Low regularity solutions for quasilinear dispersive PDEs

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Quasilinear dispersive PDEs

Quasilinear PDEs:

- Textbook definition: Coefficients of terms with highest order depend on lower order derivatives
- Better definition: The evolution depends only continously on the initial data.
- At a technical level: cannot be viewed as a perturbation of a linear equation (i.e. contraction principle approach does not work)

Example: Nonlinear (quasilinear) wave equations in $\mathbb{R}^{1+n}$

$$g^{\alpha\beta}(u)\partial_\alpha\partial_\beta u = N(u, \partial u) \qquad (NLW)$$

with initial data:

$$u(t=0) = u_0, \qquad u_t(t=0) = u_1$$

Fundamental questions about PDEs

- **Local well-posedness** in Sobolev spaces [Hadamard, enhanced]

$$(u_0, u_1) := u[0] \in \mathcal{H}^s \quad (:= H^s \times H^{s-1} \ \text{ for (NLW)})$$

  - existence of solutions u in the class $C(0, T; \mathcal{H}^s)$
  - uniqueness of solutions, either directly or as unique limits of smooth solutions
  - continuous dependence in $\mathcal{H}^s$, i.e. continuity of the data to solution map

$$\mathcal{H}^s \ni u(0) \to u \in C(0, T; \mathcal{H}^s)$$

  - weak Lipschitz dependence, i.e. for two $\mathcal{H}^s$ solutions u and v we have the difference bound

$$\|u - v\|_{C(0,T;\mathcal{H}^{s_0})} \lesssim \|u(0) - v(0)\|_{\mathcal{H}^{s_0}}$$

- **Long time behavior**
  - Extended lifespan of solutions for small data
  - Global solutions: scattering solutions, solitons, superpositions

Low regularity well-posedness: the scaling threshold

What is the lowest value of s for which an equation is Hadamard well-posed for initial data $u(0)$ in the Sobolev space $\mathcal{H}^s$?

Scaling symmetry:

$$(NLW): \qquad u(x,t) \to u(\lambda x, \lambda t)$$

Scaling threshold: Critical (scale invariant) Sobolev space for the initial data

$$s_c = \frac{n}{2}$$

Open question: Are nonlinear wave equations well-posed in $\mathcal{H}^s$ for all $s > s_c$?

Two classical methods

- **Energy estimates**
  - measure the growth of energy of the solutions in time
- **Strichartz estimates**
  - measure averaged decay of solutions to dispersive equations.

Energy Estimates and the classical threshold

Theorem (Hughes-Kato-Marsden ('76))

Nonlinear wave equations are locally well-posed in $\mathcal{H}^s$ for

$$s > s_c + 1.$$

Classical energy estimates for (NLW):

$$\frac{d}{dt} E^\sigma(u) \lesssim \|\nabla u\|_{L^\infty} E^\sigma(u), \qquad E^\sigma(u[t]) \approx \|u[t]\|^2_{\mathcal{H}^\sigma}$$

or by Gronwall

$$\|u[t]\|^2_{\mathcal{H}^\sigma} \lesssim e^{\int_0^t \|\nabla u(t_1)\|_{L^\infty} dt_1} \|u[0]\|^2_{\mathcal{H}^\sigma}$$

- Sobolev embeddings for $s > s_c + 1$: $\|\nabla u(t)\|_{L^\infty} \lesssim \|u[t]\|_{\mathcal{H}^s}$.
- Continuation criteria: Solutions can be continued as long as

$$\int_0^T \|\nabla u(t)\|_{L^\infty} dt < \infty.$$

The next step: Strichartz estimates

For the homogeneous wave equation $\Box u = 0$:

$$\|\nabla u\|_{L^2L^\infty} \lesssim \|u[0]\|_{\mathcal{H}^{\frac{n}{2}+\frac{1}{2}}}, \qquad n \geq 3$$

$$\|\nabla u\|_{L^4L^\infty} \lesssim \|u[0]\|_{\mathcal{H}^{\frac{n}{2}+\frac{3}{4}}}, \qquad n = 2$$

- If true for $\Box_g$, would give LWP for NLW for
    - $s > s_c + \frac{1}{2}$ $(n \geq 3)$

    - $s > s_c + \frac{3}{4}$ $(n = 2)$.
    - But we only have $\nabla g \in L^2L^\infty$! Is this enough ?

