$.

Proof:

- Critical spaces (critical weights),
- extrapolation scale,

Global existence for two-dimensional surfaces

Theorem (S, Wilke. 2021)

Suppose $d = 2$ *and* $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Then the solution exists globally and converges exponentially fast to the equilibrium $u_* = P_{\mathcal{E}} u_0$ *in the topology of* $H^2_q(\Sigma, \mathsf{T}\Sigma)$ *for any fixed* $q \in (1, \infty)$.

Proof:

- Critical spaces (critical weights),
- extrapolation scale,
- Korn's inequality (factoring out the Killing vector fields),

Global existence for two-dimensional surfaces

Theorem (S, Wilke. 2021)

Suppose $d = 2$ *and* $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Then the solution exists globally and converges exponentially fast to the equilibrium $u_* = P_{\mathcal{E}} u_0$ *in the topology of* $H^2_q(\Sigma, \mathsf{T}\Sigma)$ *for any fixed* $q \in (1, \infty)$.

Proof:

- Critical spaces (critical weights),
- extrapolation scale,
- Korn's inequality (factoring out the Killing vector fields),
- energy estimate,

Global existence for two-dimensional surfaces

Theorem (S, Wilke. 2021)

Suppose $d = 2$ and $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Then the solution exists globally and converges exponentially fast to the equilibrium $u_* = P_{\mathcal{E}} u_0$ in the topology of $H^2_q(\Sigma, \mathsf{T}\Sigma)$ for any fixed $q \in (1, \infty)$.

Proof:

- Critical spaces (critical weights),
- extrapolation scale,
- Korn's inequality (factoring out the Killing vector fields),
- energy estimate,
- parabolic regularization.

Global existence for two-dimensional surfaces

Theorem (S, Wilke. 2021)

Suppose $d = 2$ and $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Then the solution exists globally and converges exponentially fast to the equilibrium $u_* = P_{\mathcal{E}} u_0$ in the topology of $H^2_q(\Sigma, \mathsf{T}\Sigma)$ for any fixed $q \in (1, \infty)$.

Proof:

- Critical spaces (critical weights),
- extrapolation scale,
- Korn's inequality (factoring out the Killing vector fields),
- energy estimate,
- parabolic regularization.

Numerical simulations: Courtesy of Maxim Olshanskii, University of Houston.

- Webpage
- Sphere I
- Sphere I
- Torus

Some steps in the proof

Let

$$V_2^j(\Sigma) := \{v \in H^j_{2,\sigma}(\Sigma, \mathsf{T}\Sigma) \mid (v|z)_{L_2} = 0 \text{ for all } z \in \mathcal{E}\}, \quad j \in \{0,1\}.$$

Note that $H^j_{2,\sigma}(\Sigma, \mathsf{T}\Sigma) = \mathcal{E} \oplus V_2^j(\Sigma)$.

Some steps in the proof

Let

$$V_2^j(\Sigma) := \{v \in H^j_{2,\sigma}(\Sigma, \mathsf{T}\Sigma) \mid (v|z)_{L_2} = 0 \text{ for all } z \in \mathcal{E}\}, \quad j \in \{0,1\}.$$

Note that $H^j_{2,\sigma}(\Sigma, \mathsf{T}\Sigma) = \mathcal{E} \oplus V_2^j(\Sigma)$.

Proposition (S, Wilke. 2021)

Let $d = 2$. Suppose $v_0 \in V_2^0(\Sigma)$ and let v be the solution with initial value v_0. Then

(a) $v(t) \in V_2^0(\Sigma)$ *for* $t \in [0, t^+(v_0))$ *and* $v(t) \in V_2^1(\Sigma)$ *for* $t \in (0, t^+(v_0))$.

Some steps in the proof

Let
$$V_2^j(\Sigma) := \{v \in H^j_{2,\sigma}(\Sigma, \mathrm{T}\Sigma) \mid (v|z)_{L_2} = 0 \text{ for all } z \in \mathcal{E}\}, \quad j \in \{0,1\}.$$
Note that $H^j_{2,\sigma}(\Sigma, \mathrm{T}\Sigma) = \mathcal{E} \oplus V_2^j(\Sigma)$.

