# Large global solutions of the parabolic-parabolic Keller–Segel system in higher dimensions

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The parabolic-parabolic Keller–Segel system reads

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \varphi), \\ \tau \varphi_t = \Delta \varphi + u, & x \in \mathbb{R}^d, \ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{PP}$$

Biological motivations: **chemotaxis**.

- $u(x,t) \geq 0$: density of the population of micro-organisms
- $\varphi(x,t) \geq 0$: density of chemoattractant.

In many interersting biological situations, $0 < \tau \ll 1$. This means that diffusion for the chemoattractant is much faster than for cells.

The limit case $\tau = 0$ is also of interest in astrophysics (e.g. for the dynamics of nebulae). It leads to the parabolic-elliptic Keller–Segel system:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \varphi), \\ \Delta \varphi + u = 0, & x \in \mathbb{R}^d, \ t > 0, \\ u(0) = u_0. \end{cases} \tag{PE}$$

**Examples of finite time blowup results for (PE).**

Let $u_0 \geq 0$, Under one of the following conditions, the local solution to (PE) **cannot exist globally:**

- (d=2, beyond critical mass). $u_0 \in L^1(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} u_0(x)\,\mathrm{d}x > 8\pi \qquad \text{(sharp threshold)}$$

[Jäger, S. Luckhaus, 1992]
[Herrero, Velázquez, 1997]
[Blanchet, Laurençot, Perthame, 2006].

- (High concentration, $d \geq 3$). $u_0 \in L^1(\mathbb{R}^d, (1+|x|^\gamma)\,\mathrm{d}x)$ for some $1 < \gamma < 2$ and

$$\frac{\int_{\mathbb{R}^d} |x|^\gamma u_0(x)\,\mathrm{d}x}{\int_{\mathbb{R}^d} u_0(x)} \leq \epsilon_d \Big(\int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x\Big)^{\gamma/(d-2)}$$

[Biler 1995].

Other criteria, e.g. [Biler, Zienkiewicz 2019], [Naito 2021].

**Finite time blowup for (PP).**

Results available only for a class of radial solutions [Winkler, 2020].

**Scale invariance:** $u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t), \quad \varphi_\lambda(x,t) = \varphi(\lambda x, \lambda^2 t).$
$u_{0,\lambda}(x) = \lambda^2 u_0(\lambda x), \qquad \varphi_{0,\lambda}(x) = \varphi(\lambda x).$

It is often assumed, in the literature, that $\varphi_0 = 0$.

Examples of scale invariant spaces for $u_0$ where the **global well-posedness** problem for **small** $u_0$ was addressed (or local well-posedness for large data):

- $L^{d/2}$ [Corrias, Perthame 2006].
  When $d = 2$, measure space $\mathcal{M}(\mathbb{R}^2)$ is an alternative to $L^1(\mathbb{R}^2)$.
  If $u_0 \geq 0$, $\int u_0$ (the total mass) is conserved.
- $\mathcal{PM}^{d-2}$ [Biler,Cannone,Guerra,Karch, 2004]
- $L^{d/2,*}$ and $\dot{H}_p^{-2+d/p}$ [Kozono, Sugiyama, 2008, 2009, 2010]
- $\dot{B}^{-2+d/p}_{p,\infty}$, $p < \infty$ [Iwabuchi, 2011]

A few of the above references dealt only with (PE).

One motivation to deal with rough spaces is to include Chandrasekhar stationary solution of (PE)

$$u_C(x) = 2(d-2)/|x|^2, \qquad (d \geq 3)$$

or to study self-similar solutions.

## Well-posedness in the largest space.

The scale-invariant function spaces considered before are all embedded into

$$\dot{B}^{-2}_{\infty,\infty}.$$

In fact, this is the maximal translation-invariant, homogeneous Banach space of tempered distributions such that $\|f(\lambda\cdot)\| = \lambda^{-d/2}\|f\|$.