Strichartz estimates for variable coefficient metrics:

- $g \in C^\infty$: Lindblad-Sogge, Kapitanskii
- $g \in C^2$: Smith $(n = 2, 3)$
- $\nabla^2 g \in L^1L^\infty$: Tataru (all n)
- counterexamples below: Smith-Tataru

Strichartz estimates with loss of derivatives

Work with paradifferential form of the equation:

$$g^{\alpha\beta}_{<\lambda^\gamma}\partial_\alpha\partial_\beta u_\lambda = \text{perturbative}, \qquad \frac{1}{2} \leq \gamma \leq 1.$$

- For each frequency λ, find "semiclassical" time $\delta t \approx \lambda^{-\delta}$ scales where loss-less Strichartz holds.
- Add these bounds to get Strichartz with derivative losses on unit time.

Multiple iterations:

- Bahouri-Chemin ('98-'99), $s_c + \frac{1}{2} + [\frac{1}{5}, \frac{1}{4}]$
- Tataru ('98-'99), $s_c + \frac{1}{2} + [\frac{1}{6}, \frac{1}{4}]$

Sharp LWP for generic (NLW)

Theorem (Smith-Tataru ('01))

Nonlinear wave equations are locally well-posed in $\mathcal{H}^s$ for

$$s > s_c + \frac{1}{2}\ (n \geq 3), \qquad s > s_c + \frac{3}{4}\ (n = 2)$$

- Classical energy + (nearly) loss-less Strichartz estimates
- This result is sharp (Lindblad's counterexample)
  Heuristic: self-interaction of wave packets.
- intermediate step, Klainerman-Rodnianski: wave equation for g; then same result for Einstein equation.

Question:

Can this result be improved for equations with a "better structure"?

The null condition conjecture

Null condition: kills the self-interaction of wave packets along null geodesics

$$g^{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u = 0 \qquad (DNLW)$$

Definition

(DNLW) satisfies the *nonlinear null condition* if

$$\frac{\partial g^{\alpha\beta}(p)}{\partial p_\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0 \qquad \text{in} \qquad g^{\alpha\beta}(p)\xi_\alpha\xi_\beta = 0.$$

Conjecture (Tataru, ICM '02)

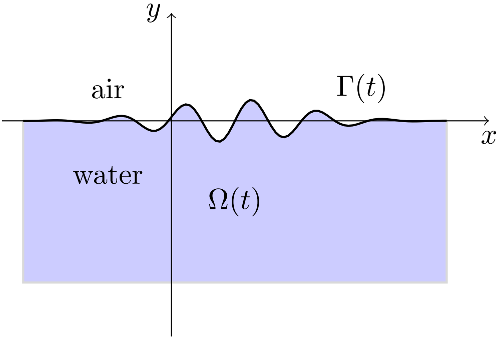
The LWP regularity threshold can be improved over the Smith-Tataru result for nonlinear wave equations satisfying the nonlinear null condition.

- Same conjecture can be applied to other nonlinear PDEs satisfying a *nonlinear null condition*.

