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On Energy Conservation for the hydrostatic Euler equations: an Onsager Conjecture

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Hydrostatic Euler

The hydrostatic Euler equations (inviscid Primitive equations of Oceanic and Atmospheric Dynamics) are given by

$$\partial_t \mathbf{u}_h + (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h + w\partial_z \mathbf{u}_h + \nabla p = 0,$$
$$\nabla \cdot \mathbf{u}_h + \partial_z w = 0, \quad \partial_z p = 0.$$

∇ is 2-dimensional and w is no longer an independent quantity.
The formally conserved quantity is $\|u\|_{L^2}^2 + \|v\|_{L^2}^2$.
Onsager's conjecture for the Euler equations:

- If $\mathbf{u}(\cdot, t) \in C^{0,\theta}$ with $\theta > \frac{1}{3}$ implies conservation of energy.
- If $\mathbf{u}(\cdot, t) \in C^{0,\theta}$ with $\theta < \frac{1}{3}$, energy dissipation is possible.

We looked at an analogue of this conjecture for the hydrostatic Euler equations.

Previous work I

For the Euler equations:

- First half of Onsager conjecture: [Eyink, 1994; Constantin, E, and Titi, 1994; Duchon and Robert, 2000; Cheskidov, Constantin, Friedlander, and Shvydkoy, 2008; Cheskidov, Lopes Filho, Nussenzveig Lopes, and Shvydkoy, 2016; Robinson, Rodrigo, and Skipper, 2018; Bardos, Gwiazda, Świerczewska-Gwiazda, Titi, and Wiedemann, 2019]
- Second half of Onsager conjecture: [De Lellis and Székelyhidi, 2009, 2010; Buckmaster, De Lellis, Isett, and Székelyhidi, 2015; Daneri and Székelyhidi, 2017; Isett, 2018; Buckmaster, De Lellis, Székelyhidi, and Vicol, 2018]

For the viscous primitive equations:

- Derivation: [Richardson, 1922; Lions, Temam, and Wang, 1992]

Previous work II

- Short-time existence: [Guillén-González, Masmoudi, and Rodríguez-Bellido, 2001]
- Global existence: [Cao and Titi, 2007; Kobelkov, 2006; Kukavica and Ziane, 2007; Hieber and Kashiwabara, 2016]
- Small-aspect ratio: [Azérad and Guillén, 2001; Bresch, Guillén González, Masmoudi, and Rodríguez Bellido, 2001; Li and Titi, 2019]

For the inviscid primitive equations:

- Ill-posedness in Sobolev spaces: [Renardy, 2009; Han-Kwan and Nguyen, 2016]
- Finite-time singularity: [Wong, 2015; Cao, Ibrahim, Nakanishi, and Titi, 2015]

Previous work III

- Local well-posedness for analytic data: [Kukavica, Temam, Vicol, and Ziane, 2011; Gerard-Varet, Masmoudi, and Vicol, 2020]
- With rotation: [Ibrahim, Lin, and Titi, 2021; Ghoul, Ibrahim, Lin, and Titi, 2022]
- Nonuniqueness of weak solutions for the inviscid case: [Feireisl, 2016; Chiodaroli and Michálek, 2017]

Main differences with the Euler equations

The equation for w can be written as

$$w = -\int_0^z \left(\partial_x u + \partial_y v\right) dz'.$$

- Nonlocality
- Anisotropy in regularity

Two different notions for weak solutions

- Assume that $w \in L^2(L^2)$ and $u, v \in L^\infty(L^2)$ (weak solution)
- Assume that $w \in L^2(B^{-s}_{2,\infty})$ and $u, v \in L^4(B^{s+}_{4,2})$ for $0 < s < \frac{1}{2}$ (very weak solution)

Results

Theorem (DB-Markfelder-Titi)

Energy is conserved under any of the following conditions.

- *If* $u, v \in L^4(B^{1/2+}_{4,\infty})$ *and* $w \in L^2(L^2)$
- *If* $w \in L^3(C^\beta)$ *and* $u, v \in L^3(C^\alpha)$ *with* $\alpha > 1 - \frac{1}{2}\beta$
- *If* $w \in L^2(L^2)$ *and u and v have Besov regularity* $B^\alpha_{3,\infty}$ *vertically and* $B^\beta_{3,\infty}$ *horizontally if* $\alpha > \frac{1}{3}, \beta > \frac{2}{3}$ *and* $\beta + 2\alpha > 2$
- *For very weak solutions with* $w \in L^2(B^{-s}_{2,\infty})$*, if* $u, v \in L^4(B^{s+1/2+}_{4,\infty})$
- *If* $u, v \in L^3(B^{3/4+}_{3,\infty})$ *(with no conditions on w)*

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