No existence theory is known if one only assumes $u_0 \in \dot{B}^{-2}_{\infty,\infty}$. But, for $u_0 \geq 0$,

$$c_d\|u_0\|_{\dot{M}^{d/2}} \leq \|u_0\|_{\dot{B}^{-2}_{\infty,\infty}} \leq C_d\|u_0\|_{\dot{M}^{d/2}}.$$

Here $\dot{M}^q(\mathbb{R}^d)$ is the Morrey space of locally finite measures $\mathrm{d}\mu$ such that

$$\|\mathrm{d}\mu\|_{\dot{M}^q} = \sup_{x\in\mathbb{R}^d,\, r>0}\left(r^{d(\frac{1}{q}-1)}\int_{B(x,r)}\mathrm{d}|\mu|\right) < \infty.$$

**Theorem (Lemarié-Rieusset, 2013)**

*Let $d \geq 2$, $\tau \geq 0$, $\varphi_0 = 0$ and $u_0 \in \dot{M}^{d/2}(\mathbb{R}^d)$. There exists $\delta > 0$ (only dependent on $d$) such that if*

$$\|u_0\|_{\dot{M}^{d/2}} < \delta,$$

*(PP) and (PE) has a mild solution $u \in E_\beta$.*

Here, $\frac{1}{2} < \beta < 1$ and

$$E_\beta = \{u \colon \sup_{t>0} t^{\beta/2} u(x,t) \in \dot{M}^{d/(2-\beta)}_{2/(2-\beta),*}\},$$

where $\dot{M}^{d/(2-\beta)}_{2/(2-\beta),*}$ is the Morrey-Marcinkiewics space.

- Bilinear estimates are uniform in $\tau$
- Convergence results (PP)$\to$ (PE) in the $E_\beta$-norm as $\tau \to 0$ do hold. (first result in this direction by [Raczinski 2009]).

**Is it possible to improve the size condition on $u_0$, when $\tau$ is large ?**

Some explicit conditions leading to the **global solvability** of (PP) (in the case $\varphi_0 = 0$ and $\tau$ large):

- $\|u_0\|_{\dot{M}^{d/2}(\mathbb{R}^d)} \lesssim 1$, $d \geq 2$, (independent on $\tau$) [Lemarié 2013].
- $\|u_0\|_{L^1(\mathbb{R}^2)} \lesssim \tau^{1/2-\epsilon}$, $d = 2$, [Corrias, Escobedo, Matos, 2014].
- $\|u_0\|_{\mathcal{M}(\mathbb{R}^2)} \lesssim \tau^{1/2-\epsilon}$, $d = 2$, [Biler, Guerra, Karch, 2015].

**Our first contribution for $\tau \gg 1$:**

- *We extend [BGK] and [CEM] to $d \geq 2$.*
- *We drop the $\epsilon$. So $\|u_0\| \approx \sqrt{\tau}$ is admissible.*

## Analysis of (PP) in Besov spaces

For sake of simplicity we limit ourselves to the case $\varphi_0 = 0$.

Let us introduce the linear and bilinear operators:

$$\mathbb{L}u(t) := \tau^{-1}\int_0^t \nabla \mathrm{e}^{\tau^{-1}(t-s)\Delta} u(s)\,\mathrm{d}s \qquad (\text{so that } \nabla\varphi = \mathbb{L}u),$$

$$B(u,v)(t) := -\int_0^t \nabla \mathrm{e}^{(t-s)\Delta}\cdot\big(u(s)\,\mathbb{L}v(s)\big)\,\mathrm{d}s.$$

Then the integral formulation of (PP) is

$$u = U_0 + B(u,u), \qquad \text{with} \quad U_0(t) = e^{t\Delta}u_0.$$

For $d = 2$ [BGK] and [CEM], studied this by **fixed point** in a subspace of $C_w([0,\infty),\mathcal{M}(\mathbb{R}^2))$. We choose a **different space:**

$$\mathcal{E}_p := \Big\{ u \in L^\infty_{\mathrm{loc}}(0,\infty;L^p(\mathbb{R}^d)),\ |||u|||_p := \operatorname{ess\,sup}_{t>0} t^{1-d/(2p)}\|u(t)\|_p < \infty \Big\}.$$

Besov spaces naturally appear:

$$u_0 \in \dot{B}^{-(2-d/p)}_{p,\infty}(\mathbb{R}^d) \iff \mathrm{e}^{t\Delta}u_0 \in \mathcal{E}_p \qquad (p > d/2).$$

Moreover,

$$\left|\left|\left| \mathrm{e}^{t\Delta}u_0 \right|\right|\right|_p \approx \|u_0\|_{\dot{B}^{-(2-d/p)}_{p,\infty}}.$$

The relevant bilinear estimate is

$$|||B(u,z)|||_p \le C\tau^{-1/2+d/2(1/p-1/q)}|||u|||_p|||z|||_p.$$

with $p$ and $q$ in an appropriate range. When $\tau \gg 1$, the best admissible choice is $q = p$ leads to:

**Theorem**

*Let $d \ge 2$, $d < p < 2d$, $u_0 \in \dot{B}^{-(2-d/p)}_{p,\infty}$ and $\varphi_0 = 0$. There exist constants $C_p, \kappa_{p,q} > 0$, independent of $\tau$ and $u_0$, such that if*

$$\|u_0\|_{\dot{B}^{-(2-d/p)}_{p,\infty}} < C_p\sqrt{\tau},$$

*then (PP) has a unique mild solution in a ball of $\mathcal{E}_p$.*

**Is $\|u_0\| \approx \sqrt{\tau}$ the largest possible size, in general, for the global existence, as $\tau \gg 1$ ?**

For an **heuristic answer** we introduce a couple of toy models.