A story of two equations

1. Water waves equation
2. Minimal surface equation in Minkowski space

Water waves: A free boundary problem



- Fluid flows inside the domain $\Omega(t)$ (infinite depth)
- Free boundary motion $\Gamma(t)$

Water waves: A free boundary problem

- Euler equations (on fluid domain $\Omega(t)$):

$$\begin{cases} (\partial_t + v \cdot \nabla) v = -\nabla p - g\mathbf{j} & \text{(Newton's law)} \\ \nabla \cdot v = 0 & \text{(incompressibility)} \end{cases}$$

- Boundary conditions (on interface $\Gamma(t)$)

$$\begin{cases} \partial_t + v \cdot \nabla \text{ is tangent to } \bigcup \Gamma(t) & \text{(kinematic)} \\ p = 0 \; (= -2\sigma H) \quad \text{on } \Gamma(t) & \text{(dynamic)} \end{cases}$$

- Irrotationality:

$$\nabla \times v = 0 \qquad \text{(propagated along the flow)}$$

Water waves: Reduction to surface

- Irrotationality ⇒ scalar velocity potential:

$$v = \nabla\phi$$

- Incompressibility ⇒ Laplace equation:

$$\Delta\phi = 0 \quad \text{in } \Omega(t)$$

- Reduction to equation of motion on free boundary (Zakharov '68) for W=surface parametrization, Q=holomorphic velocity potential

$$\begin{cases} W_t + F(1 + W_\alpha) = 0, \\ Q_t + FQ_\alpha + P\left[\dfrac{|Q_\alpha|^2}{J}\right] - igW = 0. \end{cases}$$

where

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |1 + W_\alpha|^2.$$

P=Projection onto negative wavenumbers

Minimal surfaces: nonlinear waves with null condition

- A time-like submanifold $\Sigma \subseteq \mathbb{R}^{n+2}$ of Minkowski space, critical point of
$$\int_\Sigma dA$$
- Euler-Lagrange equation (submanifold is viewed as a graph):
$$-\frac{\partial}{\partial t}\left(\frac{u_t}{\sqrt{1-u_t^2+|\nabla_x u|^2}}\right)+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1-u_t^2+|\nabla_x u|^2}}\right)=0$$
- Re-express using trace of Minkowski metric on Σ:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta u = 0, \qquad g_{\alpha\beta} = m_{\alpha\beta} + \partial_\alpha u \partial_\beta u$$

- aka. Born-Infeld in electromag., aka. zero mean curvature flow, aka. relativistic membrane equation, aka. branes in string theory

The improved low regularity results for null structure problems.

LWP for 2d gravity waves

Theorem (A. Ai, M. I., D. Tataru '19, '22 (in progress))

The 2 d gravity water waves system in infinite depth locally well-posed in $\mathcal{H}^s$ for

$$s > s_c + \frac{1}{4} \ \left(\frac{1}{8}\right).$$

- First low regularity result where the null structure is used
- loss-less Strichartz estimates due to Ai '18
- improves Alazard-Burq-Zuily by 5/24 derivatives, and Ai by 1/8 derivatives.

LWP for nonlinear waves with null condition

Theorem (A. Ai, M. I., D.Tataru '21)

The time-like minimal surface equation is locally well-posed in $\mathcal{H}^s$ for

$$s > s_c + \frac{1}{4}, \qquad n \geq 3$$

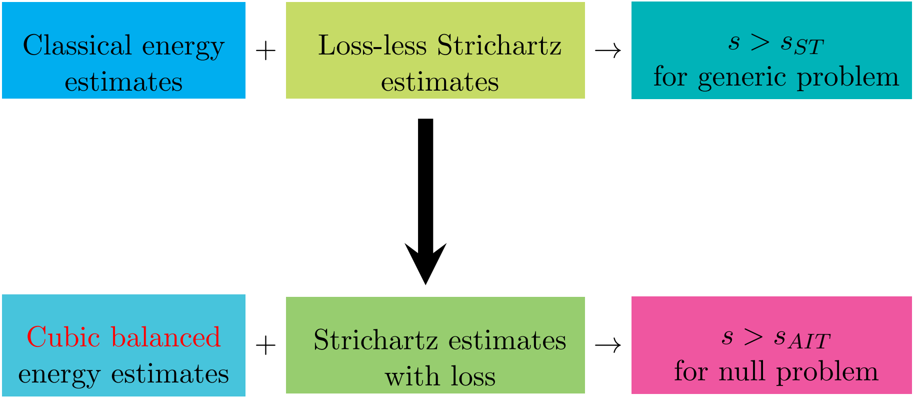
$$s > s_c + \frac{3}{8}, \qquad n = 2$$

- First result proving the null condition LWP conjecture
- improves Smith-T. by 1/4 derivatives if $n \geq 3$, and by 3/8 derivatives if $n = 2$.
- Prior ϵ-removal results by Klainerman-Rodnianski-Szeftel (GR) and Ettinger (minimal surface)

