Energy balance for 2D incompressible fluid flow

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Collaborators:

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Roman Shvydkoy (Univ. Illinois, Chicago)

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Smooth inviscid flows ($\nu = 0$) conserve kinetic energy

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Brief history...

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  Involves studying optimal conditions for energy flux to vanish.

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Note:

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- Smooth vorticity transported in 2D, L^p bounds preserved by evolution
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2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

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Note S_q is a convolution with a mollifier.

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The family $\{u^\nu\}$ is called a *physical realization* of u.

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The proof is concluded.

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Physically realizable weak solutions: $F^\nu \rightharpoonup F$ weakly $L^1(0,T;L^2(\mathbb{T}^2))$

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Proposition

Under the hypotheses of the Theorem,

$$\lim_{\nu \to 0^+} \nu \int_0^t \|\omega^\nu(s,\cdot)\|_{L^2}^2 \, ds \to 0.$$

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This is enough to conclude the proof of the Proposition.

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