In such toy models the nonlinear term $\nabla \cdot (u \nabla \varphi)$ is replaced by a nonlinearity of the same order and scaling, but without drift structure. Namely, by

1. $u\Delta\varphi$ for first toy model (TM), or
2. $(\Delta\varphi)^2$ for (TM').

We get in this way

$$\begin{cases} u_t = \Delta u - u\Delta\varphi, \\ \tau\varphi_t = \Delta\varphi + u, \qquad\qquad x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{TM}$$

The second model is

$$\begin{cases} u_t = \Delta u + (\Delta\varphi)^2, \\ \tau\varphi_t = \Delta\varphi + u, \qquad\qquad x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{TM'}$$

- Both models degenerate into the (quadratic) nonlinear heat equation in the parabolic elliptic limit $\tau = 0$.
- The steady states of (TM) and (TM') agree with those of (NLH).
- For both toy models the existence theory valid for (PP) goes through with some changes.

For example, we studied (TM) as a perturbation of (PP): the nonlinear term is rewritten as

$$u\Delta\varphi = \nabla\cdot(u\nabla\varphi) - \nabla u\cdot\nabla\varphi.$$

The bilinear estimate is now done in the space

$$\mathcal{F}_p := \Big\{ u \in \mathcal{C}((0,\infty), W^{1,p}(\mathbb{R}^d)) : \ [\![u]\!]_p := |||u|||_p + |||u|||_{1,p} < \infty \Big\},$$

where, as before, $|||u|||_p = \operatorname{ess\,sup}_{t>0} t^{1/2-d/2p}\|u(t)\|_p$. Moreover,

$$|||u|||_{1,p} := \operatorname{ess\,sup}_{t>0} t^{3/2-d/2p}\|\nabla u(t)\|_p.$$

*We establish the global existence of solutions to (TM) in $\mathcal{F}_p$ assuming, as before,*

$$\|u_0\|_{\dot B^{-(2-d/p)}_{p,\infty}} < C_p\sqrt{\tau}.$$

The additional gradient estimate brings the technical restriction $d \geq 3$.

## Finite time blowup for (TM)

Our approach is inspired by a blowup result by [Montgomery-Smith] for the "*cheap Navier–Stokes equation*".

**Theorem**

*Let* $\tau \geq 1$, $A > 0$, $\varphi_0 = 0$ *and* $u_0 \in \mathscr{S}(\mathbb{R}^d)$, *such that*

$$\widehat{u}_0 \geq A\mathbf{1}_{|\xi|\leq 1}.$$

*Let* $T^*$ *be the maximal lifetime of the (unique) classical solution to (TM). There exists a constant* $\kappa_d > 0$ *(only dependent on* $d$*) such that if*

$$A > \kappa_d\,\tau,$$

*then* $T^* < \infty$.

**Idea of the proof.**

Notice that

$$\widehat{\varphi}(\xi,t)=\tau^{-1}\int_0^t \mathrm{e}^{-\tau^{-1}(t-s)|\xi|^2}\widehat{u}(\xi,s)\,\mathrm{d}s.$$

Taking the Fourier transform in (TM) we get

$$\begin{aligned}\widehat{u}(\xi,t)&=\mathrm{e}^{-t|\xi|^2}\widehat{u}_0(\xi)-\int_0^t \mathrm{e}^{-(t-s)|\xi|^2}\widehat{u\Delta\varphi}(\xi,s)\,\mathrm{d}s\\ &=\mathrm{e}^{-t|\xi|^2}\widehat{u}_0(\xi)+(2\pi)^{-d}\int_0^t\int_{\mathbb{R}^d}\mathrm{e}^{-(t-s)|\xi|^2}\widehat{u}(\xi-\eta,s)|\eta|^2\widehat{\varphi}(\eta,s)\,\mathrm{d}\eta\,\mathrm{d}s.\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{u}(\xi,t)&=\mathrm{e}^{-t|\xi|^2}\widehat{u}_0(\xi)\\ &\quad+(2\pi)^{-d}\int_0^t\int_0^s\int_{\mathbb{R}^d}\frac{|\eta|^2}{\tau}\mathrm{e}^{-(t-s)|\xi|^2}\mathrm{e}^{-\frac{1}{\tau}(s-\sigma)|\eta|^2}\widehat{u}(\xi-\eta,s)\widehat{u}(\eta,\sigma)\,\mathrm{d}\eta\,\mathrm{d}\sigma\,\mathrm{d}s.\end{aligned}$$