In a nutshell (for minimal surface)

Classical energy estimates $+$ Loss-less Strichartz estimates $\rightarrow$ $s > s_{ST}$ for generic problem

In a nutshell (for minimal surface)



Classical energy estimates + Loss-less Strichartz estimates → $s > s_{ST}$ for generic problem

Cubic balanced energy estimates + Strichartz estimates with loss → $s > s_{AIT}$ for null problem

...

And now for something completely different ...

The long time existence problem: baseline

Objective: Lifespan bounds for small data

- Equations with quadratic nonlinearities,

$$u_t + Au = B(u,u)$$

$$\frac{d}{dt}E(u) \lesssim \|u\| E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$, Gronwall provides lifespan $T_\epsilon \approx \epsilon^{-1}$

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- Equations with cubic nonlinearities:

$$u_t + Au = Q(u,u,u)$$

$$\frac{d}{dt}E(u) \lesssim \|u\|^2 E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$, Gronwall provides lifespan $T_\epsilon \approx \epsilon^{-2}$

The long-time existence problem: normal forms

Objective: Improved lifespan for small initial data

- Normal form method (Shatah '85): Transform equation with quadratic nonlinearities into one with cubic nonlinearities via

  $$u \mapsto v = u + B(u, u)$$

  New, cubic equation:

  $$v_t + Av = Q(u, u, u)$$

  - Requirement: Nonresonant or null resonant quadratic interactions
  - Difficulty: Not invertible for quasilinear problems

- Space-time resonances (Germain-Masmoudi-Shatah '09)

  - Requirement: Nonresonant or null resonant quadratic interactions
  - Some applications to quasilinear problems with stronger dispersion

The long-time existence problem: normal forms +

Objective: Improved lifespan for small data, quasilinear problems

- Paradiagonalization (Alazard-Delort '13) Combines a partial normal form with a paradifferential symmetrization

$$v_t + Av + T_{L(u)}v = Q(u,u,u)$$

  - microlocal based approach

- Modified energy method (Hunter-I.-Tataru '12-'14) Modify the energy functional rather than the unknown:

$$E^s(u) \approx \|u\|^2_{H^s}$$

$$\frac{d}{dt}E^s(u) \lesssim \|u\|^2 E^s(u)$$

  - Modified energy easily computed from normal form transform
  - Work on Burgers-Hilbert, gravity waves, capillary waves, and several other water wave models

Balanced energy estimates for gravity waves

- Baseline quadratic estimates Alazard-Burq-Zuily '11-15:

$$\frac{d}{dt} E^s(u) \lesssim \|u\|_{C^{\frac{1}{2}}} E^s(u)$$

- Modified energy, cubic estimates (Hunter-I.-Tataru '14):

$$\frac{d}{dt} E^{s,3}(u) \lesssim A_0 A_{1/2} E^{s,3}(u), \qquad A_\sigma = \|u\|_{BMO^\sigma}$$

- Modified energy, balanced cubic estimates (Ai-I.-Tataru '19):

$$\frac{d}{dt} E^{s,3}_{bal}(u) \lesssim A^2_{1/4} E^{s,3}_{bal}(u)$$

  - (variable coeff.) normal form for balanced frequency interactions
  - modified energy for the paradifferential problem

Bony's paradifferential formalism (expanded)