Proposition (S, Wilke. 2021)

Let $d = 2$. Suppose $v_0 \in V_2^0(\Sigma)$ and let v be the solution with initial value v_0. Then

(a) *$v(t) \in V_2^0(\Sigma)$ for $t \in [0, t^+(v_0))$ and $v(t) \in V_2^1(\Sigma)$ for $t \in (0, t^+(v_0))$.*

(b) *There exists a universal constant $M > 0$ such that*
$$|v(t)|^2_{L_2(\Sigma)} + \int_0^t |v(s)|^2_{H_2^1(\Sigma)}\, ds \le M|v_0|^2_{L_2(\Sigma)}, \quad t \in (0, t^+(v_0)).$$

Let

$$V_2^j(\Sigma) := \{v \in H^j_{2,\sigma}(\Sigma, \mathrm{T}\Sigma) \mid (v|z)_{L_2} = 0 \text{ for all } z \in \mathcal{E}\}, \quad j \in \{0,1\}.$$

Note that $H^j_{2,\sigma}(\Sigma, \mathrm{T}\Sigma) = \mathcal{E} \oplus V_2^j(\Sigma)$.

Proposition (S, Wilke. 2021)

Let $d = 2$. Suppose $v_0 \in V_2^0(\Sigma)$ and let v be the solution with initial value v_0. Then

(a) $v(t) \in V_2^0(\Sigma)$ *for* $t \in [0, t^+(v_0))$ *and* $v(t) \in V_2^1(\Sigma)$ *for* $t \in (0, t^+(v_0))$.

(b) *There exists a universal constant* $M > 0$ *such that*

$$|v(t)|^2_{L_2(\Sigma)} + \int_0^t |v(s)|^2_{H^1_2(\Sigma)}\, ds \le M|v_0|^2_{L_2(\Sigma)}, \quad t \in (0, t^+(v_0)).$$

(c) $t^+(v_0) = \infty$ *and there exists a constant* $\alpha > 0$ *such that*

$$|v(t)|_{L_2(\Sigma)} \le e^{-\alpha t}|v_0|_{L_2(\Sigma)}, \quad t \ge 0.$$

Proof:

(b) Energy estimate & Korn's inequality for functions in $V_2^1(\Sigma)$.

(c) Global existence: H^∞-calculus, critical spaces & result in [Prüss, S, Wilke '18]. □

Theorem (Main Theorem for 2D-surfaces)

Suppose $d = 2$ *and* $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Then the solution exists globally and converges exponentially fast to the equilibrium $u_* = P_{\mathcal{E}} u_0$ *in the topology of* $H^2_q(\Sigma, \mathsf{T}\Sigma)$ *for any fixed* $q \in (1, \infty)$, *where* $P_{\mathcal{E}}$ *is the (orthogonal) projection of* u_0 *onto* $\mathcal{E}$.

Proof: Let $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$ be given. Then

$$u_0 = u_* + v_0$$

with $u_* = P_{\mathcal{E}} u_0$ and $v_0 \in V^0_2(\Sigma)$. Let v be the global solution with initial value v_0 and let

$$u(t) = u_* + v(t), \quad t \geq 0.$$

We know that

$$|v(t)|_{L_2(\Sigma)} \leq e^{-\alpha t} |v_0|_{L_2(\Sigma)}, \quad t \geq 0.$$

Using L_q-L_q maximal regularity and reiteration, we can show exponential convergence in the topology of $H^2_q(\Sigma, \mathsf{T}\Sigma)$ for any fixed $q \in (1, \infty)$.

References

- G. Simonett, M. Wilke.
  H^∞-calculus for the surface Stokes operator and applications.
  arXiv:2111.12586. Submitted for publication.

- J. Prüss, G. Simonett, M. Wilke.
  On the Navier-Stokes equations on surfaces.
  J. Evol. Equ. **21** (2020), 3153-3179.

- J. Prüss, G. Simonett, M. Wilke.
  Critical spaces for quasilinear parabolic evolution equations and applications.
  J. Differential Equations **264** (2018), 2028-2074.

- J. Prüss, G. Simonett.
  Moving Interfaces and Quasilinear Parabolic Evolution Equations.
  Monographs in Mathematics **105**, Birkhäuser 2016.

Outlook

- Manifolds with boundary.
- Free boundary problems on manifolds.
- Navier Stokes equations on moving surfaces.