The important feature is that the inequalities $\widehat{u}_0\ge\widehat{w}_0\ge 0$ are preserved by the (TM) flow.

An appropriate choice is

$$\widehat{w}_0(\xi) = \mathbf{1}_{B_0}(\xi),$$

where $B_0$ is the ball with center $\frac{3}{4}(1, 0 \ldots, 0)$ and radius $\frac{1}{4}$. Thus, the support of $\widehat{w}_0$ is contained in the annulus $E_0 = \{\frac{1}{2} \leq |\cdot| \leq 1\}$.

By contradiction, assume $T > 1$ and let $u \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$ be a solution to (TM) with $\widehat{u} \geq 0$. Let, for any integer $k \geq 1$,

$$\widehat{w}_k = (2\pi)^{-d} \widehat{w}_{k-1} * \widehat{w}_{k-1},$$

and consider the dyadic ball

$$B_k = B_{k-1} + B_{k-1}.$$

We have

$$\operatorname{supp} \widehat{w}_k \subset B_k \subset \left\{\xi \in \mathbb{R}^d \colon 2^{k-1} \leq |\xi| \leq 2^k\right\}.$$

The key step is

**Lemma**

*If* $u_0 \in \mathscr{S}(\mathbb{R}^d)$, $\widehat{u}_0 \geq \widehat{w}_0$, *then for all* $k = 0, 1, 2, \ldots$, *and* $t \in [0, T]$, *we have*

$$\widehat{u}(\xi, t) \geq \beta_k \mathrm{e}^{-2^k t} \mathbf{1}_{\{1-4^{-k} \leq t < T\}}(t) \widehat{w}_k(\xi),$$

*for* $\beta_k \gtrsim (cA/\tau)^{2^k}$.

The blowup result $T^* < 1$ follows when $A$ (=the size of $\|u_0\|$) is $\gtrsim \tau$.

**How can we reduce the gap for $\tau \gg 1$ ?**

So far:

- Global existence for $\|u_0\| \lesssim \sqrt{\tau}$ (PP), (TM), (TM')
- Possible blowup for $\|u_0\| \gtrsim \tau$ (TM)

Let us try to work in a space smaller than $\dot{B}^{-(2-d/p)}_{p,\infty}$.

Let us introduce the pseudomeasure space, for $a \geq 0$,

$$\mathcal{PM}^a = \{f \in \mathscr{S}'(\mathbb{R}^d)\colon \|f\|_{\mathcal{PM}^a} = \operatorname{ess\,sup}_{\xi \in \mathbb{R}^d} |\xi|^a |\widehat{f}(\xi)| < \infty\},$$

In particular, for any $\lambda > 0$,

$$\|\lambda^2 u_0(\lambda\cdot)\|_{\mathcal{PM}^{d-2}} = \|u_0\|_{\mathcal{PM}^{d-2}}$$

The idea of using pseudomeasure spaces $\mathcal{PM}^a$ for Keller–Segel goes back to [Biler, Cannone, Guerra, Karch 2004]. They constructed, for (PE), the solutions directly in

$$\mathcal{X} = L^\infty(0,\infty;\mathcal{PM}^{d-2}).$$

We will rather construct our solutions in

$$\mathscr{Y}_a = \{u \in L^\infty_{\mathrm{loc}}(0,\infty;\mathscr{S}'(\mathbb{R}^d)) \colon \|u\|_{\mathscr{Y}_a} = \operatorname{ess\,sup}_{t>0,\,\xi\in\mathbb{R}^d} t^{1+(a-d)/2}|\xi|^a|\widehat{u}(\xi,t)| < \infty\}.$$