Original nonlinear equation:

$$u_t + N(u) = 0$$

Linearized equation:

$$v_t + DN(u)v = 0$$

Linear paradifferential equation:

$$w_t + T_{DN(u)}w = 0$$

Original equation in paradifferential formulation

$$u_t + T_{DN(u)}u = R(u)$$

Linearized equation in paradifferential formulation

$$v_t + T_{DN(u)}v = R_{lin}(u)v$$

Balanced normal form analysis

$$u_t + N(u) = 0 \qquad \Longleftrightarrow \qquad (\partial_t + T_{DN(u)})u = R(u)$$

Terms in $R(u)$:

- Quadratic $Q_2(u, u)$
  - Low-high $Q_2(u_{lo}, u_{hi})$, belongs to the paradiff. part.
  - $Q_2(u_{hi}, u_{hi})$, apply quadratic NFT, turns to cubic.
- Cubic $Q_3(u, u, u)$
  - Low–low-high $Q_3(u_{lo}, u_{lo}, u_{hi})$, goes into the paradiff. part.
  - Low-high-high $Q_3(u_{lo}, u_{hi}, u_{hi})$, apply quadratic NFT with coeff.
  - High-high-high $Q_3(u_{hi}, u_{hi}, u_{hi})$, perturbative (balanced).

Further difficulties:

- Also needed for the linearized equation: symmetry loss e.g. in $Q_3(v_{lo}, u_{med}, u_{hi})$, go to quartic order (WW) and/or Strichartz (MS).

Progression of LWP results: gravity waves 2D

		$s - s_c$
Wu ('97)	Energy	4
Alazard-Burq-Zuily ('12)	Energy	$\frac{1}{2} + \epsilon$
Hunter-Ifrim-Tataru ('14)	Cubic energy	$\frac{1}{2}$
Alazard-Burq-Zuily ('14)	Energy + Strichartz w. loss	$\frac{1}{2} - \frac{1}{24} + \epsilon$
Ai ('18)	Energy + sharp Strichartz	$\frac{3}{8} + \epsilon$
Ai-Ifrim-Tataru ('19)	Balanced cubic energy	$\frac{1}{4}$
(in progress) ('22)	Balanced cubic energy + Strichartz	$\frac{1}{8}$ (?)

Balanced energy estimates for minimal surface equation

- Classical energy estimates for (DNLW):

$$\frac{d}{dt} E^s(u) \lesssim \|\partial^2 u\|_{L^\infty} E^s(u)$$

- Cubic energy estimates for minimal surface:

$$\frac{d}{dt} E^s(u) \lesssim \|\partial u\|_{L^\infty} \|\partial^2 u\|_{L^\infty} E^s(u)$$

- Ai-I.-Tataru. '21: Balanced cubic energy estimates,

$$\frac{d}{dt} E^s_{bal}(u) \lesssim \|\partial^{\frac{3}{2}} u\|^2_{L^\infty} E^s_{bal}(u)$$

- Difficulties:
  - Normal form structure is weaker than for water waves
  - Need similar balanced estimates for the linearized equations, further weakening structure

Progression of results: Nonlinear wave equation 3D

		$s - s_c$
Hughes-Kato-Marsden ('76)	Energy	$1 + \epsilon$
Bahouri-Chemin ('98-'99)	Energy + Strichartz w. loss	$\frac{3}{4}$ to $\frac{7}{10}$
Tataru ('98-'99)	Energy + Strichartz w. loss	$\frac{3}{4}$ to $\frac{2}{3}$
Klainerman-Rodnianski ('00)	Energy + Strichartz w. loss	$\frac{3-\sqrt{3}}{2}$
Smith-Tataru ('01)	Energy + sharp Strichartz	$\frac{1}{2} + \epsilon$
Kl.-Rod.-Szeftel ('15) [*GR*]	Energy + sharp Strichartz	$\frac{1}{2}$
Ai-Ifrim-Tataru ('21) [*null*]	Balanced cubic energy + Strichartz w. loss	$\frac{1}{4} + \epsilon$

Thank you!

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