## Theorem

(i) *Let $d \geq 3$ and $\tau > 0$. If $u_0 \in \mathcal{PM}^{d-2}(\mathbb{R}^d)$, $\varphi_0 \in \mathscr{S}'(\mathbb{R}^d)$ satisfy one of the following size conditions*

$$\begin{aligned} \|u_0\|_{\mathcal{PM}^{d-2}} &< \kappa_d, \\ \sqrt{\tau}\,\|\nabla\varphi_0\|_{\mathcal{PM}^{d-1}} &< \tilde{\kappa}_d \end{aligned} \qquad (0 < \tau \leq 1)$$

*or otherwise,*

$$\begin{aligned} \|u_0\|_{\mathcal{PM}^{d-2}} &< \kappa_d\, b^3 \tau^{1-b}, \\ \|\nabla\varphi_0\|_{\mathcal{PM}^{d-1}} &< \tilde{\kappa}_d\, b^2 \end{aligned} \qquad (\text{for } \tau \geq 1 \text{ and some } 0 < b \leq 1),$$

*then (PP), (TM) and (TM') possess a global mild solution $u \in \mathcal{X}$.*

(ii) *There exists $a \in [d-2, d)$ and $R > 0$ such that $u$ belongs to $\{v \in \mathscr{Y}_a \colon \|v\|_{\mathscr{Y}_a} < R\}$, and is uniquely defined in this ball.*

**Remark** When $d = 2$, the above result holds for (PP), provided the smallness condition on $\varphi$ is strenghened, for $0 < \tau \leq 1$ as follows:

$$|\ln \tfrac{\tau}{\mathrm{e}}|\sqrt{\tau}\,\|\nabla\varphi_0\|_{\mathcal{PM}^{d-1}} < \tilde{\kappa}_2, \qquad (d = 2, \quad 0 < \tau \leq 1.)$$

**Remark**

- In the above theorem, the parameter $b \in (0, 1]$ can be tuned as we like: we can choose a function $b = b(d, \tau, \varphi_0, u_0)$. When $\tau \gg 1$ an interesting choice is $b = 3/\ln\tau$. Indeed, this is the choice allowing the weakest possible size condition for $u_0$ when $\tau$ is large.

For example, in the model case $\varphi_0 = 0$ we get the following result:

**Corollary**

*Let $d \geq 3$, $u_0 \in \mathcal{PM}^{d-2}$ and $\varphi_0 = 0$. If $\tau \geq \mathrm{e}^3$, then (PP), (TM) and (TM') possesses a global solution under the smallness condition*

$$\|u_0\|_{\mathcal{PM}^{d-2}} \leq \kappa'_d\, \tau/(\ln \tau)^3.$$

*Such a solution belongs to $\mathcal{X} \cap \mathscr{Y}_{d-4/\ln \tau}$ and is unique in a ball of $\mathscr{Y}_{d-4/\ln \tau}$ centered at the origin, with radius $0 < r \lesssim \tau/(\ln \tau)^3$.*

*For (PP) the assertion holds true also when $d = 2$.*

**Sketch of the proof for (PP) and $\varphi_0 = 0$.**

The key step is a pointwise integral estimate:

$$\int_0^t\int_0^s\int_{\mathbb{R}^d} \frac{|\xi|}{\tau}\mathrm{e}^{-(t-s)|\xi|^2}\mathrm{e}^{-\frac{1}{\tau}(s-\sigma)|\eta|^2}s^{-1+(d-a)/2}\sigma^{-1+(d-a)/2}|\xi-\eta|^{-a}|\eta|^{-a+1}\,\mathrm{d}\eta\,\mathrm{d}\sigma\,\mathrm{d}s \lesssim \frac{\tau^{b-1}}{(d-a)^2(d-a-b)}t^{-1+(d-a)/2}|\xi|^{-a},$$

valid for all $a$ and $b$ such that

$$\begin{cases} d-2b\le a<d-b \\ 0<b\le 1,\ a\neq 1 \end{cases} \quad (d\ge 3) \qquad \text{or} \qquad \begin{cases} \frac{3}{2}-b<a<2-b \\ 2-2b\le a \\ 0<b\le 1 \end{cases} \quad (d=2)$$

For $\tau \gg 1$ we should pick $0<b\ll 1$. A possible choice is $a = d-\frac{4}{3}b$. The above leads to the bilinear estimate:

$$\|B(u,v)\|_{\mathscr{Y}_{d-\frac{4}{3}b}} \le b^{-3}\tau^{b-1}\|u\|_{\mathscr{Y}_{d-\frac{4}{3}b}}\|v\|_{\mathscr{Y}_{d-\frac{4}{3}b}}.$$

The conclusion follows optimizing on $b$.

Thanks